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CHARACTERIZATIONS OF UNIQUE MAXIMAL SUBMODULE AND STRONG m-SYSTEM

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Abstract. In this paper we have introduced the concept of strong m-system in modules over non-commutative rings. Using the strong m-system, the existence of unique maximal submodule is proved. In fact we have shown that if I is a submodule and S is a strong m-system with $I \cap S = \emptyset$ then there exists a unique maximal submodule P with $I \subset P$ such that $P \cap S = \emptyset$. We have also obtained a characterization for unique maximal submodule.

Keywords: unique maximal submodule; strong m-system.

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1. Introduction

Throughout this paper R stands for a ring with identity and M stands for a unital left R-module. Prime submodules over the rings which are not necessarily commutative have been studied in a number of paper, for example [5],[1]. The notion of prime submodules was first introduced by J. Dauns in [2]. A proper submodule P of M is called a prime submodule, if for any ideal A of R and for any submodule N of M, $AN \subseteq P$ implies either $N \subseteq P$ or $AM \subseteq P$. David Ssevviiri[[4],Proposition 4.1.1] has shown that as in ring theory, a prime submodule P of M, is defined equivalently as, for any a in R and for any m in M, $aRm \subseteq P$ implies $m \in P$ or

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$aM \subseteq P$. So as in ring theory an m-system in a module should have been defined as follows. A nonempty subset S of M is said to be an m-system if for any m in S and for any a in R with $ak \in S$ for some k in M then there exists an r in R such that $arm \in S$. One can easily prove that a submodule P of M is prime submodule if and only if $C(P)$, the compliment of P is an m-system. But David Ssevviiri[4] has defined m-system in modules by taking sum of two submodules instead of taking two elements. Ofcourse they have shown that a submodule is prime if and only if it's compliment is an m-system. Following David Ssevviiri[4] we have the definition of an m-system for modules as follows. A subset $S \subseteq M \setminus \{0\}$ of M is an m-system if for any submodules K, L and if $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$ then $(K+AL) \cap S \neq \emptyset$. In this paper, we have introduced the notion of strong m-system in modules.

We have shown that a strong m-system in modules is always an m-system but the converse need not to be true. We have given an example of an m-system which is not a strong m-system. Using strong m-system we have proved the existence of a unique maximal submodule.

We have shown that if I is a submodule and S is a strong m-system with $I \cap S = \emptyset$ then there exists a unique maximal submodule P with $I \subset P$ such that $P \cap S = \emptyset$. If A, B are the submodules of M, one can easily check that $(A:B) = \{r \in R : rB \subseteq A\}$ is an ideal of R. For any $a \in R$, $< a >$ denotes the ideal generated by a.

2. PRELIMINARIES

If R is any ring with unity, we say that M is a unital left R-module if, for any $r \in R$ and $m \in M$, an element $rm \in M$ is defined such that the following conditions hold for all m, $n \in M$ and r, $s \in R$:

$$r(m+n) = rm + rn$$

$$(r+s)m = rm + sm$$

$$(rs)m = r(sm)$$

$$1m = m$$

A subset N of M is called an R-submodule if the following conditions are satisfied:

N is a subgroup of the (additive, abelian) group M.

rn is in N for all $r \in R$ and $n \in N$.

A proper submodule P of M is called a prime submodule, if for any ideal A of R and for any submodule N of M, $AN \subseteq P$ implies either $N \subseteq P$ or $AM \subseteq P$.

The definition of an m-system for modules as follows[4]. A subset $S \subseteq M \setminus \{0\}$ of M is an m-system if for any submodules K, L and if $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$ then $(K+AL) \cap S \neq \emptyset$.

3. Main Results

Theorem 3.1. *For a proper submodule P of M, the axioms that follows are equivalent*

(i) P is the unique maximal submodule.

(ii) For any ideal A of R and for submodules $K \neq M$, N of M such that $K \cap N = \{0\}$ and if $A(K+N) \subseteq P$, then either $K+N \subseteq P$ or $K+AM \subseteq P$.

(iii) For all $a \in R$ and for a submodule $K \neq M$ and for all m in M such that $K \cap Rm = \{0\}$ and if $a(K+Rm) \subseteq P$, then either $K+Rm \subseteq P$ or $K+aM \subseteq P$.

(iv) For all $a \in R$, submodule $K \neq M$ and for all m in M such that $K \cap Rm = \{0\}$ and if $< a > (K+Rm) \subseteq P$, then either $K+Rm \subseteq P$ or $K+ < a > M \subseteq P$.

(v) For every left ideal $A \subseteq R$, submodule $K \neq M$ and for all m in M such that $K \cap Rm = \{0\}$ and if $A(K+Rm) \subseteq P$, then either $K+Rm \subseteq P$ or $K+AM \subseteq P$.

(vi) For every right ideal $B \subseteq R$, submodule $K \neq M$ and for all m in M such that $K \cap Rm = \{0\}$ and if $B(K+Rm) \subseteq P$, then either $K+Rm \subseteq P$ or $K+BM \subseteq P$.

Proof $(i) \Longrightarrow (ii)$ Let A be an ideal of R and let $K \neq M$, N be the submodules of M such that $K \cap N = \{0\}$ and $A(K+N) \subseteq P$. As P is the unique maximal submodule, we have $K \subseteq P$. Suppose $K+N \nsubseteq P$. Then $(K+N)+P = M$. Let $m \in M$. Then there exists $k \in K$, $n \in N$ and $p \in P$ such that $m = (k+n)+p$. Let $a \in A$ and $k_1 \in K$. Then $k_1 + am = k_1 + a(k+n) + ap \in P$ since $k_1 \in K \subseteq P$, $a(k_1+n) \in A(K+N) \subseteq P$ and $ap \in AP \subseteq P$. Hence $K+AM \subseteq P$.

$(ii) \implies (i)$ First let us show that P is a maximal submodule. Suppose there exists a submodule K such that $P \subseteq K \subseteq M$ with $K \neq M$. If $A = \{0\}$ and $N = \{0\}$ then $A(K+N) \subseteq P$ and hence by assumption $K \subseteq P$. Thus $P = K$.

Suppose L is any other maximal submodule of M. By taking $A = \{0\}$ and $N = \{0\}$ we have $\{0\}(L+\{0\}) \subseteq P$. Hence by (ii) $L \subseteq P$. Hence P is the unique maximal submodule.

$(ii) \implies (iii)$ Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K+Rm) \subseteq P$. Now, $RaR(K+Rm) \subseteq Ra(K+Rm) \subseteq RP \subseteq P$. Hence $(K+Rm) \subseteq P$ or $K+(RaR)M \subseteq P$. Thus $(K+Rm) \subseteq P$ or $K+aM \subseteq P$.

$(iii) \implies (ii)$ Let A be an ideal of R. Suppose $K \neq M$, N are submodules of M in such a way that $K \cap N = \{0\}$ and $A(K+N) \subseteq P$. Let $a \in A$ and $n \in N$. Clearly $K \cap Rn = \{0\}$ and we have $a(K+Rn) \subseteq A(K+N) \subseteq P$. This implies that $K+Rn \subseteq P$ or $K+aM \subseteq P$. Hence $K+N \subseteq P$ or $K+AM \subseteq P$.

$(iii) \implies (iv)$ Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ such that $K \cap Rm = \{0\}$. Suppose $< a > (K+Rm) \subseteq P$. Then for all $x \in < a >$, $x(K+Rm) \subseteq < a > (K+Rm) \subseteq P$ and from (iii), $K+Rm \subseteq P$ or $K+xM \subseteq P$. This is true for all x in $< a >$. Hence (iv) holds.

$(iii) \implies (v)$ and $(iii) \implies (vi)$ are similar to the above proof.

$(iv) \implies (iii)$ Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K+Rm) \subseteq P$. Then $a \in (P : K+Rm)$. As P and $K+Rm$ are submodules of M, $(P : K+Rm)$ is an ideal of R. It follows that $< a > \subseteq (P : K+Rm)$ and hence $< a > (K+Rm) \subseteq P$. Hence $K+Rm \subseteq P$ or $K + < a > M \subseteq P$. Hence it is clear that $K+Rm \subseteq P$ or $K+aM \subseteq P$.

$(v) \implies (iii)$ Let $a \in R$. Let $K \neq M$ be a submodule and let $m \in M$ be such that $K \cap Rm = \{0\}$. Suppose $a(K+Rm) \subseteq P$. Then $a \in (P : K+Rm)$. It follows that $< a > \subseteq (P : K+Rm)$ since

$(P : K+Rm)$ is an ideal of R. Then $< a > (K+Rm) \subseteq P$.

As $< a >$ is an ideal of R, $< a >$ is a left ideal also. Hence $K+Rm \subseteq P$ or $K+ < a > M \subseteq P$. Hence $K+Rm \subseteq P$ or $K+aM \subseteq P$.

$(vi) \Longrightarrow (iii)$ Similar to the above proof.

This completes the proof .

4. UNIQUE MAXIMAL SUBMODULES IN TERMS OF STRONG m-SYSTEM

A nonempty set $S \subseteq M \setminus \{0\}$ is said to be a m-system in the sense of David Ssevviiri[4] if, for each ideal $A \subseteq R$ and for all submodules K, L of M, if $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$ then $(K+AL) \cap S \neq \emptyset$.

Now we define the notion of strong m-system.

Definition 4.1. A nonempty set $S \subseteq M \setminus \{0\}$ is called strong m-system if for each ideal $A \subseteq R$ and for all submodules K, L of M such that $K \cap L = \{0\}$ and if $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$ then $A(K+L) \cap S \neq \emptyset$.

For let us show that every strong m-system is an m-system. Let S be a strong m-system. Suppose A is an ideal of R and K, L are submodules of M such that $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$. If $K \cap L = \{0\}$, then $A(K+L) \cap S \neq \emptyset$. Since $A(K+L) \subseteq (K+AL)$, we have $(K+AL) \cap S \neq \emptyset$.

If $K \cap L \neq \{0\}$, let $K' = K+L$ and $L' = \{0\}$. Then $K' \cap L' = \{0\}$, $(K'+L') \cap S \neq \emptyset$ and $(K'+AM) \cap S \neq \emptyset$. Since S is a strong m-system $A(K'+L') \cap S \neq \emptyset$. Since $A(K+L) \subseteq K+AL$ we have $(K+AL) \cap S \neq \emptyset$ and it follows that S is an m-system.

Hence every strong m-system is an m-system. But the converse need not to be true. The following example shows that a m-system need not be a strong m-system.

Let $R = \mathbb{Z}_6$ be the ring and let the R-module be R_R. Then the subset $S = \{1,3,5\}$ is an m-system but not a strong m-system. Since $A(K+L) \cap S = \emptyset$ where $A = \{0,2,4\}$ is an ideal of $R = \mathbb{Z}_6$, $K = \{0,3\}$ and $L = \{0,2,4\}$ are submodules of $M = R_R$ with $K \cap L = \{0\}$ and $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$.

In the case of prime submodules, a submodule P of M is prime and $M \setminus P$ is an m-system are both equivalent. Now, let us extend the result to unique maximal submodules.

Theorem 4.2. *Let M be an R-module and let P be a submodule of M. Then P is a unique maximal submodule if and only if $M \setminus P$ is a strong m-system.*

Proof Let P be a unique maximal submodule. Let $S = M \setminus P$. Let A be an ideal of R. Let K, L be the submodules of M with $K \cap L = \{0\}$ and $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$. If $A(K+L) \cap S = \emptyset$, then $A(K+L) \subseteq P$. By Theorem 3.1, we have $K+L \subseteq P$ or $K+AM \subseteq P$. Hence $(K+L) \cap S = \emptyset$ or $(K+AM) \cap S = \emptyset$, leads a contradiction.
Thus $A(K+L) \cap S \neq \emptyset$ and hence S is a strong m-system.

Conversly, let $S = M \setminus P$ be a strong m-system. Suppose $A(K+L) \subseteq P$ where A is an ideal of R and $K \neq M$, L are the submodules of M be such that $K \cap L = \{0\}$.
If $K+L \nsubseteq P$ and $K+AM \nsubseteq P$, then $(K+L) \cap S \neq \emptyset$ and $(K+AM) \cap S \neq \emptyset$. As S is a strong m-system, we have $A(K+L) \cap S \neq \emptyset$. Then $A(K+L) \nsubseteq P$, leads a contradiction. Hence P is a unique maximal submodule.

The well known fact is, if $S \subseteq M$ is an m-system and if P is a submodule of M such that $P \cap S = \emptyset$ is maximal in concert with this property, then P is prime submodule. A similar result does hold for strong m-system.

Theorem 4.3. *Let $S \subseteq M$ be non-void strong m-system in M and I, a submodule of M with $I \cap S = \emptyset$. Then I is contained in a unique maximal submodule P with $P \cap S = \emptyset$.*

Proof Let $\mathscr{A}$ = $\{J : J$ is a submodule of M with $I \subseteq J$ and $J \cap S = \emptyset\}$. Clearly $I \in \mathscr{A}$. Then by Zorn's lemma, $\mathscr{A}$ contains a maximal element say P with $P \cap S = \emptyset$. Now to claim that P is unique maximal submodule of M.
If $A(K+L) \subseteq P$ where A is an ideal of R and $K \neq M$, L are the submodules of M in such a manner that $K \cap L = \{0\}$. Suppose $K+L \nsubseteq P$ and $K+AM \nsubseteq P$. Then by the maximality of P, we have $P+(K+L)$ and $P+(K+AM)$ are submodules and $P+(K+L) \cap S \neq \emptyset$ and $P+(K+AM) \cap S \neq \emptyset$. Now, let $L' = \{0\}$, the zero submodule of M. Then $((P+K+L)+L') \cap S \neq \emptyset$. Since $(P+K+AM) \cap S \neq \emptyset$, this implies $(P+K+L+AM) \cap S \neq \emptyset$. Let $P+K+L = K'$. Then $(K'+L') \cap S \neq \emptyset$ and $(K'+AM) \cap S \neq \emptyset$ with $K' \cap L' = \{0\}$. Since S is a strong m-system,

$A(K' + L') \cap S \neq \emptyset$. Thus $A((P+K)+L) \cap S \neq \emptyset$.

Since $A(P+(K+L)) \subseteq AP + A(K+L) \subseteq P$, a contradiction to the fact that $P \cap S = \emptyset$.

Hence P is a unique maximal submodule of M containing I.

CONCLUSION

In this paper, the properties of unique maximal submodules related with m-system is studied. We have estabilished the concept of strong m-system in modules over non-commutative rings and the existence of unique maximal submodule using the strong m-system. As a future work we will extend the results in semimodules over semirings.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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