Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 3027-3036
https://doi.org/10.28919/jmcs/5655
ISSN: 1927-5307

CONFORMABLE FRACTIONAL DERIVATIVE AND ITS APPLICATION TO PARTIAL FRACTIONAL DERIVATIVES

SABER T.R. SYOURI[1], IBRAHIM MOHAMMED SULAIMAN[1,*], MUSTAFA MAMAT[1,2],
SITI SABARIAH ABAS[1], MUHAMMAD ZAINI AHMAD[2]
[1]Faculty of Informatics and Computing, Universiti Sultan Zainal Abidin, 21300, Terengganu, Malaysia
[2]Institute of Engineering Mathematics, Universiti Malaysia Perlis, Arau, 02600, Malaysia

Abstract: Recently, the study on conformable fractional derivative has been gaining more attention from various researchers. This is due to its importance in numerous practical applications. In this paper, we introduce new definition of conformable fractional derivation to find partial fractional derivatives. The fractional derivatives are taken in the sense of Conformable fractional derivative. This approach is considered as a powerful tool for solving linear or nonlinear equations. The new formulation of fractional calculus is applied to solve many applied problems including fractional partial derivatives. The results obtained through the conformable fractional approach have been evaluated and presented to illustrate the efficiency and good accuracy. It has been observed that the proposed approach is prevailing for the solutions of partial fractional derivatives.

Keywords: α-partial differentiable; chain rules; conformable fractional partial derivatives; partial fractional calculus; Caputo; Riemann–Liouville; positive solution.

2010 AMS Subject Classification: 35B05, 35R11, 34K37.

---

[*]Corresponding author
E-mail address: sulaimanib@unisza.edu.my
Received March 7, 2021

1. INTRODUCTION

The theory of fractional derivatives is as old as the classical calculus and has been in existence since late 17th century [1]. This form of calculus is classified as a generalized fractional integrals or derivatives [2]. However, the subject just became more popular in recent years due to its wide application in numerous practical applications. Recently, there has been an increasing progress by several researchers to define the fractional derivative [3]. Most of these researchers are focusing on the fractional calculus due to the significant of fractional order derivatives in describing numerous real world physical phenomena [4]. Some of these studies include the application of fractional derivatives in modeling the fluid-dynamics capable of eliminating the deficiency developing from the occurrence of continuum flow of traffic [5]. More so, in would be interesting to know that the idea of a fractional derivative has advanced and playing essential role in the mathematical analysis of applied problems. Phenomena of a complex nature usually depend on more than one variable. These literatures and many more make the area of fractional calculus to becomes more convenient approach for describing the mathematical models of different aspects of physical and dynamical systems [1].

Recently, Khalil et al. [6] define fractional derivative using a new concept that obviously well-suited with the classical derivative definition. In contrast to the previous definitions, the authors present a definition that possess similar formulas of derivative of product and quotient of two functions with an even easier chain rule [7]. More so, these authors gave a new definition for the Mean value theorem for conformable fractional differentiable functions, Rolle theorem, and conformable fractional integral. Based on the above idea, [8] studied the fractional integrals of higher orders concepts and further presented left and right conformable fractional derivative. These studies discussed above has led to rigorous research on studies related to this new fractional derivative definition was done [9].

More recent literatures tend to studied the fractional calculus for functions with one variable

[6,10,11]. More so, [3,10] studied various skills for finding the most accurate solution of fractional derivative using Newton's fractional method by expanding the idea and knowledge of fractional calculus for all continuous functions $f(x)$.

In this paper, we present a new definition of partial fractional derivative and further show that the partial fractional derivative arises geometrically. This idea is extended to show how the partial fractional derivative is been calculated using certain defined rules. We begin with functions of two independent variables.

2. CONFORMABLE PARTIAL FRACTIONAL DERIVATIVE

Some definitions of fractional calculus have been introduced including partial derivative and partial integral. For fractional derivatives, there are actually many different definitions, and in general, these definitions are not equivalent to each other.

Definition 1 [6,11]: Let $f(x,y)$ be a function $f:[0,\infty)\to\mathbb{R}^2$. Then, the "conformable partial fractional derivative $\frac{\partial^\alpha}{\partial x^\alpha}$, D_x^α, f_x^α " of f with order α is defined by

$$\frac{\partial^\alpha}{\partial x^\alpha}f(x\,,y_0)=\frac{d^\alpha}{dx^\alpha}f(x,y_0)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon x^{1-\alpha},y_0)-f(x\,,y_0)}{\varepsilon}\,,\forall x>0\,,0<\alpha<1.$$

Definition 2 [6,11]: If f is a continuous function on $[a,\infty)$, $a>0$. Then,

$$I_\alpha^a(f(x))=I_1^a(x^{\alpha-1}f(x))=\int_a^x\frac{f(t)}{t^{1-\alpha}}dt.$$

Property 1.

If $f(x)$ is continuous on $(0,\infty)$ and α and β are positive real numbers then

$$D_{0,x}^{-\alpha}D_{0,x}^{-\beta}f(x)=D_{0,x}^{-\alpha-\beta}f(x).$$

Property 2.

If $f^{(m)}(x)$ is continuous on $(0,\infty)$ and $m-1<\alpha\le m$ then

$$D_{0,x}^{-\alpha}\, D_{0,x}^{-\alpha} f(x) = f(x) \ .$$

Property 3.

If $f^{(m)}(x)$ is continuous on $(0,\infty)$ and $m-1<\alpha\le m$ then

$$D_{0,x}^{-\alpha}\, D_{0,x}^{-\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!}\, x^k,$$

provided the limit exists.

Essentially, $\frac{\partial^\alpha}{\partial x^\alpha}$ is just the fractional derivative of f with respect to x, keeping the y variable fixed such that $\alpha \in (0,1)$. It can be expressed in various forms that achieve the same goal the partial fractional derivative of $f\,(x, y, \dots)$ is

$$[\ \frac{\partial^\alpha}{\partial x^\alpha}\ ,\ D_x^\alpha,\ f_x^\alpha\ ,\frac{\partial^\alpha}{\partial y^\alpha}\ ,\ D_x^\alpha,\ f_y^\alpha, \dots]$$

In the above description, we only give some basic properties about conformable fractional derivative and fractional integral which would be extended in the subsequent section. For more reference on other properties of conformable fractional derivative and related definitions of fractional calculus, we can refer to [4-22].

Based on the above definitions and the definition of fractional differentiation, we present the following definition which is generalized to partial fractional differentiation.

Definition 3. For m to be the smallest integer that exceeds α, the conformable partial fractional derivative operator of order $\alpha > 0$ is defined as

$$D_x^m u(y,x) = \frac{\partial^\alpha u(y,x)}{\partial x^\alpha} = \begin{cases} \lim\limits_{\varepsilon\to 0} \dfrac{f(x+\varepsilon x^{1-\alpha}, y_0) - f(x\,, y_0)}{\varepsilon}, & \forall x>0\,, 0<\alpha<1 \\ \dfrac{\partial^\alpha}{\partial x^\alpha} f(x\,, y_0) & \alpha = m \in \mathbb{N} \end{cases}$$

Next, using conformable fractional derivative approach on the new definition, we show that the product and quotient rules for partial fractional differentiation are defined such that $\alpha \in (0,1)$ as follows:

1. $\frac{\partial^\alpha}{\partial x^\alpha}(uv) = \frac{\partial^\alpha u}{\partial x^\alpha} v + \frac{\partial^\alpha v}{\partial x^\alpha} u$

2. $\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{u}{v}\right) = \frac{\frac{\partial^\alpha u}{\partial x^\alpha}v - \frac{\partial^\alpha v}{\partial x^\alpha}u}{v^2}$
3. $\frac{\partial^\alpha}{\partial y^\alpha}(uv) = \frac{\partial^\alpha u}{\partial y^\alpha}v + \frac{\partial^\alpha v}{\partial y^\alpha}u$
4. $\frac{\partial^\alpha}{\partial y^\alpha}\left(\frac{u}{v}\right) = \frac{\frac{\partial^\alpha u}{\partial y^\alpha}v - \frac{\partial^\alpha v}{\partial y^\alpha}u}{v^2}.$

Theorem 1.

Let c be any arbitrary real number and f be a function with two variables as $f(x, y)$, then, $D_x^\alpha f$ or $\frac{\partial^\alpha}{\partial x^\alpha} f$ as partial with respect to x, hence y is a constant.

1. $D_x^\alpha(e^{cx}) = cx^{1-\alpha}e^{cx}$.
2. $D_x^\alpha(\sin(cx)) = cx^{1-\alpha}\cos(cx)$.
3. $D_x^\alpha(\cos(cx)) = -cx^{1-\alpha}\sin(cx)$.
4. $D_x^\alpha(\tan(cx)) = cx^{1-\alpha}\sec^2(cx)$.
5. $D_x^\alpha(\cot(cx)) = -cx^{1-\alpha}\csc^2(cx)$.
6. $D_x^\alpha(\sec(cx)) = cx^{1-\alpha}\sec(cx)\tan(cs)$.
7. $D_x^\alpha(\csc(cx)) = -cx^{1-\alpha}\csc(cx)\cot(cs)$.

The proof of this theorem follows from [11].

3. MIXED FRACTIONAL PARTIAL DERIVATIVES

In this section, we define and denote the $\alpha-$ fractional partial derivatives with respect to x and likewise $\beta-$ fractional partial derivatives to same function f with respect to x, knowing that the real numbers α and β are subject to the following condition; $0 < \alpha < 1$ and $0 < \beta < 1$, as follows: -

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f}{\partial x^\beta}\right) = \frac{\partial^{\alpha+\beta} f}{\partial x^{\alpha+\beta}}$$

which can be rewritten as

$$f_{xx}^{\alpha+\beta} = D_{xx}^{\alpha+\beta}$$

Denoting the α–partial fractions derivatives with respect to x, then β–partial fractions derivatives of the same function f with respect to y as follows: -

$$f_{xy}^{\alpha+\beta} = \frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f}{\partial x^\alpha}\right) = \frac{\partial^{\beta+\alpha} f}{\partial y^\beta \partial x^\alpha}$$

which is equivalently written as

$$f_{xy}^{\alpha+\beta} = D_{xy}^{\alpha+\beta}$$

Next, we can denote the α–partial fractions derivatives with respect to y then β–partial fractions derivatives of the same function f with respect to x as follows: -

$$f_{yx}^{\alpha+\beta} = \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\alpha f}{\partial y^\alpha}\right) = \frac{\partial^{\beta+\alpha} f}{\partial x^\beta \partial y^\alpha}$$

and rewritten as

$$f_{yx}^{\alpha+\beta} = D_{yx}^{\alpha+\beta}$$

This explains why the order is opposite of what we expect – the factional derivative 'operates on the left'.

4. APPLICATION

In order to verify the accuracy of the method, this section present the solutions of some benchmark problems.

Example 1: Finding a partial fractions derivatives with respect to y. Find $\frac{\partial^\alpha}{\partial y^\alpha}$ if $f(x, y) = y\sin(xy)$, when $\alpha = \frac{1}{4}$.

Solution

We fixe x as a constant and f as product of y and $\sin(xy)$. By theorem 1, we have

$$\frac{\partial^{\frac{1}{4}} f}{\partial y^{\frac{1}{4}}} = \frac{\partial^{\frac{1}{4}}}{\partial y^{\frac{1}{4}}}(y\sin(xy)) = y\frac{\partial^{\frac{1}{4}}}{\partial y^{\frac{1}{4}}}(\sin(xy)) + \sin(xy)\frac{\partial^{\frac{1}{4}}}{\partial y^{\frac{1}{4}}}(y)$$

$$\frac{\partial^{\frac{1}{4}}}{\partial y^{\frac{1}{4}}}(y\sin(xy)) = y\left(xy^{\frac{3}{4}}\cos(xy)\right) + \sin(xy) \times y^{\frac{3}{4}}$$

$$\frac{\partial^{\frac{1}{4}}}{\partial y^{\frac{1}{4}}}(y\sin(xy)) = \sqrt[4]{y^3}(yx\cos(xy) + \sin(xy)) \quad \blacksquare$$

Example 2: Solve the following initial value problem

$$y^{\frac{1}{2}} + 2y = \begin{cases} 2, & 0 \le x < 1 \\ -2, & x \ge 0 \end{cases}$$

having initial condition $y(0) = 2$.

I. Find the explicit solution on the interval $0 \le x < 1$.

II. Find the explicit solution on the interval $x \ge 1$.

Solution

In this case, we divide into two cases. For the first case, let $F = e^{2x}$, then, we have

$$y^{\frac{1}{2}} = 2(1 - e^{2x}) \qquad 0 \le x < 1$$

$$I_\alpha^a(f(x)) = I_1^a(x^{\alpha-1}f(x)) = \int_a^x \frac{f(t)}{t^{1-\alpha}}dt$$

$$y.e^{2x} = I_{\frac{1}{2}}^1(f(x)).$$

Example 3: If $f(x,y) = x^2y + y^4$, solve for $\frac{\partial^{0.5} f}{\partial x^{0.5}}$ and $\frac{\partial^{0.5} f}{\partial y^{0.5}}$.

Solution

We begin by finding $\frac{\partial^{0.5} f}{\partial x^{0.5}}$. Let y_0 be constant, then we have

$$\frac{d^{0.5}}{dx^{0.5}} f(x, y_0) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-0.5}, y_0) - f(x, y_0)}{\varepsilon}$$

$\forall x > 0$ and $\alpha = \frac{1}{2}$,

$$= \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-0.5}, y_0) - f(x, y_0)}{\varepsilon}, \qquad \forall x > 0, \qquad \alpha = \frac{1}{2}.$$

$$= \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{0.5}, y_0) - f(x, y_0)}{\varepsilon}.$$

Suppose that $h = \varepsilon x^{0.5}$, then, $h \to 0$ when $\varepsilon \to 0$ and $\varepsilon = hx^{0.5}$ and suppose y is constant. Now using theorem 1, part 2,5, we have $D_{0.5}(x^p) = px^{p-\alpha}$, $\forall\, a, b \in \mathbb{R}$

$D_{0.5}(y_0) = 0$, for all constant function y_0

$$\frac{\partial^{0.5}}{\partial x^{0.5}} f(x, y_0) = 2x^{1.5}$$

to find

$$\frac{d^{0.5}}{dy^{0.5}} f(x_0, y) = \lim_{\varepsilon\to 0} \frac{f(x_0\,, y + \varepsilon y^{1-0.5}) - f(x_0\,, y)}{\varepsilon} \;\;,\;\; \forall x > 0\,, y_0 \text{ is costant}\,, \alpha = \frac{1}{2}.$$

Then, using part 2,3,5 of theorem 1, it follows that

$$D_{0.5}(x^p) = px^{p-\alpha}, \;\; \forall\, a, b \in \mathbb{R}$$

$$D_{0.5}(f \times g) = f \times D_{0.5}(g) + g \times D_{0.5}(f)$$

$$D_{0.5}(x_0) = 0, \forall x_0,$$

where x_0 is the constant function, then

$$\frac{\partial^{0.5}}{\partial y^{0.5}} f(x_0, y) = x^2 y^{0.5} + 4\, y^{3.5}$$

Example 4: This example aims to compare the fractional partial derivative to the ordinary partial derivative. If $f(x, y) = x^3 + 2x^2y^2 + y^3$. Find $f_{yx}(1, 2)$ and calculate $f_{yx}^{\frac{1}{2}+\frac{1}{3}}(1{,}2)$ or $f^{1/3}{}_y\left(f^{1/2}{}_x\right)\Big|_{(1,2)}$.

Solution

$$f_x = 3x^2 + 4x\,y^2 \;\;\text{, and}\;\;\;\; f_{yx} = 8xy\;\; .$$

So

$$f_{yx}(1, 2) = 8 \times 1 \times 2 = 16$$

Now; by using $D_x^{\alpha}(x^p) = px^{p-\alpha}$

$$D_x^{1/2}(f) \;=2x^{\frac{5}{2}} + 4x^{\frac{3}{2}}y^2 \;\text{ and }\; D_y^{1/3}\left(D_x^{1/2}(f)\right) \;=\; 8x^{\frac{3}{2}}y^{\frac{5}{3}}$$

$$f^{1/3}{}_y\left(f^{1/2}{}_x\right)\Big|_{(1,2)} = 8\times 1\times\sqrt[3]{32} = 16\sqrt[3]{4}$$

Finally, the α –partial fractions derivatives with respect to y then β –partial fractions derivatives of the same function f with respect to y as follows:

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}\left(\frac{\partial^{\beta} f}{\partial y^{\beta}}\right)=\frac{\partial^{\alpha+\beta} f}{\partial y^{\alpha+\beta}}$$

which is further rewritten as

$$f_{yy}^{\alpha+\beta}=D_{yy}^{\alpha+\beta}$$

5. Conclusion

The objective of this study is to implement the conformable fractional derivative on fractional order partial differential and optimal numerical solution of fractional order partial differential equations with initial conditions. The obtained solution demonstrates the reliability of the proposed method and shows the applicability of this method to solve large amount of non-linear fractional equations. The proposed method is free from assumptions and rounding-off errors. It offers more realistic series solutions whose continuity depends on the fractional derivative. Convergence of approximate solutions is observed as the number of decomposed terms is increased. Consistent accuracy throughout the domain of the problem is also witnessed.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] M. Abul-Ez, M. Zayed, A. Youssef, M. De la Sen, On conformable fractional Legendre polynomials and their convergence properties with applications, Alexandria Eng. J. 59 (2020), 5231–5245.

[2] S. Chen, Y. Liu, L. Wei, B. Guan, Exact solutions to fractional Drinfel'd–Sokolov–Wilson equations, Chin. J. Phys. 56 (2018), 708–720.

[3] M. Mamat, S. Syouri, I.M. Alghrouz, et al. Conformable fractional differential transform method for solving fractional derivatives, Int. J. Adv. Sci. Technol. 29 (2020), 1734-1743.

[4] I. Podlubny, Fractional differential equations, Academic Press, London, 1999.

[5] J.H. He, Some application of nonlinear fractional differential equations and their approximations. Bull. Sci. Technol. 15 (1999), 86-90.

[6] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014), 65–70.

[7] E. Ünal, A. Gökdoğan, Solution of conformable fractional ordinary differential equations via differential transform method, Optik. 128 (2017), 264–273.

[8] T. Abdeljawad, On conformable fractional calculus, J. Comput. Appl. Math. 279 (2015), 57–66.

[9] T.R.S. Saber, E. Mostafa, H. Rezazadeh, The first integral method: Fractional Derivative. M. Sc.Thesis. AL-Quds University, Jerusalem, Palestine, 2009.

[10] S.T.R. Syouri, I.M. Sulaiman, I.M.M. Alghrouz, M. Mamat, A Modified differential transform method for solving nonlinear systems, Int. J. Adv. Sci. Technol. 29 (2020), 240-247.

[11] S.T.R. Syouri, M. Mamat, I.M.M. Alghrouz, I.M. Sulaiman, Conformable fractional differintegral method for solving fractional equations, Int. J. Sci. Technol. Res. 9 (2020), 292- 295,.

[12] R. Gorenflo, F. Mainardi, Fractional calculus: integral and differential equations of fractional order, in Fractals and Fractional Calculus in Continuum Mechanics (Ed. A. Carpintefi and F. Mainardi), Springer Verlag, Wien 1997.

[13] Y. Luchko, R. Gorenflo, The initial value problem for some fractional differential equations with the Caputo derivative. Preprint Series A0898, Freic Universitat Berlin, 1998.

[14] J.-H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000), 115–123.

[15] S. Al-Ahmad, I.M. Sulaiman, M. Mamat, L.G. Puspa, A modification of differential transform method for solving systems of second order ordinary differential equations, Math. Stat. 8 (2020),471-478.

[16] I. Shingareva, C. Lizarraga-Celaya, Solving nonlinear differential equations with Maple and Mathematica, Springer-Verlag, New York, 2011.

[17] S. Momani, An explicit and numerical solutions of the fractional KdV equation, Math. Comput. Simul. 70 (2005), 110–118.

[18] S. Momani, N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, Appl. Math. Comput. 182 (2006), 1083–1092.

[19] S. Al-Ahmad, I.M. Sulaiman, M. Mamat, K. Kamfa, Solutions of classes of differential equations using modified differential transform method, J. Math. Comput. Sci. 10 (2020), 2360-2382

[20] S. Momani, R. Qaralleh, Numerical approximations and Padé approximants for a fractional population growth model, Appl. Math. Model. 31 (2007), 1907–1914.

[21] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals. 31 (2007), 1248–1255.

[22] S. Al-Ahmad, M. Mamat, R. Alahmad, et al. On new properties of differential transform via difference equations, Int. J. Eng. Technol. 7 (2018), 321-324.