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THE CONCEPT OF RHOTRIX TYPE A SEMIGROUPS

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Abstract: Several articles exist on rhotrices as associative rhotrix theory and non-associative rhotrix theory. In this paper, we describe the structure of rhotrix type A semigroups. Ultimately, we extend the work of Mohammed et al on rhotrix semigroup.

Keywords: rhotrix; matrix; Green's *-relations; type A semigroups; congruences.

Mathematics Subject Classification: 20M10.

1. INTRODUCTION

In [5], Atanassov and Shanaon discussed arrays of numbers that are in some way, between two-dimensional vectors and (2×2)-dimensional matrices in their paper titled matrix-tertions and noitrets. As an extension, Ajibade [1] in 2003 introduced objects which are in some ways, between (2×2)-dimensional and (3×3)-dimensional matrices. This new paradigm of science now known as rhotrix theory was defined in [1] for dimension three as:

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \mathbb{R} \right\},$$

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where $c = h(R)$ is called the heart of any rhotrix R and $\mathbb{R}$ is the set of real numbers.

It is worthy to note that these heart-oriented rhotrices are always of odd dimension. Thereafter, Mohammed [21] in his PhD thesis extended the idea to rhotrix set of size n.

It is known in [1] that addition and multiplication of two heart-oriented rhotrices are as follows:

$$R + Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

and $$R \circ Q = \left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

respectively. Furthermore, Mohammed [21] and Ezegwu et al [9] gave a generalization of this heart-oriented rhotrices.

A row-column multiplication of heart-oriented rhotrices was given by Sani [28] as:

$$R \circ Q = \left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle.$$

Sani [29] also gave a generalization of this row-column multiplication of heart-oriented rhotrices as:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \langle \sum_{i_2 j_{1=1}}^{t} \left(a_{i_1 j_1} b_{i_2 j_2} \right), \sum_{l_2 k_1 = 1}^{t-1} \left(c_{l_i k_1} d_{l_2 k_2} \right) \rangle \text{ , } t = \frac{n+1}{2},$$

where R_n and Q_n denote n-dimensional rhotrices (with n rows and n columns).

Mohammed [20] classified the heart-oriented rhotrices as abstract structures of rings, fields, integral domains and unique factorization domain. The necessary and sufficient condition under which a linear map can be represented over a heart-oriented rhotrix was carried out in [22] by Mohammed et al. More so, Mohammed [24] and Isere [15] gave a new technique for expressing rhotrices in a general form. Another method of rhotrix representation was given by Chinedu in [6]. In [30], the algebraic properties of singleton, coiled and modulo rhotrices were presented. A study of finite fields over rhotrices were carried out in [31] and [35]. Tudunkaya and makanjuola [32] gave the structure of rhotrices having entries from the set of integers modulo P and their properties. The rhotrix quadratic polynomial presentation as part of a note on rhotrix exponent rule and its applications in [19] was extended in [33]. Rhotrix polynomial and its extension to construction of rhotrix polynomial ring was studied in [34]. An investigation of rhotrix sets and rhotrix sets and rhotrix spaces categorized over numbers in real and complex fields was presented in [23]. A system of linear equation arising from the rhotrix equation $A \circ X = C$ was carried out in [2] and the conditions for their solvability were determined. A note on rhotrix system of equations was presented in [3] as an extension to earlier work done in [2].

In [4], there was an introduction to the concept of paraletrix as a generalization of rhotrix Mohammed and Balarabe [25] gave the first review of articles on rhotrix theory since its inception. Also in [26], some construction of rhotrix semigroup was given. In 2018, Isere and Adeniran [17] introduced the concept of quasigroups and rhotrix loops as non-associative rhotrix theory. In the same year, Isere [16] gave a description of even dimensional rhotrix.

In generalizing regular semigroups, Fountain [10] considered the Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ instead of trying to weaken the regularity of a semigroup. Let a, b be elements of a semigroup S, we define $a\,\mathcal{R}^*b$ if and only if for all $x, y \epsilon S^1,\ xa = ya \Leftrightarrow xb = yb$. Dually we define the relation $\mathcal{L}^*$.

Utilizing the Green *- relations, Fountain called a semigroup S abundant if any $\mathcal{L}^*$ and $\mathcal{R}^*$-classes of S contain at least one idempotent. Following Fountain [11], an abundant semigroup S is said to be adequate if the set of idempotents of S $(E(S))$ forms a semilattice. It is obvious that regular semigroups are abundant semigroups while inverse semigroups are adequate semigroups.

A semigroup S is said to be lpp if every principal left ideal of S, regarded as an S-system is projective. An rpp semigroup is defined dually. In [11], a semigroup S is lpp if and only if every $\mathcal{R}^*$-class of S contains at least one idempotent. Thus, an lpp semigroup S is said to be left adequate if $E(S)$ forms a semilattice. Right adequate semigroups are defined dually. It is obvious that a semigroup is adequate if and only if it is both left and right adequate. Furthermore, it is obvious that each $\mathcal{R}^*$-class $\mathcal{R}^*_a$ of a left adequate semigroup contains a unique idempotent which is denoted by $a^\dagger$. A left adequate semigroup is said to be left type A if for all $e \epsilon E(S)$ and $a \epsilon S,\ ae = (ae)^\dagger a$ (see [12]), and dually for right type A semigroups. A semigroup S is said to be a type A if it is both left and right type A.

It is worthy to note that all articles so far existing on rhotrix theory are classified into associative rhotrix theory and non-associative rhotrix theory. Therefore, the objective of this work is to give a description of associative rhotrix theory in terms of type A semigroups. Our results extend that of rhotrix semigroup given in [26].

2. PRELIMINARIES

In this section we recall some definitions as well as some known results which will be useful in this work. For notation and terminologies not mentioned in this paper, the reader is referred to [1], [26], [9] and [28] respectively.

Throughout this paper, we will use R to denote any rhotrix while R_n is n-dimensional rhotrix.

Definition 2.1. Suppose $R_n = \langle a_{ij}, c_{lk} \rangle$ is an n-dimensional rhotrix, then the determinant of R_n is given by $\det(R_n) = \det(A_t)\det(C_{t-1})$ where A_t and C_{t-1} are two square matrices of dimension $(t \times t)$ and $(t-1) \times (t-1)$ respectively which make up the rhotrix R_n with $t = \frac{n+1}{2}$ and $n \epsilon\, 2\mathbb{Z}^+ + 1$.

Definition 2.2. The inverse of the rhotrix $R_n = \langle a_{ij}, c_{lk} \rangle$ is the rhotrix $R_n^{-1} = \langle q_{ij}, r_{lk} \rangle$ such that $R_n \circ R_n^{-1} = \langle a_{ij}, c_{lk} \rangle \circ \langle q_{ij}, r_{lk} \rangle = \langle I_{ij}, I_{lk} \rangle$ where $[q_{ij}]_{t\times t}$ and $[r_{ij}]_{t-1\times t-1}$ are the inverses of the two square matrices $[a_{ij}]_{t\times t}$ and $[c_{ij}]_{t-1\times t-1}$ respectively, which make up the rhotrix R_n with $t = \frac{n+1}{2}$ and $n \epsilon\, 2\mathbb{Z}^+ + 1$.

Remark 2.3. A rhotrix R_n is said to be invertible or non-singular if the determinant is non-zero. That is R_n is invertible if $\det(R_n) \neq 0$.

Theorem 2.4 [1]**.** For any rhotrix $R \neq 0$, $R^2 = 0$ if and only if $h(R) = 0$ where 0 is the zero rhotrix.

Theorem 2.5 [19]. Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be any rhotrix of size 3, then for any integer m,

$R^m = (h(R))^{m-1} \left\langle \begin{matrix} & ma & \\ mb & h(R) & md \\ & me & \end{matrix} \right\rangle$. In particular, R^0 and R^{-1} are the identity and inverse of R respectively, provided $h(R)$ is non-zero.

Proposition 2.6 [2]. Let A, B and C be three rhotrices of the same size with entries in $\mathbb{R}$, then the system of linear equations resulting from $A \circ B = C$ has

i) a unique solution if and only if $h(A) \neq 0$ and $h(C) \neq 0$.

ii) an infinite solution if and only if $h(A) = h(C) = 0$.

iii) no solution if and only if $h(A) = 0$ and $h(C) \neq 0$.

Theorem 2.7 [26]. The rhotrix semigroup $(R_n(\mathrm{F}),\circ)$ is embedded in the matrix semigroup $(\mathcal{M}_n(\mathrm{F}),.)$

Remark 2.8 [26]. Given the map $\theta : R_n(\mathrm{F}) \to \mathcal{M}_n(\mathrm{F})$, the image set $\langle R_n(\mathrm{F})\rangle\theta$ is a subsemigroup of $\mathcal{M}_n(\mathrm{F})$ consisting of all filled coupled $n \times n$ matrices. Since $\mathcal{M}_n(\mathrm{F})$, the semigroup of all square matrices over F is regular, then it is not difficult to see that $\langle R_n(\mathrm{F})\rangle\theta$ is a regular semigroup. Such a semigroup is denoted by $R_n^*(\mathrm{F})$.

Using the Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ defined in [14], the following results was obtained

Theorem 2.9 [26]. Suppose $A, B \in R_n^*(\mathrm{F})$, then

i) $A \,\mathcal{L}\, B$ if and only if $im\ (A) = im\ (B)$.

ii) $A \,\mathcal{R}\, B$ if and only if $\ker(A) = \ker(B)$.

iii) $A \,\mathcal{H}\, B$ if and only if $im\ (A) = im\ (B)$ and $\ker(A) = \ker(B)$.

Theorem 2.10 [26]. i) Suppose $A = \langle a_{ij}, c_{lk}\rangle$ and $B = \langle b_{ij}, d_{lk}\rangle$ belong to the semigroup $R_n^*(\mathrm{F})$ then $A \,\mathcal{D}\, B$ if and only if $\mathrm{rank}\,(a_{ij}) = \mathrm{rank}\,(b_{ij})$ and $\mathrm{rank}\,(c_{lk}) = \mathrm{rank}\,(d_{lk})$.

ii) In $R_n^*(\mathrm{F})$, $\mathcal{D} = \mathcal{J}$.

The following Lemma is due to [11].

Lemma 2.11 [11]. Let S be a semigroup and e be an idempotent in S. Then the following are equivalent in S

i) $a\,\mathcal{R}^* e$

ii) $ea = a$ and for all $x, y \in S^1$, $xa = ya \Rightarrow xe = ye$.

Definition 2.12. Let S be a semigroup and let $x \in S$. Then x is said to be coregular and y its coninverse if $x = xyx = yxy$. S is coregular if all its elements are coregular. S is said to be orthodox if it is regular and the set $E(S)$ of idempotents forms a subsemigroup.

Lemma 2.13 [10]. Suppose a, b are elements of an adequate semigroup S. Then we have:

i) $a\,\mathcal{R}^* b$ if and only if $a^\dagger = b^\dagger$ and $a\,\mathcal{L}^* b$ if and only if $a^* = b^*$.

ii) $(ab)^* = (a^*b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$.

iii) $a^\dagger(ab)^\dagger = (ab)^\dagger$ and $(ab)^* b^* = (ab)^*$.

Definition 2.14 [7]. Let S be an adequate semigroup and let ρ be a congruence on S. Then ρ is said to be admissible if $ax\,\rho\,ay = a^*x\,\rho\,a^*y$ and $xa\,\rho\,ya \Longrightarrow xa^\dagger\,\rho\,ya^\dagger$ for all $a \in S$ and $x, y \in S^1$.

Lemma 2.15 [8]. If ρ is an admissible congruence on the adequate semigroup S and if a, b are elements of S such that $a\ \rho\ b$, then $a^*\ \rho\ b^*$ and $a^\dagger\ \rho\ b^\dagger$.

Remark 2.16 [8]. If ρ is an admissible congruence on a type A semigroup S, then S/ρ is a type A semigroup when * and † are defined on S/ρ by putting

$$(a\rho)^\dagger = a^\dagger\rho \text{ and } (a\rho)^* = a^*\rho .$$

For more knowledge on admissible congruences, the reader is referred to [13] and [27].

Definition 2.17. A homomorphism $\theta : S \to T$ of an adequate semigroup is admissible if $a\ \mathcal{R}^*(S) b$ implies $a\theta\ \mathcal{R}^*(T)\ b\theta$ and $a\ \mathcal{L}^*(S)\ b$ implies $a\theta\ \mathcal{L}^*(T)\ b\theta$.

Definition 2.18. The relation σ on a type A semigroup S is defined by the rule that:

$$(a, b)\ \epsilon\ \sigma \text{ if and only if } ae = be \text{ for some } e\ \epsilon\ E(S).$$

It is known in [18] that σ is the minimum cancellative congruence on S. It is important to note that σ can also be written as $a\ \sigma\ b$ if and only if $fa = fb$ for some $f\ \epsilon\ E(S)$.

3. RHOTRIX TYPE A SEMIGROUP

This section focuses on the construction of a rhotrix type A semigroup and the properties embedded in the semigroup constructed.

Now let $R_n(\mathrm{F})$ be a set of all rhotrices of size n with entries from an arbitrary field F. For any $A_n, B_n\ \epsilon\ R_n(\mathrm{F})$, define a binary operation $\circ$ on $R_n(\mathrm{F})$ by the rule:

$$A_n \circ B_n = \langle a_{i_1 j_1}, c_{l_i k_1}\rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2}\rangle = \langle \sum_{i_2 j_{1}=1}^{t}\left(a_{i_1 j_1} b_{i_2 j_2}\right), \sum_{l_2 k_1=1}^{t-1}\left(c_{l_i k_1} d_{l_2 k_2}\right)\rangle, t = \frac{n+1}{2},$$

where A_n and B_n denote n-dimensional rhotrices.

Theorem 3.1. $S = (R_n(\mathrm{F}), \circ)$ is a semigroup.

Proof. Let $A_n, B_n\ \epsilon\ S$, we have that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$, so that $A_n \circ B_n\ \epsilon\ S$, since $\det(A_n \circ B_n) = \det\ (A_n) \times \det\ (B_n) \neq 0$. It follows that S is closed under the binary operation. Next is to show that S is associative. Suppose $A_n = \langle a_{i_1 j_1}, c_{l_i k_1}\rangle$, $B_n = \langle b_{i_2 j_2}, d_{l_2 k_2}\rangle$, $C_n = \langle u_{i_3 j_3}, v_{l_3 k_3}\rangle$, then we have that

$$\begin{aligned}
A_n \circ (B_n \circ C_n) &= \langle a_{i_1 j_1}, c_{l_i k_1}\rangle \circ \left(\langle b_{i_2 j_2}, d_{l_2 k_2}\rangle \circ \langle u_{i_3 j_3}, v_{l_3 k_3}\rangle\right) \\
&= \langle a_{i_1 j_1}, c_{l_i k_1}\rangle \circ \left(\langle \sum_{i_3 j_2=1}^{t}\left(b_{i_2 j_2} u_{i_3 j_3}\right), \sum_{l_3 k_2=1}^{t-1}\left(d_{l_2 k_2} v_{l_3 k_3}\right)\rangle\right) \\
&= \langle \sum_{i_2 j_1=1}^{t} a_{i_1 j_1}\left[\sum_{i_3 j_2=1}^{t}\left(b_{i_2 j_2} u_{i_3 j_3}\right)\right], \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1}\left[\sum_{l_3 k_2=1}^{t-1}\left(d_{l_2 k_2} v_{l_3 k_3}\right)\right]\rangle \\
&= \langle \sum_{i_2 j_1=1}^{t} \sum_{i_3 j_2=1}^{t} a_{i_1 j_1}\left(b_{i_2 j_2} u_{i_3 j_3}\right), \sum_{l_2 k_1=1}^{t-1} \sum_{l_3 k_2=1}^{t-1} c_{l_1 k_1}\left(d_{l_2 k_2} v_{l_3 k_3}\right)\rangle
\end{aligned}$$

$$= \langle \sum_{i_2 j_1=1}^{t} \sum_{i_3 j_2=1}^{t} a_{i_1 j_1} b_{i_2 j_2} u_{i_3 j_3} , \sum_{l_2 k_1=1}^{t-1} \sum_{l_3 k_2=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2} v_{l_3 k_3} \rangle.$$

Similarly, we have that

$$\begin{aligned}(A_n \circ B_n) \circ C_n &= \left(\langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle\right) \circ \langle u_{i_3 j_3}, v_{l_3 k_3} \rangle \\ &= \left(\langle \sum_{i_2 j_1=1}^{t} a_{i_1 j_1} b_{i_2 j_2},\ \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2} \rangle\right) \circ \langle u_{i_3 j_3}, v_{l_3 k_3} \rangle \\ &= \langle \sum_{i_3 j_2=1}^{t} u_{i_3 j_3} \left[\sum_{i_2 j_1=1}^{t} a_{i_1 j_1} b_{i_2 j_2}\right], \sum_{l_3 k_2=1}^{t-1} v_{l_3 k_3} \left[\sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2}\right] \rangle \\ &= \langle \sum_{i_3 j_2=1}^{t} \sum_{i_2 j_1=1}^{t} \left(a_{i_1 j_1} b_{i_2 j_2}\right) u_{i_3 j_3},\ \sum_{l_3 k_2=1}^{t-1} \sum_{l_2 k_1=1}^{t-1} \left(c_{l_1 k_1 d_{l_2 k_2}}\right) v_{l_3 k_3} \rangle \\ &= \langle \sum_{i_3 j_2=1}^{t} \sum_{i_2 j_1=1}^{t} a_{i_1 j_1} b_{i_2 j_2} u_{i_3 j_3},\ \sum_{l_3 k_2=1}^{t-1} \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1 d_{l_2 k_2}} v_{l_3 k_3} \rangle.\end{aligned}$$

Consequently,

$$A_n \circ (B_n \circ C_n) = (A_n \circ B_n) \circ C_n \ .$$

Therefore S is a semigroup.

Lemma 3.2. Let $\langle a_{ij}, c_{lk} \rangle, \langle b_{ij}, d_{lk} \rangle \in S = (R_n(\mathrm{F}), \circ)$. Then we have

i) $\langle a_{ij}, c_{lk} \rangle \, \mathcal{R}^* \, \langle b_{ij}, d_{lk} \rangle$ if and only if $a_{ij} \, \mathcal{R}^* \, b_{ij}$ and $c_{lk} \, \mathcal{R}^* \, d_{lk}$.

ii) $\langle a_{ij}, c_{lk} \rangle \, \mathcal{L}^* \, \langle b_{ij}, d_{lk} \rangle$ if and only if $a_{ij} \, \mathcal{L}^* \, b_{ij}$ and $c_{lk} \, \mathcal{L}^* \, d_{lk}$.

iii) $\langle a_{ij}, c_{lk} \rangle \, \mathcal{H}^* \, \langle b_{ij}, d_{lk} \rangle$ if and only if $\langle a_{ij}, c_{lk} \rangle \, \mathcal{R}^* \, \langle b_{ij}, d_{lk} \rangle$ and $\langle a_{ij}, c_{lk} \rangle \, \mathcal{L}^* \, \langle b_{ij}, d_{lk} \rangle$.

Proof. i) Suppose $\langle a_{ij}, c_{lk} \rangle \, \mathcal{R}^* \, \langle b_{ij}, d_{lk} \rangle$, then for $\langle x_{ij}, x_{lk} \rangle, \langle y_{ij}, y_{lk} \rangle \in S$ we have

$$\langle x_{ij}, x_{lk} \rangle \langle a_{ij}, c_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle a_{ij}, c_{lk} \rangle \Leftrightarrow \langle x_{ij}, x_{lk} \rangle \langle b_{ij}, d_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle b_{ij}, d_{lk} \rangle$$

$$\Longrightarrow \langle x_{ij} a_{ij}, x_{lk} c_{lk} \rangle = \langle y_{ij} a_{ij}, y_{lk} c_{lk} \rangle \Leftrightarrow \langle x_{ij} b_{ij}, x_{lk} d_{lk} \rangle = \langle y_{ij} b_{ij}, y_{lk} d_{lk} \rangle.$$

Consequently, we have

$x_{ij} a_{ij} = y_{ij} a_{ij}$, $x_{lk} c_{lk} = y_{lk} c_{lk} \Leftrightarrow x_{ij} b_{ij} = y_{ij} b_{ij}$, $x_{lk} d_{lk} = y_{lk} d_{lk}$.

This implies that $a_{ij} \, \mathcal{R}^* \, b_{ij}$ and $c_{lk} \, \mathcal{R}^* \, d_{lk}$.

Conversely, let $a_{ij} \, \mathcal{R}^* \, b_{ij}$ and $c_{lk} \, \mathcal{R}^* \, d_{lk}$, then there exists arbitrary elements x_{ij}, $y_{ij} \in \mathcal{M}_t(\mathrm{F})$ and x_{lk}, $y_{lk} \in \mathcal{M}_{t-1}(\mathrm{F})$ such that $x_{ij} a_{ij} = y_{ij} a_{ij} \Leftrightarrow x_{ij} b_{ij} = y_{ij} b_{ij}$ and $x_{lk} c_{lk} = y_{lk} c_{lk} \Leftrightarrow x_{lk} d_{lk} = y_{lk} d_{lk}$.

It follows that

$$\langle x_{ij}, x_{lk} \rangle \langle a_{ij}, c_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle a_{ij}, c_{lk} \rangle \Leftrightarrow \langle x_{ij}, x_{lk} \rangle \langle b_{ij}, d_{lk} \rangle = \langle y_{ij}, y_{lk} \rangle \langle b_{ij}, d_{lk} \rangle.$$

Thus $\langle a_{ij}, c_{lk} \rangle \, \mathcal{R}^* \, \langle b_{ij}, d_{lk} \rangle$.

ii) The proof is similar to i).

iii) The proof is a routine check.

Lemma 3.3. Let $\langle a_{ij}, c_{lk}\rangle \in S = (R_n(\mathrm{F}), \circ)$. Then $\langle a_{ij}, c_{lk}\rangle \in E(S)$ if and only if $a_{ij} \in E(\mathcal{M}_t(\mathrm{F}))$ and $c_{lk} \in E(\mathcal{M}_{t-1}(\mathrm{F}))$.

Proof. Let $\langle a_{ij}, c_{lk}\rangle \in E(S)$, then we have that

$$\langle a_{ij}, c_{lk}\rangle \circ \langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle$$

$$\Longrightarrow \langle a_{ij}a_{ij}, c_{lk}c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle.$$

Consequently,

$$\left(a_{ij}\right)^2 = a_{ij} \qquad \text{(where } a_{ij} \in \mathcal{M}_t(\mathrm{F}) \text{ is a square matrix)}$$

$$(c_{lk})^2 = c_{lk} \qquad \text{(where } c_{lk} \in \mathcal{M}_{t-1}(\mathrm{F}) \text{ is a square matrix).}$$

Thus $a_{ij} \in E(\mathcal{M}_t(\mathrm{F}))$ and $c_{lk} \in E(\mathcal{M}_{t-1}(\mathrm{F}))$.

The converse of the proof can be easily verified.

Example 3.4. The following rhotrices are idempotents in $R_5(\mathrm{F})$;

$$\left\langle \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & -1 & & 0 & & -2 & & \\ 1 & & 0 & & 3 & & 0 & & -4 \\ & & -2 & & 0 & & 4 & & \\ & & & & -3 & & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle,$$

$$\left\langle \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & -1 & & 3 & & -2 & & \\ 1 & & 1 & & 3 & & -6 & & -4 \\ & & -2 & & -2 & & 4 & & \\ & & & & -3 & & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix} \right\rangle,$$

$$\left\langle \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & -1 & & 4 & & -2 & & \\ 1 & & 12 & & 3 & & -1 & & -4 \\ & & -2 & & -3 & & 4 & & \\ & & & & -3 & & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \right\rangle.$$

It is obvious that $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \in E(\mathcal{M}_t(\mathrm{F}))$ while

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \in E(\mathcal{M}_{t-1}(\mathrm{F}))$.

Theorem 3.5. $S = (R_n(\mathrm{F}), \circ)$ is a type A semigroup.

Proof. We only prove that S is a left type A as the proof for right type A is dual. Now let $a = \langle a_{ij}, c_{lk}\rangle \in R_n(\mathrm{F})$, $e = \langle I_{ij}, c_{lk}\rangle \in E(S)$, then we have that

$$ae = \langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle.$$

Suppose $(ae)^{\dagger} = \langle a_{ij}, I_{lk}\rangle$, then we have that

$$(ae)^{\dagger}a = \langle a_{ij}, I_{lk}\rangle\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle.$$

Thus $ae = (ae)^{\dagger}a$.

That S is adequate follows from the fact that

$$\langle a_{ij}, I_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle I_{ij}, c_{lk}\rangle\langle a_{ij}, I_{lk}\rangle$$

$$= \langle a_{ij}, c_{lk}\rangle \qquad \text{(where } a_{ij} \epsilon E(\mathcal{M}_t(\mathrm{F})) \text{ and } c_{lk} \epsilon E(\mathcal{M}_{t-1}(\mathrm{F})).$$

Remark 3.6. It is important to note that $\langle a_{ij}, I_{lk}\rangle$ is the idempotent in the $\mathcal{R}^*$-class as well as the $\mathcal{L}^*$-class. For the sake of ambiguity, we will use $\langle a_{ij}, c_{lk}\rangle^{\dagger} = \langle a_{ij}, I_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle^{*} = \langle a_{ij}, I_{lk}\rangle$ to denote the respective idempotents.

It has been shown in [26] that S is a regular semigroup and in [27] that a particular kind of type A semigroup is coregular namely *-bisimple type A I-semigroup. In our next Lemma, we show that S belong to the class of type A semigroup that is not coregular.

Lemma 3.7. $S = (R_n(\mathrm{F}), \circ)$ is not a coregular semigroup.

Proof. Now let $A = \langle a_{ij}, c_{lk}\rangle$, $B = \langle b_{ij}, d_{lk}\rangle \epsilon R_n(\mathrm{F})$. It is known in [26] that

$$A \circ B \circ A = \big(\langle a_{ij}, c_{lk}\rangle \circ \langle b_{ij}, d_{lk}\rangle\big) \circ \langle a_{ij}, c_{lk}\rangle$$

$$= \langle a_{ij}, c_{lk}\rangle = A.$$

Since $(b_{ij}), (a_{ij}) \epsilon (\mathcal{M}_t(\mathrm{F}))$ and $(c_{lk}), (d_{lk}) \epsilon (\mathcal{M}_{t-1}(\mathrm{F}))$, where $t = \frac{n+1}{2}, n \epsilon 2\mathbb{Z}^+ + 1$ and

$$(b_{ij})(a_{ij})(b_{ij}) = (b_{ij}),$$

$$(d_{lk})(c_{lk})(d_{lk}) = (d_{lk}).$$

It follows that

$$B \circ A \circ B = \big(\langle b_{ij}, d_{lk}\rangle \circ \langle a_{ij}, c_{lk}\rangle\big) \circ \langle b_{ij}, d_{lk}\rangle$$

$$= \langle b_{ij}, d_{lk}\rangle = B.$$

So $\langle a_{ij}, c_{lk}\rangle \neq \langle b_{ij}, d_{lk}\rangle$. Thus S is not coregular by definition.

It is important to note that Lemma 3.7 above is an example of a class of type A semigroup that is not coregular while the class of type A semigroup given in [27] is coregular.

Lemma 3.8. $S = (R_n(\mathrm{F}), \circ)$ is an orthodox semigroup.

Proof. The proof is a routine check.

4. THE STRUCTURE THEOREM

In this section S will denote a rhotrix type A semigroup while $\mathcal{M}_t(\mathrm{F})$ and $\mathcal{M}_{t-1}(\mathrm{F})$ will denote set of two square matrices over an arbitrary field F of sizes $(t \times t)$ and $(t-1)\times(t-1)$ respectively.

$$\text{Let } S_1 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ & - - - & - - - & - - - & \\ a_{t_1} & - - - & - - - & - - - & a_{1_t} \\ & - - - & - - - & - - - & \\ & a_{t(t-1)} & w & y & \\ & & a_{tt} & & \end{array} \right\rangle, S_2 = \left\langle \begin{array}{ccccc} & & b_{11} & & \\ & b_{21} & c_{11} & b_{12} & \\ & - - - & - - - & - - - & \\ b_{t_1} & - - - & - - - & - - - & b_{1_t} \\ & - - - & - - - & - - - & \\ & b_{t(t-1)} & x & z & \\ & & b_{tt} & & \end{array} \right\rangle \epsilon\, S,$$

where $w = c_{(t-1)(t-1)}$, $y = a_{(t-1)t}$, $x = d_{(t-1)(t-1)}$, $z = b_{(t-1)t}$.

$$\text{Put } S_1 S_1 = \left\langle \begin{array}{ccccc} & & s_{11} & & \\ & s_{21} & s_{11}^{*} & s_{12} & \\ & - - - & - - - & - - - & \\ s_{t_1} & - - - & - - - & - - - & s_{1_t} \\ & - - - & - - - & - - - & \\ & s_{t(t-1)} & u & v & \\ & & s_{tt} & & \end{array} \right\rangle$$

where $u = s^{*}_{(t-1)(t-1)}$, $v = s_{(t-1)t}$,

$s_{ij} = \sum_{k=1}^{t} a_{ik} b_{2k-1j}$, $s^{*}_{ij} = \sum_{k=1}^{t} c_{ik} d_{2k-1j}$ – those entries for which i assumes odd values,

$s_{ij} = \sum_{k=1}^{t-1} a_{ik} b_{2kj}$, $s^{*}_{ij} = \sum_{k=1}^{t-1} c_{ik} d_{2kj}$ – those entries for which i assumes even values.

Theorem 4.1. Let $R_n(\mathrm{F})$ be a set of all rhotrices of size n with entries from an arbitrary field F. For $A_n, B_n \,\epsilon\, R_n(\mathrm{F})$, define a binary operation $\circ$ on $R_n(\mathrm{F})$ by the rule:

$$A_n \circ B_n = \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \langle \sum_{i_2 j_{1=1}}^{t} \left(a_{i_1 j_1} b_{i_2 j_2} \right), \sum_{l_2 k_1 = 1}^{t-1} \left(c_{l_i k_1} d_{l_2 k_2} \right) \rangle \ , t = \frac{n+1}{2}$$

$a_{ij} \,\epsilon\, \mathcal{M}_t(\mathrm{F})$ and $c_{lk} \,\epsilon\, \mathcal{M}_{t-1}(\mathrm{F})$. Then $S = (R_n(\mathrm{F}), \circ)$ is a rhotrix type A semigroup. Conversely, every rhotrix type A semigroup is isomorphic to one of such construction.

Proof. We have proved the direct part in section 3, and so we will only prove the converse part. Let S be a rhotrix type A semigroup. Define a map $\theta : S \to R_n(\mathrm{F})$ by the rule that

$$s\theta = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ & - - - & - - - & - - - & \\ a_{t_1} & - - - & - - - & - - - & a_{1_t} \\ & - - - & - - - & - - - & \\ & a_{t(t-1)} & w & y & \\ & & a_{tt} & & \end{array} \right\rangle \theta = \left\langle \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1t} \\ \vdots & & \ddots & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{tt} \end{pmatrix}, \begin{pmatrix} c_{11} & \cdots & c_{1(t-1)} \\ \vdots & \ddots & \vdots \\ c_{(t-1)} & \cdots & c_{(t-1)(t-1)} \end{pmatrix} \right\rangle$$

where $t = \frac{n+1}{2}, n \,\epsilon\, 2\mathbb{Z}^{+} + 1$, $a_{tt} \,\epsilon\, \mathcal{M}_t(\mathrm{F})$ and $c_{(t-1)(t-1)} \,\epsilon\, \mathcal{M}_{t-1}(\mathrm{F})$ and θ maps each element of S to its corresponding $t \times t$ matrix and $(t-1)(t-1)$ matrix in $R_n(\mathrm{F})$ with the usual matrix multiplication. Obviously, θ is well defined. It is also one-to-one since for $s_1, s_2 \epsilon S, s_1\theta = s_2\theta$ which implies that $s_1 = s_2$. That θ is a homomorphism follows from the fact that

$$(s_1 s_2)\theta = s_1\theta s_2\theta.$$

Hence θ is an isomorphism from S onto $R_n(\mathrm{F})$. The proof of the theorem is then complete.

Example 4.2. Suppose $S = \{s_1, s_2\}$ such that $s_1 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 1 & \\ 3 & 1 & 0 & -1 & 4 \\ & -2 & 5 & 1 & \\ & & 3 & & \end{array} \right\rangle$ and

$s_2 = \left\langle \begin{array}{ccccc} & & 2 & & \\ & -1 & 4 & -2 & \\ 1 & 12 & 3 & -1 & -4 \\ & -2 & -3 & 4 & \\ & & -3 & & \end{array} \right\rangle$. Then for $\theta : S \to R_5(\mathrm{F})$, we have that

$$s_1\, s_2 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 1 & \\ 3 & 1 & 0 & -1 & 4 \\ & -2 & 5 & 1 & \\ & & 3 & & \end{array} \right\rangle \left\langle \begin{array}{ccccc} & & 2 & & \\ & -1 & 4 & -2 & \\ 1 & 12 & 3 & -1 & -4 \\ & -2 & -3 & 4 & \\ & & -3 & & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & 5 & & \\ & 5 & -12 & -7 & \\ 11 & 64 & -6 & 3 & -12 \\ & -18 & -16 & -11 & \\ & & -29 & & \end{array} \right\rangle$$

$$\Longrightarrow\ (s_1 s_2)\theta = \langle \begin{bmatrix} 5 & -7 & -12 \\ 5 & -6 & -11 \\ 11 & -18 & -29 \end{bmatrix}, \begin{bmatrix} -12 & 3 \\ 64 & -16 \end{bmatrix} \rangle.$$

Also,

$$\begin{aligned} s_1\theta s_2\theta &= \langle \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 1 \\ 3 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix} \rangle \langle \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \rangle \\ &= \langle \begin{bmatrix} 1 & 1 & 4 \\ 2 & 0 & 1 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \rangle \\ &= \langle \begin{bmatrix} 5 & -7 & -12 \\ 5 & -6 & -11 \\ 11 & -18 & -29 \end{bmatrix}, \begin{bmatrix} -12 & 3 \\ 64 & -16 \end{bmatrix} \rangle. \end{aligned}$$

Consequently, $(s_1 s_2)\theta = s_1\theta s_2\theta$.

5. CONGRUENCES

In this section, we give a description of congruences on rhotrix type A semigroup S. The results and proofs are essentially the same as for the one in [8].

Now let S be a rhotrix type A semigroup and let ρ be any congruence on S. Let $\rho_{/E(S)}$ be the restriction of ρ on $E(S)$ which we will denote by $tr\,\rho$ (trace of ρ). Obviously, $tr\,\rho$ is a congruence on $E(S)$.

Suppose $\langle I_{ij}, c_{lk}\rangle, \langle a_{ij}, 0_{lk}\rangle \epsilon E(S)$ such that $\langle I_{ij}, c_{lk}\rangle \,\rho\, \langle a_{ij}, 0_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle \epsilon S$, then we have that $\langle I_{ij}, c_{lk}\rangle\langle a_{ij}, c_{lk}\rangle \,\rho\, \langle a_{ij}, 0_{lk}\rangle\langle a_{ij}, c_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle \,\rho\, \langle a_{ij}, c_{lk}\rangle\langle a_{ij}, 0_{lk}\rangle$. It follows from Lemma 2.15 and that if ρ is admissible, then we have that

$$\left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\rho\left(\langle a_{ij}, c_{lk}\rangle\langle a_{ij}, 0_{lk}\rangle\right)^{\dagger} \text{ and } \left(\langle I_{ij}, c_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)^{*}\rho\left(\langle a_{ij}, 0_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)^{*}.$$

A congruence γ on $E(S)$ is said to be normal if for any $\langle I_{ij}, c_{lk}\rangle, \langle a_{ij}, 0_{lk}\rangle \epsilon E(S)$ and $\langle a_{ij}, c_{lk}\rangle \epsilon S$,

$\langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, 0_{lk}\rangle$ implies $\left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\gamma\left(\langle a_{ij}, c_{lk}\rangle\langle a_{ij}, 0_{lk}\rangle\right)^{\dagger}$ and $\left(\langle I_{ij}, c_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)^{*}\gamma\left(\langle a_{ij}, 0_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)^{*}$.

Lemma 5.1. Suppose γ is a normal congruence on $E(S)$, then for any $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon S$, the following conditions are equivalent:

(i) $\langle a_{ij}, c_{lk}\rangle^{*} \gamma \langle b_{ij}, d_{lk}\rangle^{*}$, $\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$ for $\langle I_{ij}, c_{lk}\rangle \epsilon E(S), \langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^{*}$.

(ii) $\langle a_{ij}, c_{lk}\rangle^{\dagger} \gamma \langle b_{ij}, d_{lk}\rangle^{\dagger}$, $\langle a_{ij}, 0_{lk}\rangle\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, 0_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ for $\langle a_{ij}, 0_{lk}\rangle \epsilon E(S), \langle a_{ij}, 0_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^{\dagger}$.

Proof. Let $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon S$ and suppose (i) is true, then S is type A and we have that

$$\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$$

$$\Rightarrow \left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\langle a_{ij}, c_{lk}\rangle = \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\langle b_{ij}, d_{lk}\rangle$$

and $$\left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger} = \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}.$$

Consequently,

$$\langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^{*}$$

$$\Longrightarrow \left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\gamma\left(\langle a_{ij}, c_{lk}\rangle\,\langle a_{ij}, c_{lk}\rangle^{*}\right)^{\dagger} \text{ or } \left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\gamma\,\langle a_{ij}, c_{lk}\rangle^{\dagger}$$

and $$\langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle b_{ij}, d_{lk}\rangle^{*}$$

$$\Longrightarrow \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\gamma\left(\langle b_{ij}, d_{lk}\rangle\,\langle b_{ij}, d_{lk}\rangle^{*}\right)^{\dagger} \text{ or } \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}\gamma\,\langle b_{ij}, d_{lk}\rangle^{\dagger}.$$

Since $\left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger} = \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)^{\dagger}$, then we have that $\langle a_{ij}, c_{lk}\rangle^{\dagger} \gamma \langle b_{ij}, d_{lk}\rangle$ and (ii) holds.

That (ii) implies (i) follows similarly.

Theorem 5.2. Let S be a rhotrix type A semigroup and let γ be a normal congruence on $E(S)$, then the relation defined by the rule that

$$\sigma_\gamma = \{(\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle) \epsilon\, S \times S :\ \langle a_{ij}, c_{lk}\rangle^* \,\gamma\, \langle b_{ij}, d_{lk}\rangle^*, \langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$$
$$= \langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle \text{ for some } \langle I_{ij}, c_{lk}\rangle \epsilon\, E(S), \langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^* \}$$

is the minimum congruence on S whose restriction to $E(S)$ is γ. Furthermore, σ_γ is an admissible congruence.

Proof. It can be easily shown that σ_γ is an equivalence relation. To show that σ_γ is a congruence, suppose $\langle a_{ij}, c_{lk}\rangle\, \sigma_\gamma \langle b_{ij}, d_{lk}\rangle$ and let $\langle g_{ij}, h_{lk}\rangle \epsilon\, S$. Then we have that $\langle a_{ij}, c_{lk}\rangle^*\, \sigma_\gamma\, \langle b_{ij}, d_{lk}\rangle^*$,

$$\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$$

for $\langle I_{ij}, c_{lk}\rangle \epsilon\, E(S), \langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^*$, $\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$.

So $\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle = \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle$,

$\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle \,\gamma\, \big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle a_{ij}, c_{lk}\rangle^*$,

$\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle a_{ij}, c_{lk}\rangle^* = \big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*$,

$\big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle \,\gamma\, \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*\langle b_{ij}, d_{lk}\rangle^*$,

$\big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*\langle b_{ij}, d_{lk}\rangle^* = \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*$.

Consequently,

$$\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\gamma\, \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^* \text{ and}$$
$$\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle$$
$$= \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)\big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle$$
$$= \big(\langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\big)\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle,$$

where $\big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*\langle I_{ij}, c_{lk}\rangle \,\gamma\, \big(\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\big)^*$.

Hence $\langle g_{ij}, h_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\, \sigma_\gamma\, \langle g_{ij}, h_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ and so σ_γ is a left congruence. That σ_γ is a right congruence follows similarly. Thus σ_γ is a congruence on S.

It is now clear that $\langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, 0_{lk}\rangle$ if and only if $\langle I_{ij}, c_{lk}\rangle\, \sigma_\gamma\, \langle a_{ij}, 0_{lk}\rangle$ and $tr\, \sigma_\gamma = \gamma$.

Let π be a congruence S such that $tr\, \pi = \gamma$ and $\langle a_{ij}, c_{lk}\rangle\, \sigma_\gamma\, \langle b_{ij}, d_{lk}\rangle$ for $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon\, S$, then we have that $\langle a_{ij}, c_{lk}\rangle^* \,\gamma\, \langle b_{ij}, d_{lk}\rangle^*$ and $\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$ for $\langle I_{ij}, c_{lk}\rangle \epsilon\, E(S)$, $\langle I_{ij}, c_{lk}\rangle \,\gamma\, \langle a_{ij}, c_{lk}\rangle^*$ and so $\langle a_{ij}, c_{lk}\rangle^*\, \pi\, \langle I_{ij}, c_{lk}\rangle$, $\langle b_{ij}, d_{lk}\rangle^*\, \pi\, \langle I_{ij}, c_{lk}\rangle$.

Consequently,

$$\langle a_{ij}, c_{lk}\rangle\, \pi = \langle a_{ij}, c_{lk}\rangle\, \langle a_{ij}, c_{lk}\rangle^*\, \pi = \langle a_{ij}, c_{lk}\rangle\pi\, \langle a_{ij}, c_{lk}\rangle^*\, \pi$$
$$= \langle a_{ij}, c_{lk}\rangle\, \pi\, \langle I_{ij}, c_{lk}\rangle\, \pi$$

$$= \left(\langle a_{ij}, c_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)\pi = \left(\langle b_{ij}, d_{lk}\rangle\langle I_{ij}, c_{lk}\rangle\right)\pi$$
$$= \langle b_{ij}, d_{lk}\rangle\,\pi\,\langle I_{ij}, c_{lk}\rangle\,\pi = \langle b_{ij}, c_{lk}\rangle\pi\,\langle b_{ij}, d_{lk}\rangle^{*}\,\pi$$
$$= \langle b_{ij}, d_{lk}\rangle\,\langle b_{ij}, d_{lk}\rangle^{*}\,\pi = \langle b_{ij}, d_{lk}\rangle\pi\,.$$

Thus $\sigma_\gamma \subseteq \pi$ and so σ_γ is the minimum congruence on S whose restriction to $E(S)$ is γ.

Lastly, we prove that σ_γ is admissible. Since σ_γ is a congruence, let $\langle a_{ij}, c_{lk}\rangle \epsilon S$ and $\langle p_{ij}, q_{lk}\rangle$, $\langle m_{ij}, n_{lk}\rangle \epsilon S$ such that left congruence condition holds, i.e. $\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\,\sigma_\gamma\langle a_{ij}, c_{lk}\rangle\langle m_{ij}, n_{lk}\rangle$.

Then we have that

$$\left(\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)^{*}\,\gamma\,\left(\langle a_{ij}, c_{lk}\rangle\langle m_{ij}, n_{lk}\rangle\right)^{*} \text{ and}$$
$$\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle\langle m_{ij}, n_{lk}\rangle\langle I_{ij}, c_{lk}\rangle$$

for some $\langle I_{ij}, c_{lk}\rangle \epsilon E(S)$, $\langle I_{ij}, c_{lk}\rangle\,\gamma\,\left(\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)^{*}$.

That is, $\left(\langle a_{ij}, c_{lk}\rangle^{*}\langle p_{ij}, q_{lk}\rangle\right)^{*}\,\gamma\,\left(\langle a_{ij}, c_{lk}\rangle^{*}\langle m_{ij}, n_{lk}\rangle\right)^{*}$ and

$$\langle a_{ij}, c_{lk}\rangle^{*}\langle p_{ij}, q_{lk}\rangle\langle I_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{*}\langle m_{ij}, n_{lk}\rangle\langle I_{ij}, c_{lk}\rangle,$$

$\langle I_{ij}, c_{lk}\rangle\,\gamma\,\left(\langle a_{ij}, c_{lk}\rangle^{*}\langle p_{ij}, q_{lk}\rangle\right)^{*}$.

Hence $\langle a_{ij}, c_{lk}\rangle^{*}\langle p_{ij}, q_{lk}\rangle\,\sigma_\gamma\,\langle a_{ij}, c_{lk}\rangle^{*}\langle m_{ij}, n_{lk}\rangle$.

Similarly, using the right congruence condition, that is $\langle p_{ij}, q_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\ \sigma_\gamma\,\langle m_{ij}, n_{lk}\rangle\,\langle a_{ij}, c_{lk}\rangle$ we have that $\langle p_{ij}, q_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{\dagger}\,\sigma_\gamma\,\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{\dagger}$. Therefore σ_γ is an admissible congruence.

Remark 5.3. It is important to note that the relation σ_γ can also be defined as follows:

$$\sigma_\gamma = \left\{\left(\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle\right) \epsilon\, S\times S : \langle a_{ij}, c_{lk}\rangle^{\dagger}\,\gamma\,\langle b_{ij}, d_{lk}\rangle^{\dagger}, \langle a_{ij}, 0_{lk}\rangle\langle a_{ij}, c_{lk}\rangle \right.$$
$$\left. = \langle a_{ij}, 0_{lk}\rangle\langle b_{ij}, d_{lk}\rangle \text{ for some } \langle a_{ij}, 0_{lk}\rangle \in E(S), \langle a_{ij}, 0_{lk}\rangle\,\gamma\,\langle a_{ij}, c_{lk}\rangle^{\dagger} \right\}.$$

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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