Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 3806-3812
https://doi.org/10.28919/jmcs/5691
ISSN: 1927-5307

DISJOINT ELEMENTS AND SEMI SOLIDS IN RIESZ $lG-$MODULE

K. SOWMYA*, SR. MAGIE JOSE

Department of Mathematics, St. Mary's College, Thrissur, Kerala, India

Abstract. Disjoint elements and semi solids in Riesz lG - module are introduced and properties are studied.

Keywords: $Rlg-$disjoint; $RlG-$disjoint; $RlG-$semi solid; Riesz $lG-$module; Riesz $lG-$submodule.

2010 AMS Subject Classification: 06F25, 06F20.

1. INTRODUCTION

Lattice ordered algebraic structures were discussed by Blyth [9] and Steinberg [8]. Based on group action dealt by Gallian [3] and Michel, Zhilinskii [5], representation theory was developed by Curtis, Reiner[2] and Steinberg [1]. This concept was studied in lattice structure which leads to the definition of lattice ordered G-modules by Ursala, Isaac [6] and Riesz lG-module by Sowmya, Magie and Ursala [4]. Disjoint elements in Riesz spaces were studied by Luxemburg, Zaanen [10] and Gloden [7]. Solid space (Ideal) of a Riesz space which acts as a black hole was also introduced in [7, 10]. In this paper, the concepts of disjoint elements and semi solids are introduced in a *Riesz $lG-$ module*.

*Corresponding author
E-mail address: sowmya.k@smctsr.ac.in
Received March 12, 2021

2. PRELIMINARIES

In this section, some basic definitions and results are reviewed.

Through out this paper, e denotes the identity element in the group G with binary operation $*$ and 0 denotes the identity element in the vector space E over the set of reals $\mathbf{R}$.

Definition 2.1. [8] A *partial order* on a non empty set L is a binary relation on L that is reflexive, anti-symmetric, and transitive. A partially ordered set or *poset* is a set in which a partial order is defined.

Definition 2.2. [8] A *Lattice* L is a poset in which the infimum $a \wedge b$ and supremum $a \vee b$ exist for any two elements a and b in L.

Definition 2.3. [9] Let $(G, *)$ be a group and $\leq$ be a partial order on it.Then G is a *lattice ordered group* or an *l-group* if $(G, \leq)$ is a lattice and the binary operation in G is order preserving.That is, $g \leq h \implies x * g * y \leq x * h * y$ for all $x, y, g, h \in G$.

Definition 2.4. [9] An *l-subgroup* of G is a subgroup of G which is a sublattice of G.

Definition 2.5. [9] Let G be a lattice-ordered group. The set $G^+ = \{g \in G : g \geq e\}$ is the *positive cone* of G, whose elements are termed as positive elements of G and the set $G^- = \{g \in G : g \leq e\}$ is the *negative cone* of G which contains all negative elements of G.

Definition 2.6. [9] Let G be a lattice-ordered group. Then for every $g \in G$ the *positive part* of g is $g^+ = g \vee e \in G^+$, and the *negative part* is $g^- = g \wedge e \in G^-$. The *absolute value* of g is $|g| = g \vee g^{-1} = g^+ * (g^-)^{-1}$ and $|g| \in G^+$.

Definition 2.7. [7] A real vector space V which is a poset is called an *ordered vector space* if for $x, y, z \in V$ and $0 \leq \alpha \in \mathbf{R}$,
$x \leq y \implies x + z \leq y + z$ and $\alpha x \leq \alpha y$.

Definition 2.8. [7] An ordered vector space which is a lattice is a *vector lattice* or *Riesz space*.

Definition 2.9. [7] Let E be a Riesz space. Two elements x and y in E are said to be disjoint (denoted as $x \perp y$) if $|x| \wedge |y| = 0$.

Theorem 2.10. [7] Let E be a Riesz space. For $x, y \in E$,

(i): If $x \perp y$, then $rx \perp y$ for every real number r.

(ii): If $x_1, x_2 \perp y$, then $x_1 + x_2 \perp y$.

(iii): If $x_0 = \sup\{x_i : i \in I\}$ and if $x_i \perp y$ for all i, then $x_0 \perp y$.

(iv): If $x \perp y$, then $|x+y| = |x| + |y|$.

Definition 2.11. [7] Let E be a Riesz space. An ideal A is a linear subspace of E such that for $x \in A$ and $|y| \leq |x| \implies y \in A$.

Definition 2.12. [4] Let G be an l-group. A Riesz space E is called a *Riesz $lG-$ module* if the group action G on E denoted by $g \circ x \in E$ for all $g \in G$ and $x \in E$ and has the following properties

(i): $e \circ x = x$

(ii): $(g * h) \circ x = g \circ (h \circ x)$

(iii): $g \circ (rx + sy) = r(g \circ x) + s(g \circ y)$

(iv): $|g| \circ (x \wedge y) = (|g| \circ x) \wedge (|g| \circ y)$

$|g| \circ (x \vee y) = (|g| \circ x) \vee (|g| \circ y)$

$(g \wedge h) \circ |x| = (g \circ |x|) \wedge (h \circ |x|)$

$(g \vee h) \circ |x| = (g \circ |x|) \vee (h \circ |x|)$ for all $g, h \in G$, $x, y \in E$, $r, s \in \mathbf{R}$.

Remark 2.13. [4] $g \circ 0 = 0$ for all $g \in G$.

Example 2.14. [4] $\mathbf{R^2}$ is a *Riesz $lG-$ module* under the action of $\mathbf{R}^+$, the set of positive real numbers, where the group action is defined by $r \circ (x, y) = (rx, ry)$, for $r \in \mathbf{R}^+$ and $(x, y) \in \mathbf{R^2}$.

Definition 2.15. [4] Let E be a *Riesz $lG-$ module*. A vector sublattice (Riesz subspace) F of E is a *Riesz $lG-$ submodule* or *$RlG-$submodule* of E if F itself is a *Riesz $lG-$ module* under the same action of G as that on E.

3. Main Results

Theorem 3.1. *Let E be a Riesz $lG-$ module. Then G^+ maps E^+ into E^+.*

Proof. Let $x, y \in E$ and $\hat{g} \in G^+$.
By condition (iv) in the definition of a *Riesz* $lG-$ *module*, $x \leq y$ shows that $\hat{g} \circ x \leq \hat{g} \circ y$. Now, $0 \leq x \implies 0 = \hat{g} \circ 0 \leq \hat{g} \circ x$. Hence, $\hat{g} \circ x \in E^+$. □

Theorem 3.2. *G^+ sends a Riesz subspace (vector sublattice) to a Riesz subspace (vector sublattice).*

Proof. Let E be a *Riesz* $lG-$ *module* and K be a Riesz subspace (vector sublattice) of E. Then for $\hat{g} \in G^+$, we show that $K' = \{\hat{g} \circ x : x \in K\}$ is a Riesz subspace (vector sublattice) of E. First, note that K' is non empty, for, $0 = \hat{g} \circ 0 \in K'$. Let $x, y \in K$, $\hat{g} \in G^+$ and $r \in \mathbf{R}$. Then $x+y$, rx, $x \wedge y$, $x \vee y \in K$. Now $\hat{g} \circ x + \hat{g} \circ y = \hat{g} \circ (x+y) \in K'$. Also, $r(\hat{g} \circ x) = \hat{g} \circ (rx) \in K'$. $\hat{g} \circ x \wedge \hat{g} \circ y = \hat{g} \circ (x \wedge y) \in K'$ and $\hat{g} \circ x \vee \hat{g} \circ y = \hat{g} \circ (x \vee y) \in K'$. Thus K' is a Riesz subspace (vector sublattice) of E. □

Definition 3.3. *A Riesz $lG-$ module E is said to be distributive $RlG-$ module, if $g \circ (x \wedge y) = g \circ x \wedge g \circ y$ and $g \circ (x \vee y) = g \circ x \vee g \circ y$ holds for all $g \in G$.*

Example 3.4. *The real plane $\mathbf{R^2}$ is a distributive $RlG-$ module under the action (as in Example 2.14) of the group $\mathbf{R}^+$.*

Theorem 3.5. *Let E be a distributive $RlG-$ module and K be a Riesz subspace of E. For $g \in G$, let $K' = \{g \circ x : x \in K\}$ is a Riesz subspace of E.*

Proof. Since E is a *distributive* $RlG-$ *module* , from theorem 3.2 it follows that K' is a Riesz subspace of E. □

Theorem 3.6. *For $g \in G, x \in E$, $|g| \circ |x| = ||g| \circ x|$. Hence, for $g \in G^+$, $g \circ |x| = |g \circ x|$.*

Proof. $|g| \circ |x| = |g| \circ (x \vee (-x)) = (|g| \circ x) \vee (|g| \circ (-x))$ (by condition (iv) in the definition of a *Riesz* $lG-$ *module*)
$= (|g| \circ x) \vee -(|g| \circ x) = ||g| \circ x|$.The second result follows immediately. □

Theorem 3.7. *Let x and y be two disjoint elements of E. Then $g \circ x$ and $g \circ y$ are disjoint for all $g \in G^+$.*

Proof. Let $x,\ y \in E$ and $g \in G^+$. Since $x,\ y$ are disjoint, $|\, x \,| \wedge |y| = 0$.
Now, $0 = g \circ 0 = (g \circ (|\, x \,| \wedge |y|)) = (g \circ |\, x \,|) \wedge (g \circ |y|) = |g \circ x| \wedge |g \circ y|$ using the theorem 3.6. Hence $g \circ x$ and $g \circ y$ are disjoint. □

Definition 3.8. Two elements x and y in a *Riesz* $lG-$ *module* E are said to be $Rlg-disjoint$ denoted by $x \perp^{Rlg} y$ if $|g \circ x| \wedge |g \circ y| = 0$ for some $g \in G^+$. That is, if $g \circ x$ and $g \circ y$ are disjoint for some $g \in G^+$. If x and y are $Rlg-disjoint$ for all $g \in G^+$, then they are called $RlG-disjoint$.

Remark 3.9. In a *Riesz* $lG-$ *module* E, the identity element 0 is $RlG-$ disjoint to all other elements in E.

Remark 3.10. *If x and y are disjoint $(x \perp y)$, then they are $RlG-disjoint$.*

Theorem 3.11. *Let E be a Riesz $lG-$ module. Let $g \in G^+$. If x and y are $Rlg-disjoint$, then $|g \circ (x+y)| = |g \circ x| + |g \circ y|$.*

Proof. If x and y are $Rlg-disjoint$, then $|\, g \circ x| \wedge |\, g \circ y| = 0$ for $g \in G^+$. That is, $g \circ x$ and $g \circ y$ are disjoint. Therefore, $|g \circ x + g \circ y| = |g \circ x| + |g \circ y|$. Hence, $|g \circ (x+y)| = |g \circ x| + |g \circ y|$. □

Theorem 3.12. *Let $x,\ y \in E$ and fix $g \in G^+$. Let $y^{\perp^{Rlg}} = \{x : x \perp^{Rlg} y\}$ denotes the set of all elements of E which are $Rlg-disjoint$ to y. Then $y^{\perp^{Rlg}}$ is a linear subspace of E.*

Proof. Note that $y^{\perp^{Rlg}}$ is nonempty as $0 \in y^{\perp^{Rlg}}$. Let $x,\ z \in y^{\perp^{Rlg}}$ and $g \in G^+$. Then $|g \circ x| \wedge |g \circ y| = 0$ and $|g \circ z| \wedge |g \circ y| = 0$. That is, $g \circ x$ and $g \circ z$ are disjoint to $g \circ y$. Then, $(g \circ x + g \circ z) \perp g \circ y$. Therefore, $g \circ (x+z) \perp g \circ y$. Hence $x+z \in y^{\perp^{Rlg}}$.
Let $r \in \mathbf{R}$. Now $x \perp y$ implies $rx \perp y$. Since, x and y are $Rlg-disjoint$, $g \circ x$ is disjoint to $g \circ y$ which in turn shows that $r(g \circ x) \perp (g \circ y)$. But, $r(g \circ x) = g \circ (rx)$. Hence, $rx \in y^{\perp^{Rlg}}$. □

Theorem 3.13. *Let E be a Riesz $lG-$ module and $y \in E$. For $g \in G^+$, the set of distinct nonzero elements which are pairwise $Rlg-disjoint$ is linearly independent.*

Proof. Let $\{x_i : i = 1, 2, \ldots, n\}$ be a set of nonzero elements that are pairwise $Rlg-disjoint$. Let $x_1 = r_2x_2 + r_3x_3 + \ldots + r_nx_n$ for $r_i \in \mathbf{R}$, $i = 2, 3, \ldots, n$. From theorem 3.12, it follows that $x_1 \perp^{Rlg} r_2x_2 + r_3x_3 + \ldots + r_nx_n$. Then $x_1 \perp^{Rlg} x_1$. That is, $|g \circ x_1| \wedge |g \circ x_1| = 0$. Hence, $|g \circ x_1| = 0$. That is, $g \circ x_1 = 0 \implies x_1 = 0$ which contradicts the choice of elements. □

The positive cone G^+ maps E^+ onto E^+ (3.1) . This made us to define the following.

Definition 3.14. $z \in E^+ \implies g \circ z \in E^+$ for all $g \in G$, then G is said to be $RlG-strict$ on z. The l-group G is said to be $RlG-strict$ on E, if G is $RlG-strict$ on x for every $x \in E^+$.

Theorem 3.15. *Let E be a Riesz $lG-$ module and $x, y \in E$. Then G is $RlG-strict$ on E if and only if $x \le y \iff g \circ x \le g \circ y$ for all $g \in G$.*

Theorem 3.16. *Let E be a Riesz $lG-$ module and I is an ideal of E. Let $g \in G^+$. Suppose that G is $RlG-strict$ on E. Then $I' = \{g \circ x : x \in I\}$ is an ideal of E.*

Proof. Theorem 3.2 shows that I' is a Riesz subspace of E. Now, let $x \in I$ and $g \in G^+$. Then $g \circ x \in I'$. Choose $y \in E$ such that $|g \circ y| \le |g \circ x|$, then, $g \circ |y| \le g \circ |x|$. Since G is $RlG-strict$ on E, $|y| \le |x|$. Since, I is an ideal, $y \in I$ and thus $g \circ y \in I'$. Thus, I' is an ideal of E. □

Definition 3.17. *Let E be a Riesz $lG-$ module and S be a vector subspace of E. Then S is called a $RlG-$ semi solid in E if for, any $g \in G^+$, $x \in S, y \in E$, $|g \circ y| \le |g \circ x| \implies y \in S$.*

Theorem 3.18. *Let E be a Riesz $lG-$ module and D be a nonempty subset of E^+. Let $D^{\perp^{Rlg}} = \{x : x \perp^{Rlg} y$ for all $y \in D\}$. Then $D^{\perp^{Rlg}}$ is a $RlG-$semi solid in E. The set $D^{\perp^{Rlg}}$ denotes the set of all elements of E that are $Rlg-$ disjoint to every $y \in D$.*

Proof. Since $0 \in D^{\perp^{Rlg}}$, it is nonempty. Theorem 3.12 shows that $D^{\perp^{Rlg}}$ is a vector subspace of E.

Let $x \in D^{\perp^{Rlg}}, y \in D, z \in E$ and $g \in G^+$. To prove $D^{\perp^{Rlg}}$ is *$RlG-$semi solid*, we prove that if $x \perp^{Rlg} y$, $|g \circ z| \le |g \circ x| \implies z \perp^{Rlg} y$. For that, let $|g \circ z| \le |g \circ x|$. Then $|g \circ z| \wedge |g \circ y| \le |g \circ x| \wedge |g \circ y| = 0$. Therefore, $|g \circ z| \wedge |g \circ y| = 0$. Thus, $z \perp^{Rlg} y$. □

Theorem 3.19. *Intersection of any two $RlG-$semi solids is again an $RlG-$ semi solid.*

Proof. Let E be *Riesz* $lG-$ *module* and I_1, I_2 be two $RlG-$*semi solids* in E. Then $I_1 \cap I_2$ is a vector subspace of E. Let $z \in E$. Suppose $x \in I_1 \cap I_2$, and $|g \circ z| \leq |g \circ x|$. Since, $x \in I_1 : |g \circ z| \leq |g \circ x| \implies z \in I_1$. Since, $x \in I_2 : |g \circ z| \leq |g \circ x| \implies z \in I_2$. Therefore, $z \in I_1 \cap I_2$. □

Definition 3.20. Let D be a nonempty subset of E. The intersection of $RlG-$*semi solids* in E containing D is an $RlG-$*semi solid* in E and contains D. It is called an $RlG-$*semi solid* generated by D. If D contains only one element, then it is called a principal $RlG-$*semi solid*.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] B. Steinberg, Representation theory of finite groups. an introductory approach, Universitext, Springer, New York, 2012.

[2] C.W. Curtis, I. Reiner, Representation theory of finite groups and associative algebras, Interscience, New York, 1962.

[3] J.A. Gallian, Contemporary Abstract Algebra, Narosa Publishing House, New Delhi, (1999).

[4] K. Sowmya, J. Magie, P. Ursala, Introduction to *Riesz* $lG-$ *module*, Malaya J. Mat. 8 (2020), 561-564.

[5] L. Michel, B. Zhilinskii, Introduction to Lattice geometry through Group action, EDP Sciences, Les Ulis, France, (2015).

[6] P. Ursala, P. Isaac, An introduction to lattice ordered G-modules, J. Glob. Res. Math. Arch. 4 (2017), 76-82.

[7] R.F. Gloden, (ed.), Lectures on ”Riesz Spaces” given by Prof. A. C. Zaanen, Euratom, (1966).

[8] S.A. Steinberg, Lattice-ordered Rings and Modules, Springer, New York, (2010).

[9] T.S. Blyth, Lattices and ordered algebraic structures, Springer, New York, (2005).

[10] W.A.J. Luxemburg, A.C. Zaanen, Riesz Spaces I, North-Holland, Amsterdam, 1971.