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DIGITAL IMAGE DIMENSION AND SPACE REDUCTION WITH CONTRACTION MAPPING

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Abstract: A digital image can be represented as two dimensional arrays of pixels. In this paper a simple but representative system of contractive mapping on Euclidean plane, called the digital plane is used for image processing to generate images with reduced dimension occupying less storage space and can be efficiently transmitted. With suitable matrix metric and contraction mapping, size of the original image matrix is diminished significantly by reducing the order of sub matrices and hence images of reduced size without much compromise to the quality of image is obtained. The variations between the original and contracted image are not too pronounced when the images are of large size and are seen on small screen (mobile, tablets etc.)

Keywords: contraction mapping; digital images; data compression.
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1. INTRODUCTION

Contraction is a mapping which reduces the distance (function). In 1922 an important theorem in metric space theory called Contraction Mapping Principle was given by Banach [1]. The concept is useful in the existence and uniqueness theory and considerably forms the foundation of range of image processing tools. The properties of digital images are characterized with tools from

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algebraic topology. The field of digital topology was founded by Rosenfeld [2] and contributed fundamentally to variety of applications like image processing, pattern recognition, developed the notion of digital continuity for 2D and 3D digital images. The topological properties of digital images are studied as discretized arrays of two or more dimensions [3]. A digital metric space is complete [4]. Boxer [5] gave the digital versions of several notions from topology and studied a variety of digital continuous functions. Lefschetz fixed point theorem for digital images was proved by Ege and Karaca [6, 7], also showed that sphere-like digital images have the fixed point property. Han [8] refined and improved various notions and assertions given in [6]. With advanced technology in digital cameras, digital imaging system has proliferated in the past few years. The demand of storing huge amount of large sized image files, the call for space to store them is increasing. Various lossless and lossy compression techniques are available [9] which facilitates in reducing the size of image data files. The authors put forward contraction method to reduce the size of image files as an application to contractive mappings. By reducing the total number of pixels, image resizing (reduction) can be achieved. An input image is of size $m \times n$ and objective is to obtain an image of proportionately reduced size. The original digital image is processed to reduce its size and the image when finally displayed on the screen is not exactly same as the original because it is subjected to distortions in trade off the size reduction, but will be the good representative of the original image. The quality of the processed image is confirmed not only by subjective method which involve human being to evaluate the quality of the image but also by the objective image quality metrics which numerically calculates the quality measure as PSNR [10].

2. PRELIMINARIES

Definition 2.1: (Contraction): A mapping $f: X \to Y$ is called contraction if

$d\big(f(x), f(y)\big) \leq k\, d(x, y)\ \forall\, x, y \in X$ and $0 \leq k < 1$; d is the distance metric on space X.

Definition 2.2: (Banach Contraction Principle) [1]: Consider a complete metric space (X, d). If mapping $f: X \to X$ be a contraction on a complete metric space then f has a unique fixed point in X, the sequence of iterates $x, f(x), f\big(f(x)\big) \dots \dots..$, for $x \in X$, converges to the fixed point of function f.

Definition 2.3: Frobenius norm of matrix A_{mn} is given by $\rho_F(A) = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \big(a_{ij}\big)^2}$

Definition 2.4: Peak Signal-to-noise ratio (PSNR) calculates the difference between two images and is defined as

$$PSNR = 10log_{10}\left(\frac{b \times b}{rms}\right)$$

In this paper 8 bit test image is used for illustration, therefore $b = 255$.

3. PROPOSED ALGORITHM FOR IMAGE CONTRACTION

Various partitioning techniques are available in literature [9, 11] which facilitates the image compression process. The readers can find the comprehensive study regarding hierarchical development of image representations in recent past in [4]. Image data reduction; primarily utilize diminution of not pertinent information in an image called redundancy. An image can have different type of redundancies viz., psycho visual, inter pixel and coding redundancy. In the presented scheme the limitation of the human visual system is utilized along with inter pixel redundancy which is due to similarities in the neighboring pixels. Digital images can be represented as rectangular arrays of pixel values. Any 8 bit grayscale digital image can be represented as $m \times n$ matrix of pixel values, where each element represents the gray value of the pixel of the corresponding index. In 8 bit gray scale these pixels have 256 shades 0 to 255, ‘0’ is for black, ‘255’ is for white, in between there are 254 shades of gray.

Consider X , the digital image space, f be the contraction mapping defined on it along with distance function d . Here the contraction mapping Principle is given by:

$$d[f(A_1), f(A_2)] \leq k\ d(A_1, A_2)\ \forall\ A_1, A_2\ \in X$$

such that $f: X \to X$; $f(A_i) = \not{f}(\not{g}(A_i)) = B_i \quad \forall\ A_i \in A, B_i \in B \subset X$ (1)

$\not{g}(A_i) = \frac{1}{N}\sum_1^N x_i \quad \forall\ x_i \in A_i$

$\not{f}$ = floor function

d= matrix distance function is taken as Frobenius norm (Def. 2.3)

$$\rho_F(A, B) = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij} - b_{ij}\right)^2}$$

k =contractivity factor, $0 < k < 1$

The steps of the proposed algorithm is listed below which involves local operations on each square fixed width block which are performed in parallel on every element (pixel value) and to its immediate neighbours.

1. Input the gray scale image I.
2. Get $\mathcal{A} = [a_{i,j}]$, $1 \le i \le r; 1 \le j \le c$ as the pixel values of the input image.
3. Partition $\mathcal{A}$ into sub-matrices $\mathcal{A}_p^n$ (p = number of sub matrices obtained after partitioning) with fixed block of size $n \times n$ $(n \ge 2)$.
4. Each sub-matrix A_p is subjected to the contraction mapping f defined by (1) and corresponding contracted sub-matrix B_p^1 of dimension 1×1 is obtained. Arrange each contracted sub matrix in the same sequence as that of the parent sub matrix, subsequently obtain contracted matrix B.
5. Get contracted output image I'.

The above steps can be understood as

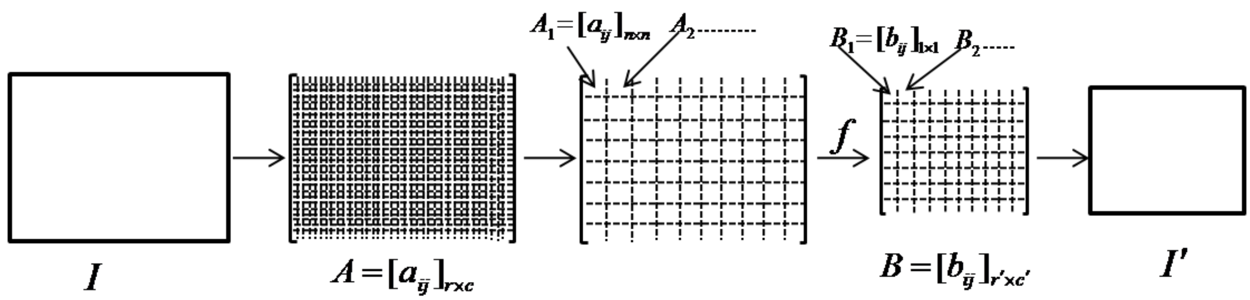

Figure 1: Schematic for I to I'

Each $A_s^n = n \times n$, is a non-overlapping square sub-matrix of fixed dimension $(n \ge 2)$ which then is converted to square sub-matrix B_s^1 of dimension 1×1 by using composition of two functions g, f as given in (1). All such B_s^1 matrices are arranged sequentially in place of each A_s^n to obtain contracted matrix $\mathcal{B} = \bigcup_{i=1}^{s} B_s^1$, which is with reduced dimension than that of original matrix $\mathcal{A}$. Finally convert contracted matrix B to contracted image I' occupying less space than *I.* The method matches to fixed width partitioning.

In Banach contraction mapping theorem (Definition 2.2), the contractivity constant k remains same in all the iterations which also forms the framework of the proof of the theorem and is theoretically possible. Here, when the distance (d) for each fixed width block along with its contractivity factor is considered then for overall scheme $k = \min\{k_i; i = 1, 2, \dots.\}$

4. ILLUSTRATION

Python 3.7 on spyder IDE is used to implement the proposed algorithm. The image I (Fig. 2) is used as an input to the algorithm. Input reference image I is converted to subsequent $r \times c$ matrix $\mathcal{A}$ where each element correspond to the intensity of that pixel. Matrix $\mathcal{A}$ is then partitioned into several blocks (sub-matrices) of fixed block size $n \in \mathbb{I}$ $(n \geq 2)$. This is image segmentation. Each block is worked on without the reference to the others, independently and if programmed can be processed concurrently. The various steps involved are illustrated below. Consider a small 16×16 pixel region of the original image I (Fig. 2). The first matrix is corresponding to the pixels of I , 64 non-overlapping blocks A_s^2 , $s = 1, 2, \ldots 64$, each of them is of block size 2×2, when subjected to mapping (1), 64 non-overlapping blocks B_s^1 , $s = 1, 2, \ldots .64$, each of size 1×1 are obtained, from their systematic union matrix B with reduced dimension is obtained and can be converted to image I'. For the same input, $n > 2$, the resulting contracted matrix is of different size and values. Now we show the sequence of iterates obtained on mapping (1) as below:

Figure 2: Original image I

$$A = \begin{bmatrix} 144 & 146 & 144 & 143 & 143 & 143 & 144 & 146 & 146 & 154 & 157 & 152 & 150 & 145 & 133 & 127 \\ 141 & 146 & 149 & 150 & 152 & 150 & 150 & 153 & 153 & 146 & 138 & 130 & 129 & 120 & 105 & 106 \\ 146 & 150 & 151 & 150 & 148 & 140 & 134 & 134 & 137 & 123 & 117 & 116 & 102 & 80 & 71 & 97 \\ 145 & 143 & 137 & 134 & 132 & 127 & 120 & 119 & 106 & 97 & 93 & 94 & 81 & 70 & 88 & 136 \\ 130 & 122 & 112 & 105 & 103 & 99 & 93 & 94 & 95 & 98 & 92 & 91 & 80 & 88 & 114 & 148 \\ 102 & 100 & 95 & 94 & 94 & 94 & 94 & 97 & 94 & 100 & 90 & 91 & 88 & 118 & 150 & 167 \\ 88 & 89 & 90 & 95 & 97 & 94 & 93 & 96 & 98 & 92 & 78 & 98 & 118 & 150 & 170 & 162 \\ 107 & 99 & 97 & 97 & 97 & 96 & 95 & 98 & 99 & 86 & 79 & 128 & 157 & 160 & 171 & 168 \\ 97 & 99 & 102 & 100 & 98 & 97 & 93 & 96 & 93 & 72 & 77 & 145 & 178 & 170 & 179 & 168 \\ 96 & 99 & 99 & 93 & 92 & 92 & 94 & 102 & 85 & 77 & 96 & 165 & 192 & 175 & 168 & 162 \\ 97 & 94 & 97 & 97 & 96 & 98 & 95 & 98 & 83 & 93 & 125 & 182 & 191 & 172 & 162 & 156 \\ 104 & 98 & 102 & 103 & 99 & 102 & 96 & 85 & 88 & 109 & 150 & 190 & 188 & 169 & 160 & 155 \\ 111 & 109 & 109 & 102 & 97 & 104 & 99 & 83 & 94 & 133 & 174 & 190 & 180 & 171 & 164 & 160 \\ 83 & 94 & 108 & 110 & 110 & 107 & 98 & 89 & 109 & 157 & 186 & 180 & 167 & 171 & 168 & 167 \\ 36 & 60 & 87 & 104 & 112 & 102 & 89 & 95 & 132 & 162 & 176 & 168 & 162 & 169 & 167 & 170 \\ 22 & 40 & 61 & 85 & 101 & 90 & 84 & 104 & 153 & 172 & 176 & 175 & 162 & 166 & 161 & 162 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 145 & 143 & 143 & 146 & 149 & 151 & 150 & 139 \\ 145 & 150 & 147 & 142 & 139 & 125 & 107 & 94 \\ 135 & 122 & 115 & 106 & 99 & 92 & 79 & 121 \\ 94 & 93 & 94 & 95 & 96 & 89 & 118 & 162 \\ 100 & 99 & 97 & 95 & 87 & 107 & 166 & 169 \\ 96 & 96 & 94 & 97 & 84 & 142 & 182 & 162 \\ 105 & 104 & 100 & 90 & 106 & 176 & 177 & 159 \\ 68 & 102 & 107 & 92 & 140 & 177 & 167 & 168 \end{bmatrix} = B \quad ; f^2(A) = \begin{bmatrix} 145 & 144 & 141 & 122 \\ 111 & 102 & 94 & 120 \\ 97 & 95 & 105 & 169 \\ 94 & 97 & 149 & 167 \end{bmatrix} = B';$$

$$f^3(A) = \begin{bmatrix} 125 & 119 \\ 95 & 147 \end{bmatrix}; \quad f^4(A) = [121]; \quad f^5(A) = [121]$$

$A_1 = \begin{bmatrix} 144 & 146 \\ 141 & 146 \end{bmatrix}$; $A_2 = \begin{bmatrix} 144 & 143 \\ 149 & 150 \end{bmatrix}$; $f(A_1) = [145]$; $f(A_2) = [143]$.

For the reconstructed image as the PSNR can be calculated only for the images of same dimension, therefore we take $f(A_1) = \begin{bmatrix} 145 & 145 \\ 145 & 145 \end{bmatrix}$ and $f(A_2) = \begin{bmatrix} 143 & 143 \\ 143 & 143 \end{bmatrix}$. Hence,

$$d[f(A_1), f(A_2)] \leq k \, d(A_1, A_2) \; \forall \, A_1, A_2 \; \in A \subset X, 0 < k < 1$$

Here, $d[f(A_1), f(A_2)] = 4$; $d(A_1, A_2) = 9.434$ (approx.) and $0 < k = 0.424 < 1$.

4. RESULT AND ANALYSIS

The image I (Fig. 2) is used as an input to the algorithm and figures 3(a), 3(b) and 3(c) are the obtained contracted images with reduced dimensions taking the fixed block size of the sub matrix as $n = 2, 3, 4$ respectively.

Figure 3: Image with block size (a) $n = 2$ (b) $n = 3$ (c) $n = 4$

Using the pixels of reduced output image, the reconstructed images (Fig. 4a, 4b and 4c) are obtained to match the dimension of I. Quality of reconstructed images is given by the PSNR (listed in table 1). But using the subjective quality measure it is clear from Fig. 4a, 4b and 4c that as the n increases, the quality of reconstructed images minifies.

The properties of contracted images are summarized in Table 1. Space saved is calculated with reference to image I, repetitive application of the preset function can be implemented to further diminish the image size. The 114×85 size (Fig. 4c) can be achieved either by considering partitioning with block size 4×4 or by repeated application of the scheme (iteratively) twice by considering block size 2×2 in each step. In both the cases the contracted image will be approximately the same. The quality of processed digital image can be done subjectively and objectively. PSNR is one of the widely used objective quality measure [10].

Figure 4 (a): Image reconstructred to original size after size reduction with 2×2 fixed size blocks, PSNR=25.36

Figure 4 (b): Image reconstructred to original size after size reduction with 3×3 fixed size blocks, PSNR=25.95

Figure 4 (c): Image reconstructred to original size after size reduction with 4×4 fixed size blocks, PSNR=24.15

Table 1: Summary of input test image and output images

	Figure	Dimension	Size (KB)	Block size	% space saved (as compared to I)	PSNR
Test Image I	2	453×340	124	-	-	-
Contracted	3a	227×170	50.0	2×2	59.6	-
	3b	151×113	23.1	3×3	81.3	-
	3c	114×85	7.43	4×4	94.0	-
Reconstructed	6a	453×340	35.5	2×2	71.3	25.36
	6b	453×340	19.3	3×3	84.4	25.95
	6c	453×340	13.5	4×4	89.1	24.15

5. CONCLUSION

The metric is defined suitably on a continuous digital space which is then discretized as pixels for image processing. The concept of Contraction mapping on composition of functions is defined and has been used to get contracted images. It is noticed the original image size contracted without much compromise in image quality. Thus, the obtained contracted image is of reduced size and utilizes less space for storage and therefore comparatively easier to transmit. The scheme is implemented to various block size to examine the extent an image can be contracted. Also, the designed function is iterated repeatedly on input image, the obtained final contracted image is of considerable quality if the input image is of large size and/or is an image with less variation in colors.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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