Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 3377-3390
https://doi.org/10.28919/jmcs/5705
ISSN: 1927-5307

ON (m,n)-QUASI-GAMMA-IDEALS IN ORDERED *LA*-GAMMA-SEMIGROUPS

WICHAYAPORN JANTANAN[1], RONNASON CHINRAM[2], PATTARAWAN PETCHKAEW[3,*]

[1]Department of Mathematics, Faculty of Science, Buriram Rajabhat University, Mueang, Buriram 31000, Thailand

[2]Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand

[3]Mathematics Program, Faculty of Science and Technology, Songkhla Rajabhat University, Songkhla 90000, Thailand

Abstract. In this paper, we introduce the concepts of m-left-Γ-ideals, n-right-Γ-ideals and (m,n)-quasi-Γ-ideals in ordered LA-Γ-semigroups and investigate about some properties of those ideals. We show that the intersection of an m-left-Γ-ideal and an n-right-Γ-ideal of an ordered LA-Γ-semigroup is an (m,n)-quasi-Γ-ideal. In addition, we introduce the notion of the (m,n) intersection property in ordered LA-Γ-semigroups and prove that every (m,n)-quasi-Γ-ideal in an ordered LA-Γ-semigroup with left identity has the (m,n) intersection property.

Keywords: ordered LA-Γ-semigroups; m-left-Γ-ideals; n-right-Γ-ideals; (m,n)-quasi-Γ-ideals; (m,n) intersection property.

2010 AMS Subject Classification: 06F05, 20M99.

1. INTRODUCTION

The notion of quasi-ideals in semigroups is a generalization of the notion of one-sided-ideals in semigroups. It was introduced by Steinfeld [12] in 1956. The notion of (m,n)-quasi-ideals in

*Corresponding author
E-mail address: pattarawan.pe@gmail.com
Received March 16, 2021

semigroups was introduced by Ansari, Khan and Kaushik [3]. In 1972, Kazim and Naseerudin [6] gave the concept of an *LA*-semigroup (left almost semigroup). This algebraic structure is a generalization of commutative semigroups. An *LA*-semigroup is also widely known as an Abel-Grassmann's groupoid or *AG*-groupoid [7, 10]. Later, an ordered *LA*-semigroups or ordered *AG*-groupoids were considered in [11]. The notion of Γ-semigroups was introduced by Sen [9] in 1981. The concept of an *LA*-Γ-semigroup (Γ-*AG*-groupoid) was introduced by Shah and Rehman [10] in 2010. Every *LA*-semigroup under the operation $\{\circ\}$ is an *LA*-Γ-semigroup if consider in case $\Gamma = \{\circ\}$. Moreover, Khan, Amjid, Zaman and Yaqoob [7] gave the concept of ordered *LA*-Γ-semigroups (ordered Γ-*AG*-groupoid) in 2014. This algebraic structure is a generalization of *LA*-Γ-semigroups [2, 4]. Then the structure of ordered *LA*-Γ-semigroups is also a generalization of commutative semigroups and *LA*-semigroups.

In this study, we introduce and examine the concept of m-left-Γ-ideals, n-right-Γ-ideals and (m,n)-quasi-Γ-ideals in ordered *LA*-Γ-semigroups. We characterize m-left-Γ-ideals and n-right-Γ-ideals in ordered *LA*-Γ-semigroups and explore the properties of (m,n)-quasi -Γ-ideals in ordered *LA*-Γ-semigroups. Moreover, the properties of (m,n)-quasi-Γ-ideals in regular ordered *LA*-Γ-semigroups are investigated.

2. PRELIMINARIES

For the sake of completeness, we recall some necessary definitions, notations and properties which are used throughout the paper.

Definition 2.1. [10] Let S and Γ be non-empty sets, then S is called an *LA*-Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$, written (a,γ,b) and denoted by $a\gamma b$ such that S satisfied the left invertive law $(a\gamma b)\beta c = (c\gamma b)\beta a$ for all $a,b,c \in S$ and $\gamma, \beta \in \Gamma$.

Definition 2.2. [10] An element e of an *LA*-Γ-semigroup S is called a *left identity* if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

Lemma 2.1. [10] If S is an *LA*-Γ-semigroup with left identity e, then $S\Gamma S = S$ and $S = e\Gamma S = S\Gamma e$.

Definition 2.3. [1] An LA-Γ-semigroup S is called a *locally associative LA-Γ-semigroup* if $(a\gamma a)\beta a = a\gamma(a\beta a)$ for all $a \in S$ and $\gamma, \beta \in \Gamma$.

Proposition 2.2. [1] Let S be an LA-Γ-semigroup.

(i) Every LA-Γ-semigroup with left identity satisfy the equalities $a\gamma(b\beta c) = b\gamma(a\beta c)$ and $(a\gamma b)\beta(c\alpha d) = (d\gamma c)\beta(b\alpha a)$ for all $a,b,c,d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.

(ii) An LA-Γ-semigroup S is Γ-medial, i.e., $(a\gamma b)\beta(c\alpha d) = (a\gamma c)\beta(b\alpha d)$ for all $a,b,c,d \in S$ and $\gamma, \beta, \alpha \in \Gamma$.

Definition 2.4. [2] An *ordered LA-Γ-semigroup (po-LA-Γ-semigroup)* is a structure $(S, \Gamma, \cdot, \leq)$ in which the following conditions hold.

(i) $(S, \Gamma, \cdot)$ is an LA-Γ -semigroup.

(ii) $(S, \leq)$ is a poset (i.e. reflexive, anti-symmetric and transitive).

(iii) For all a, b and $x \in S$, $a \leq b$ implies $a\alpha x \leq b\alpha x$ and $x\alpha a \leq x\alpha b$ for all $\alpha \in \Gamma$.

Throughout this paper, unless stated otherwise, S stands for an ordered LA-Γ-semigroup. For a non-empty subsets A and B of an ordered LA-Γ-semigroup S, we defined

$$A\Gamma B = \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}$$

and

$$(A] = \{t \in S \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we shall write $(a]$. For a positive integer m, the power of A is defined

$$A^m = (\ldots((A\Gamma A)\Gamma A)\ldots)\Gamma A \ (m \text{ times}).$$

For $A = \{a\}$, we shall write a^m.

Proposition 2.3. [8] Let S be a locally associative LA-Γ-semigroup with left identity. Then $a^1 = a, a^{n+1} = a^n\Gamma a$ and $a^{m+n} = a^m\Gamma a^n$ for all $a \in S$ and m, n are positive integers.

Definition 2.5. [2] A non-empty subset A of an ordered LA-Γ-semigroup S, is called an *LA-Γ-subsemigroup* of S if $A\Gamma A \subseteq A$.

Definition 2.6. [2] A non-empty subset A of an ordered LA-Γ-semigroup S is called a *left (right) Γ-ideal* of S if

(i) $S\Gamma A \subseteq A$ $(A\Gamma S \subseteq A)$,

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

A non-empty subset A of an ordered LA-Γ-semigroup S is called a Γ-*ideal* of S if it is both a left and a right Γ-ideal of S.

Definition 2.7. [2] A non-empty subset A of an ordered LA-Γ-semigroup S is called a *quasi*-Γ-*ideal* of S if

(i) $A\Gamma S \cap S\Gamma A \subseteq A$,

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Lemma 2.4. [7] Let S be an ordered LA-Γ-semigroup, then the following are true.

(i) $A \subseteq (A]$, for all $A \subseteq S$.

(ii) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$.

(iii) $(A]\Gamma(B] \subseteq (A\Gamma B]$, for all subsets A, B of S.

(iv) $(A] = ((A]]$, for all $A \subseteq S$.

(v) For every left (resp. right) Γ-ideal T of S, $(T] = T$.

(vi) $((A]\Gamma(B]] = (A\Gamma B]$, for all subsets A, B of S.

(vii) $(A \cap B] \subseteq (A] \cap (B]$, for all subsets A, B of S.

$(viii)$ $(A \cup B] = (A] \cup (B]$, for all subsets A, B of S.

3. Main Results

In this section, we define and study m-left-Γ-ideal, n-right-Γ-ideal and (m,n)-quasi-Γ-ideal in ordered LA-Γ-semigroups.

Definition 3.1. An LA-Γ-subsemigroup A of an ordered LA-Γ-semigroup S is called an *m-left (n-right)* Γ-*ideal* of S if

(i) $S^m\Gamma A \subseteq A$ $(A\Gamma S^n \subseteq A)$, where m, n are positive integers,

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Definition 3.2. An LA-Γ-subsemigroup A of an ordered LA-Γ-semigroup S is called an (m,n)-*quasi*-Γ-*ideal* of S if

(i) $(S^m\Gamma A]\cap(A\Gamma S^n]\subseteq A$, where m,n are positive integers,

(ii) if $a\in A$ and $b\in S$ such that $b\leq a$, then $b\in A$.

Example 3.1. Consider $S=\{1,2,3\}$, $\Gamma=\{\alpha,\beta\}$ and the order $\leq$.

α	1	2	3
1	1	1	1
2	1	1	1
3	1	1	2

β	1	2	3
1	2	2	2
2	2	2	2
3	2	2	3

$\leq:=\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$.

Hence S is an ordered *LA*-Γ-semigroup because the elements of S satisfies left invertive law. Let $A=\{1,2\}$, we have $S^1\Gamma A=\{1,2\}=A$ and $A\Gamma S^2=\{1,2\}=A$. Also, for every $1,2\in A$ there exists $1,2\in S$ such that $1\leq 1$, $1\leq 2$, $2\leq 2$ implies that $1,2\in A$ or $(A]=A$. Thus A is an 1-left-Γ-ideal and A is an 2-right-Γ-ideal of S.

Let $A=\{1,2\}$, we have $(S^1\Gamma A]\cap(A\Gamma S^2]=\{1,2\}\cap\{1,2\}=\{1,2\}=A$. Also, $(A]=A$. Hence A is an $(1,2)$-quasi-Γ-ideal of S.

Lemma 3.1. Let S be an ordered *LA*-Γ-semigroup and let T_i be an *LA*-Γ-subsemigroup of S for all $i\in I$. If $\bigcap_{i\in I}T_i\neq\varnothing$, then $\bigcap_{i\in I}T_i$ is also an *LA*-Γ-subsemigroup of S.

Proof. Assume that $\bigcap_{i\in I}T_i\neq\varnothing$. Let $a,b\in\bigcap_{i\in I}T_i$. Since T_i is an *LA*-Γ-subsemigroup of S for all $i\in I$, we have $a\gamma b\in T_i$ for all $i\in I$ and $\gamma\in\Gamma$. Thus $a\gamma b\in\bigcap_{i\in I}T_i$. Hence $\bigcap_{i\in I}T_i$ is an *LA*-Γ-subsemigroup of S. □

Theorem 3.2. Let S be an ordered *LA*-Γ-semigroup and let Q_i be an (m,n)-quasi-Γ-ideal of S for all $i\in I$. If $\bigcap_{i\in I}Q_i\neq\varnothing$, then $\bigcap_{i\in I}Q_i$ is also an (m,n)-quasi-Γ-ideal of S .

Proof. Assume that $\bigcap_{i\in I}Q_i\neq\varnothing$. By Lemma 3.1, we obtain that $\bigcap_{i\in I}Q_i$ is an *LA*-Γ-subsemigroup of S. Thus $(S^m\Gamma\bigcap_{i\in I}Q_i]\cap(\bigcap_{i\in I}Q_i\Gamma S^n]\subseteq(S^m\Gamma Q_i]\cap(Q_i\Gamma S^n]\subseteq Q_i$ for all $i\in I$. Also, we get that $(S^m\Gamma\bigcap_{i\in I}Q_i]\cap(\bigcap_{i\in I}Q_i\Gamma S^n]\subseteq\bigcap_{i\in I}Q_i$. Next, let $a\in\bigcap_{i\in I}Q_i$ and $b\in S$ such that $b\leq a$. Since

$a \in \bigcap_{i\in I} Q_i$, we have $a \in Q_i$ where Q_i is an (m,n)-quasi-Γ-ideal of S for all $i \in I$. Hence $b \in Q_i$ for all $i \in I$. Therefore, $b \in \bigcap_{i\in I} Q_i$. Hence $\bigcap_{i\in I} Q_i$ is an (m,n)-quasi-Γ-ideal of S. The proof is completed. □

Theorem 3.3. Let S be an ordered LA-Γ-semigroup with left identity and $a \in S$. Then the following statements are true:

(i) $(S^m\Gamma a]$ is an m-left-Γ-ideal of S.

(ii) $(a^2\Gamma S^n]$ is an n-right-Γ-ideal of S.

(iii) $(S^m\Gamma a] \cap (a^2\Gamma S^n]$ is an (m,n)-quasi-Γ-ideal of S.

Proof. (i) First, we show that $(S^m\Gamma a]$ is an LA-Γ-subsemigroup of S. We obtain

$$\begin{aligned} (S^m\Gamma a]\Gamma(S^m\Gamma a] &\subseteq ((S^m\Gamma a)\Gamma(S^m\Gamma a)] \\ &\subseteq ((S^m\Gamma S)\Gamma(S^m\Gamma a)] \\ &= ((a\Gamma S^m)\Gamma(S\Gamma S^m)] && \text{Proposition 2.2(i)} \\ &\subseteq ((a\Gamma S^m)\Gamma S^m] \\ &= ((S^m\Gamma S^m)\Gamma a] && \text{Left invertive law} \\ &\subseteq (S^m\Gamma a]. \end{aligned}$$

Hence $(S^m\Gamma a]$ is an LA-Γ-subsemigroup of S. Next, we show that $(S^m\Gamma a]$ is an m-left-Γ-ideal of S, i.e., $S^m\Gamma(S^m\Gamma a] \subseteq (S^m\Gamma a]$. Let $x \in S^m\Gamma(S^m\Gamma a]$. Then $x = y\gamma z$ for some $y \in S^m$, $z \in (S^m\Gamma a]$ and $\gamma \in \Gamma$. Since $z \in (S^m\Gamma a]$, we have $z \le s\beta a$ for some $s \in S^m$ and $\beta \in \Gamma$. Since $S\Gamma S = S$, so let $y = b\alpha c$ for some $b, c \in S$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} x &= y\gamma z \\ &\le (b\alpha c)\gamma(s\beta a) \\ &= (a\alpha s)\gamma(c\beta b) && \text{Proposition 2.2(i)} \\ &= ((c\beta b)\alpha s)\gamma a && \text{Left invertive law} \\ &\in S^m\Gamma a. \end{aligned}$$

Thus $x \in (S^m\Gamma a]$. Next, let $x \in (S^m\Gamma a]$ and $z \in S$ such that $z \leq x$. Since $x \in (S^m\Gamma a]$, we have $x \leq y\gamma a$ for some $y\gamma a \in S^m\Gamma a$. So $z \leq y\gamma a$ for some $y\gamma a \in S^m\Gamma a$. Thus $z \in (S^m\Gamma a]$. Hence $(S^m\Gamma a]$ is an m-left-Γ-ideal of S.

(ii) First, we show that $(a^2\Gamma S^n]$ is an LA-Γ-subsemigroup of S. Consider

$$\begin{aligned}
(a^2\Gamma S^n]\Gamma(a^2\Gamma S^n] &\subseteq ((a^2\Gamma S^n)\Gamma(a^2\Gamma S^n)] \\
&\subseteq ((S\Gamma S^n)\Gamma(a^2\Gamma S^n)] \\
&\subseteq (S^n\Gamma(a^2\Gamma S^n)] \\
&= (a^2\Gamma(S^n\Gamma S^n)] \qquad \text{Proposition 2.2(i)} \\
&\subseteq (a^2\Gamma S^n].
\end{aligned}$$

Hence $(a^2\Gamma S^n]$ is an LA-Γ-subsemigroup of S. Next, we show that $(a^2\Gamma S^n]$ is an n-right-Γ-ideal of S, i.e., $(a^2\Gamma S^n]\Gamma S^n \subseteq (a^2\Gamma S^n]$. Let $x \in (a^2\Gamma S^n]\Gamma S^n$. Then $x = y\gamma z$ for some $y \in (a^2\Gamma S^n]$, $z \in S^n$ and $\gamma \in \Gamma$. Since $y \in (a^2\Gamma S^n]$, we have $y \leq (a\beta a)\alpha b$ for some $b \in S^n$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
x &= y\gamma z \\
&\leq ((a\beta a)\alpha b)\gamma z \\
&= (z\alpha b)\gamma(a\beta a) \qquad \text{Left invertive law} \\
&= (a\alpha a)\gamma(b\beta z) \qquad \text{Proposition 2.3(i)} \\
&\in (a\Gamma a)\Gamma(S^n\Gamma S^n) \\
&\subseteq a^2\Gamma S^n.
\end{aligned}$$

So $x \in (a^2\Gamma S^n]$. Next, let $x \in (a^2\Gamma S^n]$ and $z \in S$ such that $z \leq x$. Since $x \in (a^2\Gamma S^n]$, then $x \leq a^2\gamma y$ for some $a^2\gamma y \in a^2\Gamma S^n$. So $z \leq a^2\gamma y$ for some $a^2\gamma y \in a^2\Gamma S^n$. Thus $z \in (a^2\Gamma S^n]$. Hence $(a^2\Gamma S^n]$ is an n-right-Γ-ideal of S.

(iii) Consider

$$\begin{aligned}
((S^m\Gamma a] \cap (a^2\Gamma S^n])\Gamma((S^m\Gamma a] \cap (a^2\Gamma S^n]) &\subseteq (S^m\Gamma a]\Gamma((S^m\Gamma a] \cap (a^2\Gamma S^n]) \\
&= (S^m\Gamma a]\Gamma(S^m\Gamma a] \cap (S^m\Gamma a]\Gamma(a^2\Gamma S^n] \\
&\subseteq (S^m\Gamma a] \cap (S^m\Gamma a]\Gamma(a^2\Gamma S^n] \subseteq (S^m\Gamma a].
\end{aligned}$$

Then, we have that $((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])\subseteq(S^m\Gamma a]$. Next, consider

$$\begin{aligned}((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n]) &\subseteq (a^2\Gamma S^n]\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])\\ &= (a^2\Gamma S^n]\Gamma(S^m\Gamma a]\cap(a^2\Gamma S^n]\Gamma(a^2\Gamma S^n]\\ &\subseteq (a^2\Gamma S^n]\Gamma(S^m\Gamma a]\cap(a^2\Gamma S^n]\subseteq(a^2\Gamma S^n]\end{aligned}$$

Thus, we get that $((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])\subseteq(a^2\Gamma S^n]$. Now, we obtain that $((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])\subseteq(S^m\Gamma a]\cap(a^2\Gamma S^n]$. Hence $(S^m\Gamma a]\cap(a^2\Gamma S^n]$ is an LA-Γ-subsemigroup of S. Next, we show that $(S^m\Gamma a]\cap(a^2\Gamma S^n]$ is an (m,n)-quasi-Γ-ideal of S. Consider

$$\begin{aligned}&(S^m\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])]\cap(((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma S^n]\\ &\quad= (S^m\Gamma(S^m\Gamma a]\cap S^m\Gamma(a^2\Gamma S^n]]\cap((S^m\Gamma a]\Gamma S^n\cap(a^2\Gamma S^n]\Gamma S^n]\\ &\quad\subseteq ((S^m\Gamma a]\cap S^m\Gamma(a^2\Gamma S^n]]\cap((S^m\Gamma a]\Gamma S^n\cap(a^2\Gamma S^n]]\\ &\quad\subseteq ((S^m\Gamma a]]\cap((a^2\Gamma S^n]]=(S^m\Gamma a]\cap(a^2\Gamma S^n].\end{aligned}$$

So, we have that $(S^m\Gamma((S^m\Gamma a]\cap(a^2\Gamma S^n])]\cap(((S^m\Gamma a]\cap(a^2\Gamma S^n])\Gamma S^n]\subseteq(S^m\Gamma a]\cap(a^2\Gamma S^n]$. Next, let $x\in(S^m\Gamma a]\cap(a^2\Gamma S^n]$ and $y\in S$ such that $y\le x$. Since $x\in(S^m\Gamma a]\cap(a^2\Gamma S^n]$, we have $x\in(S^m\Gamma a]$ and $x\in(a^2\Gamma S^n]$. Then $x\le b\gamma a$ for some $b\gamma a\in S^m\Gamma a$ and $x\le a^2\beta c$ for some $a^2\beta c\in a^2\Gamma S^n$. So $y\le x\le b\gamma a$ for some $b\gamma a\in S^m\Gamma a$ and $y\le x\le a^2\beta c$ for some $a^2\beta c\in a^2\Gamma S^n$. Thus $y\in(S^m\Gamma a]$ and $y\in(a^2\Gamma S^n]$. Therefore, $y\in(S^m\Gamma a]\cap(a^2\Gamma S^n]$. Hence $(S^m\Gamma a]\cap(a^2\Gamma S^n]$ is an (m,n)-quasi-Γ-ideal. This completes the poof. □

Theorem 3.4. Let S be an ordered LA-Γ-semigroup. Then the following statements are true:

(i) Let L_i be an m-left-Γ-ideal of S for all $i\in I$. If $\bigcap_{i\in I}L_i\neq\varnothing$, then $\bigcap_{i\in I}L_i$ is an m-left Γ-ideal of S.

(ii) Let R_i be an n-right Γ-ideal of S for all $i\in I$. If $\bigcap_{i\in I}R_i\neq\varnothing$, then $\bigcap_{i\in I}R_i$ is an n-right Γ-ideal of S.

Proof. (i) Let L_i be an m-left-Γ-ideal of S for all $i\in I$. We obtain that $S^m\Gamma L_i\subseteq L_i$. Assume that $\bigcap_{i\in I}L_i\neq\varnothing$. By Lemma 3.1, we have $\bigcap_{i\in I}L_i$ is an LA-Γ-subsemigroup of S. Consider

$S^m\Gamma(\bigcap_{i\in I}L_i)\subseteq S^m\Gamma L_i\subseteq L_i$ for all $i\in I$. So $S^m\Gamma(\bigcap_{i\in I}L_i)\subseteq\bigcap_{i\in I}L_i$. Next, let $a\in\bigcap_{i\in I}L_i$ and $b\in S$ such that $b\leq a$. Since $a\in\bigcap_{i\in I}L_i$, we get $a\in L_i$ where L_i is an m-left Γ-ideal of S for all $i\in I$. Thus $b\in L_i$ for all $i\in I$. Therefore, $b\in\bigcap_{i\in I}L_i$. Hence $\bigcap_{i\in I}L_i$ is an m-left-Γ-ideal of S.

(ii) The proof is similar to the proof of (i). □

Lemma 3.5. Let S be an ordered LA-Γ-semigroup. Then the following statements are true:

(i) Every m-left-Γ-ideal of S is an (m,n)-quasi-Γ-ideal of S.

(ii) Every n-right-Γ-ideal of S is an (m,n)-quasi-Γ-ideal of S.

Proof. (i) Let A be an m-left-Γ-ideal of S. We have $S^m\Gamma A\subseteq A$ and $A\subseteq S$. Consider $(S^m\Gamma A]\cap(A\Gamma S^n]\subseteq(S^m\Gamma A]\subseteq(A]=A$. Clearly, if $a\in A$ and $b\in S$ such that $b\leq a$, then $b\in A$. Hence A is an (m,n)-quasi-Γ-ideal of S.This shows that every m-left-Γ-ideal of S is an (m,n)-quasi-Γ-ideal of S.

(ii) The proof of this statement is similar to the proof of (i). □

Theorem 3.6. Let S be an ordered LA-Γ-semigroup and let A be an m-left-Γ-ideal of S and B be an n-right-Γ-ideal of S. Then $A\cap B$ is an (m,n)-quasi-Γ-ideal of S.

Proof. By Lemma 3.5, we obtain that A and B are an (m,n)-quasi-Γ-ideal of S. Therefore, $A\cap B$ is an (m,n)-quasi-Γ-ideal of S by Theorem 3.2. □

Definition 3.3. An LA-Γ-subsemigroup Q of an ordered LA-Γ-semigroup S has the (m,n) *intersection property* if Q is the intersection of an m-left-Γ-ideal and n-right-Γ-ideal of S.

Theorem 3.7. Every (m,n)-quasi-Γ-ideal Q of an ordered LA-Γ-semigroup S with left identity is the intersection of some m-left-Γ-ideal of S and some n-right-Γ-ideal of S.

Proof. Let Q be an (m,n)-quasi-Γ-ideal of S. Let $L=(Q\cup S^m\Gamma Q]$ and $R=(Q\cup Q\Gamma S^n]$. Now, we show that L is an LA-Γ-subsemigroup of S.

Consider

$$\begin{aligned} L\Gamma L &= (Q\cup S^m\Gamma Q]\Gamma(Q\cup S^m\Gamma Q] \\ &\subseteq ((Q\cup S^m\Gamma Q)\Gamma(Q\cup S^m\Gamma Q)] \\ &= (Q\Gamma Q\cup Q\Gamma(S^m\Gamma Q)\cup(S^m\Gamma Q)\Gamma Q\cup(S^m\Gamma Q)\Gamma(S^m\Gamma Q)] \\ &= (Q\Gamma Q\cup S^m\Gamma(Q\Gamma Q)\cup(Q\Gamma Q)\Gamma S^m\cup(S^m\Gamma S^m)\Gamma(Q\Gamma Q)] \\ &= (Q\Gamma Q\cup S^m\Gamma(Q\Gamma Q)\cup(Q\Gamma Q)\Gamma(S^m\Gamma S^m)\cup S^m\Gamma(Q\Gamma Q)] \\ &= (Q\Gamma Q\cup S^m\Gamma(Q\Gamma Q)\cup(S^m\Gamma S^m)\Gamma(Q\Gamma Q)\cup S^m\Gamma(Q\Gamma Q)] \\ &= (Q\Gamma Q\cup S^m\Gamma(Q\Gamma Q)\cup S^m\Gamma(Q\Gamma Q)\cup S^m\Gamma(Q\Gamma Q)] \\ &\subseteq (Q\cup S^m\Gamma Q\cup S^m\Gamma Q\cup S^m\Gamma Q]=(Q\cup S^m\Gamma Q]=L. \end{aligned}$$

So $L\Gamma L\subseteq L$. Thus L is an LA-Γ-subsemigroup of S.

Consider

$$\begin{aligned} S^m\Gamma L &= S^m\Gamma(Q\cup S^m\Gamma Q] \\ &\subseteq (S^m]\Gamma(Q\cup S^m\Gamma Q] \\ &\subseteq (S^m\Gamma(Q\cup S^m\Gamma Q)] \\ &= (S^m\Gamma Q\cup S^m\Gamma(S^m\Gamma Q)] \\ &= (S^m\Gamma Q\cup(S^m\Gamma S^m)\Gamma(S^m\Gamma Q)] \\ &= (S^m\Gamma Q\cup(Q\Gamma S^m)\Gamma(S^m\Gamma S^m)] \\ &= (S^m\Gamma Q\cup(Q\Gamma S^m)\Gamma S^m] \\ &= (S^m\Gamma Q\cup(S^m\Gamma S^m)\Gamma Q] \\ &= (S^m\Gamma Q] \\ &\subseteq (Q\cup S^m\Gamma Q]=L. \end{aligned}$$

So $S^m\Gamma L\subseteq L$. Next, let $x\in L=(Q\cup S^m\Gamma Q]$ and $y\in S$ such that $y\leq x$. Since $x\in(Q\cup S^m\Gamma Q]$, then $x\leq a$ for some $a\in Q\cup S^m\Gamma Q$. We have $y\leq x\leq a$ for some $a\in Q\cup S^m\Gamma Q$. Therefore $y\in(Q\cup S^m\Gamma Q]$. Hence L is an m-left-Γ-ideal of S. In the same way, we can prove that R is an

n-right-Γ-ideal of S. Next, we show that $Q = L \cap R$. Since $Q \subseteq Q \cup (S^m\Gamma Q] = (Q] \cup (S^m\Gamma Q] = (Q \cup S^m\Gamma Q]$ and $Q \subseteq Q \cup (Q\Gamma S^n] = (Q] \cup (Q\Gamma S^n] = (Q \cup Q\Gamma S^n]$. Therefore, we obtain that $Q \subseteq (Q \cup S^m\Gamma Q] \cap (Q \cup Q\Gamma S^n]$. Hence, $Q \subseteq L \cap R$. Next, consider

$$\begin{aligned}(Q \cup S^m\Gamma Q] \cap (Q \cup Q\Gamma S^n] &= ((Q] \cup (S^m\Gamma Q]) \cap ((Q] \cup (Q\Gamma S^n]) \\ &= ((Q] \cap ((Q] \cup (Q\Gamma S^n])) \cup ((S^m\Gamma Q] \cap ((Q] \cup (Q\Gamma S^n])) \\ &= (((Q] \cap (Q]) \cup ((Q] \cap (Q\Gamma S^n])) \cup (((S^m\Gamma Q] \cap (Q]) \cup \\ &\quad ((S^m\Gamma Q] \cap (Q\Gamma S^n])) \\ &= (Q] = Q.\end{aligned}$$

Therefore $Q = L \cap R$. This shows that Q is the intersection of some m-left-Γ-ideal of S and some n-right-Γ-ideal of S. □

Finally, we investigate about (m,n)-quasi-Γ-ideal in regular ordered *LA*-Γ-semigroups.

Definition 3.4. [7] An ordered *LA*-Γ-semigroup S is called *regular* if $a \in ((a\Gamma S)\Gamma a]$ for every $a \in S$, or

(i) for every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$,

(ii) $A \subseteq ((A\Gamma S)\Gamma A]$ for every $A \subseteq S$.

Lemma 3.8. Let S be an ordered *LA*-Γ-semigroup with left identity. If S is regular and $\varnothing \neq A \subseteq S$ where A is an Γ-idempotent, then the following statements are true:

(i) $A \subseteq (S^m\Gamma A]$ for all $m \in \mathbb{Z}^+$.

(ii) $A \subseteq (A\Gamma S^n]$ for all $n \in \mathbb{Z}^+$.

Proof. (i) Let $P(m)$ be the statement $A \subseteq (S^m\Gamma A]$ for all $m \in \mathbb{Z}^+$ and let $x \in A$. Since S is regular, then there exists $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq (x\alpha y)\beta x$. We have $(x\alpha y)\beta x \in S\Gamma A$. Thus $x \in (S\Gamma A]$. Therefore, $A \subseteq (S\Gamma A]$. Hence, we get $P(1)$ holds. Let $P(k)$ holds for all $k \in \mathbb{Z}^+$.

Then $A \subseteq (S^k\Gamma A]$. Consider

$$\begin{aligned}
S\Gamma A &\subseteq S\Gamma(S^k\Gamma A] \\
&= (S]\Gamma(S^k\Gamma A] \\
&\subseteq (S\Gamma(S^k\Gamma A)] \\
&= ((S\Gamma S)\Gamma(S^k\Gamma A)] \\
&= ((A\Gamma S^k)\Gamma(S\Gamma S)] && \text{Proposition 2.2 (i)} \\
&= ((A\Gamma S^k)\Gamma S] \\
&= ((S\Gamma S^k)\Gamma A] && \text{Left invertive law} \\
&= (S^{k+1}\Gamma A].
\end{aligned}$$

Now, we have that $A \subseteq (S\Gamma A] \subseteq (S^{k+1}\Gamma A]$. So $A \subseteq (S^{k+1}\Gamma A]$. Therefore, $P(k+1)$ holds. Hence $A \subseteq (S^m\Gamma A]$ for all $m \in \mathbb{Z}^+$.

(ii) Let $P(n)$ be the statement $A \subseteq (A\Gamma S^n]$ for all $n \in \mathbb{Z}^+$ and let $x \in A$. Since S is regular, then there exists $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x \leq (x\alpha y)\beta x$. Consider

$$\begin{aligned}
x &\leq (x\alpha y)\beta x \\
&\in (A\Gamma S)\Gamma A \\
&= (A\Gamma S)\Gamma(A\Gamma A] && \Gamma\text{-idempotent} \\
&\subseteq ((A\Gamma S)\Gamma(A\Gamma A)] \\
&= ((A\Gamma A)\Gamma(S\Gamma A)] && \text{Proposition 2.2(i)} \\
&\subseteq ((A\Gamma A)\Gamma S] \\
&\subseteq ((A\Gamma A]\Gamma S] \\
&= (A\Gamma S].
\end{aligned}$$

So $x \in (A\Gamma S]$. Thus $A \subseteq (A\Gamma S]$. Hence $P(1)$ holds. Let $P(k)$ hold for all $k \in \mathbb{Z}^+$. Then $A \subseteq (A\Gamma S^k]$. Consider

$$\begin{aligned}
A\Gamma S &\subseteq (A\Gamma S^k]\Gamma S \\
&= (A\Gamma S^k]\Gamma(S] \\
&\subseteq ((A\Gamma S^k)\Gamma S] \\
&\subseteq ((S\Gamma S^k)\Gamma A] && \text{Left invertive law} \\
&= ((S\Gamma S^k]\Gamma(A\Gamma A]] && \Gamma\text{-idempotent} \\
&= ((S\Gamma S^k)\Gamma(A\Gamma A)] \\
&= ((A\Gamma A)\Gamma(S^k\Gamma S)] && \text{Proposition 2.3(i)} \\
&\subseteq ((A\Gamma A]\Gamma(S^k\Gamma S)] \\
&= (A\Gamma S^{k+1}].
\end{aligned}$$

Thus $A \subseteq (A\Gamma S] \subseteq (A\Gamma S^{k+1}]$. Therefore, $P(k+1)$ holds. Hence $A \subseteq (A\Gamma S^n]$ for all $n \in \mathbb{Z}^+$. □

Theorem 3.9. Let S be an ordered LA-Γ-semigroup with left identity and let $\varnothing \neq A \subseteq S$. Then A is an (m,n)-quasi-Γ-ideal of S if and only if it is the intersection of an m-left-Γ-ideal of S and n-right-Γ-ideal of S.

Proof. $(\Rightarrow)$ Let A be an (m,n)-quasi-Γ-ideal of S. By Theorem 3.7, A is the intersection of an m-left-Γ-ideal of S and n-right-Γ-ideal of S.

$(\Leftarrow)$ Let A be an intersection of an m-left-Γ-ideal of S and an n-right-Γ-ideal of S. By Theorem 3.6 , A is an (m,n)-quasi-Γ-ideal of S. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] C. Akin, Fuzzy LA-(m,n)-Γ-ideals in LA-Γ-semigroups. Adv. Fuzzy Sets Syst. 4(6) (2017), 211-223.

[2] M.A. Ansari, Roughness applied to generalized Γ-ideals of ordered LA Γ-ideals. Commu. Math. Appl. 10(1) (2019), 71-84.

[3] M.A. Ansari, M.R. Khan, J.P. Kaushik, A note on (m,n)-quasi-ideals in semigroups, Int. J. Math. Anal. 38 (2009), 1853-1858.

[4] A. Basar, A note on (m,n)-Γ-ideals of ordered *LA*-Γ-semigroups, Konuralp J. Math. 7(1) (2019), 107-111.

[5] T. Gaketem, On (m,n)-quasi-ideals in *LA*-semigroups, Appl. Sci. 17 (2015), 57-61.

[6] M.A. Kazim, N. Naseeruddin, On almost semigroups, Alig. Bull. Math. 2(1) (1972), 1-7.

[7] M. Khan, V.Amjid, G. Zaman, N. Yaqoob, Characterizations of ordered Γ-Aber-Grassmann's Groupoids, Discuss. Math., Gen. Algebra Appl. 34(1) (2014), 55-73.

[8] M. Khan, F. Smarandache, S. Anis, Theory of Abel Grassmann's groupoids, Educational Publisher, Columbus, 2015.

[9] M.K. Sen, On Γ-semigroups, In: Proceeding of International Symposium on Algebra and Its Applications. Decker Publication, New York, (1981), 301-308.

[10] T. Shah, I. Rehman, On Γ-ideals and Γ-bi-ideals in Γ-*AG*-groupoids, Int. J. Algebra, 4(6) (2010), 267-276.

[11] T. Shah, I. Rehman, R. Chinram, On M-systems in ordered AG-groupoids, Far East J. Math. Sci. 47(1) (2010), 13-21.

[12] O. Steinfeld, Uber die quasiideale von halbgruppen, Publ. Math. Debrecen. 4(1) (1956), 262-275.