Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 3563-3593
https://doi.org/10.28919/jmcs/5711
ISSN: 1927-5307

BIVARIATE TRANSMUTED EXPONENTIATED GUMBEL DISTRIBUTION (BTEGD) AND CONCOMITANTS OF ITS ORDER STATISTICS

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Abstract: In this article we have studied bivariate transmuted exponentiated Gumbel distribution using Morgenstern approach (Morgenstern [4]). We have also studied the shape behavior of the pdf and cdf of the bivariate transmuted exponentiated Gumbel distribution. The distribution of the concomitants of r^{th}order statistics, the moment generating function (mgf) and moments of the concomitants of r^{th} order statistics are obtained. Numerical computations have been done for the moments of the concomitants.

Keywords: bivariate transmuted exponentiated Gumbel distribution; Morgenstern family; concomitants of order statistics; moment generating function; moments of concomitants.

2021 AMS Subject Classification: 62E10.

1. INTRODUCTION

The Gumbel distribution is a very popular statistical distribution due to its extensive applicability in several areas and its wide applications have been reported by Kotz and Nadarajah [16]. The

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Received March 17, 2021

applicability of Gumbel distribution in the area of climate modeling, for example: global warming problems, offshore modeling, rainfall and wind speed modeling have been discussed by Nadarajah [18]. In several areas of engineering such as: flood frequency analysis, network space, software reliability, structural and wind engineering, the applicability of Gumbel Distribution has been reported by Cardeiro, Ortega and Cunha [5]. Due to its wide applicability, several works aimed at extending the Gumbel distribution becomes important. Some examples are mentioned in: Nadarajah and Kotz [17], Cardeiro, Ortega and Cunha [5], Andrade, Rodrigues, Bourguignon and Cordeiro [22] and Deka, Das and Baruah [3]. Thus, the interest in theory and methods about the Gumbel distribution is progressive.

A bivariate distribution $F\,(x, y)$ for a pair of random variables (X, Y) expresses the dependence between X and Y as embedded in its functional form and parameters. The Morgenstern family of distribution is a flexible system of distribution for constructing bivariate distribution using marginal probability density function. Morgenstern [4] introduced a family of bivariate distribution functions having a representation of the form

$$F_{XY}(x, y) = F_X(x)F_Y(y)\{1 + \rho[1 - F_X(x)][1 - F_Y(y)]\} \tag{1}$$

where $F_X(x)$ and $F_Y(y)$ are two univariate distribution functions and the association parameter ρ is constrained to lie in the interval $[-1, 1]$.

The corresponding *pdf* is

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)[1 + \rho\{1 - 2F_X(x)\}\{1 - 2F_Y(y)\}];\ -1 \le \rho \le 1 \tag{2}$$

Concomitants of order statistics have been used extensively by several authors using the concept of Morgenstern approach. Shahbaz and Shahbaz [19] have studied concomitants of generalized order statistics for a bivariate Weibull distribution. Tahmaseb and Jafari [21] have studied concomitants of order statistics and record values from Morgenstern type bivariate generalized exponential distribution. Khan and Kumar [14] have studied concomitants of order statistics from Weighted Marshall-Olkin Bivariate Exponential distribution. Chacko and Thomas [13] have studied estimation of a parameter of Morgenstern type bivariate exponential distribution by ranked set sampling. Athar and Nayabuddin [10], have studied concomitants of dual generalized order statistics from Farlie Gumbel Morgenstern Type Bivariate Inverse Rayleigh distribution. Concomitants are useful accompaniments in statistical modeling that may stand as random variables of interest in connection with order statistics of random samples. For such a situation, order statistics and their concomitants are taken under consideration in the current paper for the Morgenstern Type Bivariate Transmuted Exponentiated Gumbel distribution.

In the following sections, order statistics and concomitants are mentioned first and then the new Bivariate Transmuted Exponentiated Gumbel distribution is presented. A next step is the construction of the distribution of the concomitants of order statistics for the presented Bivariate Transmuted Exponentiated Gumbel distribution.

2. ORDER STATISTICS AND CONCOMITANTS

A random sample of $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ from a bivariate distribution with bivariate distribution function $F(x, y)$ yields order statistics of the first coordinate as $(X_{1:n}, X_{2:n}, \dots, X_{n:n})$ such that $X_{i:n} \leq X_{j:n}$ for $i < j$. If the pairs $(X_i, Y_i), i = 1, 2, \dots, n,$ are ordered by their X variates according to$(X_{1:n}, X_{2:n}, \dots, X_{n:n})$; then the Y variate associated with the r^{th}order statistic $X_{r:n}$ of X, denoted by $Y_{[r:n]}, 1 \leq r \leq n$, and is called the concomitant of the r^{th} order statistics. For a detailed overview of concomitants, we refer to David and Nagaraja [8], [9].

Concomitants of order statistics have several applications in statistics. Concomitants are used in many applied areas where a population characteristic Y is investigated with respect to another characteristic X of the same population. For example in selection procedures, Yeo and David [23] considered the problem of choosing the best *k* objects out of *n* candidates on the basis of auxiliary measurements *X*, while the measurements of primary interest *Y* are not available. The authors are interested in the probability that the *m* subjects with the largest *X*-values consists of the *k* objects with the largest *Y* -values. Recently, it becomes a very important research topic and discussed in several literature. For a detailed description of the theoretical aspects of concomitants of order statistics, one may refer David and Nagaraja [9], Chacko and Thomas [12], Scaria and Nair [11], Nair and Scaria [15] and Tahmasebi and Behboodian [20] etc. The distribution of r^{th}

concomitants is given by David and Nagaraja [8] as

$$g_{[r:n]}(y) = \int f(y|x) f_{r:n}(x)dx, \tag{3}$$

Where $f_{r:n}(x)$ is the *pdf* of $X_{r:n}$.

The density function of an r^{th} order statistic $(X_{r:n})$, is defined by Arnold *et al.* [2] as,

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}\{F(x)\}^{r-1}\{1-F(x)\}^{n-r} f(x), -\infty < x < \infty \tag{4}$$

The general expressions given here are used in the following sections where a new Bivariate Transmuted Exponentiated distribution is presented.

3. BIVARIATE TRANSMUTED EXPONENTIATED GUMBEL DISTRIBUTION

Deka *et al.* [3] have studied the Transmuted Exponentiated Gumbel Distribution (TEGD) along with several statistical properties and applied it to model water quality parameters data set. The *cdf* of the TEGD is

$$F(x) = 1 - \left\{1 - exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}^{\alpha}\left[1 - \lambda + \lambda\left\{1 - exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}^{\alpha}\right], -\infty < x, \mu < \infty, \quad \sigma, \alpha > 0, \quad |\lambda| \leq 1 \tag{5}$$

And its corresponding *pdf* is

$$f(x) = \frac{\alpha}{\sigma}\left\{1 - exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}^{\alpha-1}\left\{exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}\left(exp\left(-\frac{x-\mu}{\sigma}\right)\right)\left[1 - \lambda + 2\lambda\left\{1 - exp\left(-exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}^{\alpha}\right], -\infty < x, \mu < \infty, \quad \sigma, \alpha > 0, \quad |\lambda| \leq 1 \tag{6}$$

Using the marginal Transmuted Exponentiated Gumbel density functions for the random variable X and Y where $X \sim TEGD(\mu_1, \sigma_1, \alpha_1, \lambda_1)$ and $Y \sim TEGD(\mu_2, \sigma_2, \alpha_2, \lambda_2)$ in equation (1) we get the *cdf* for MTBTEGD as

$$F_{XY}(x,y) = \left\{1 - \left\{exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\left[1 - \lambda_1 + \lambda_1\left\{1 - \right.\right.\right.$$

$$exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right]\right\}\left\{1-\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1-\lambda_2+\lambda_2\left\{1-\right.\right.\right.$$

$$exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\right\}\left[1+\rho\left\{\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\left[1-\lambda_1+\lambda_1\left\{1-\right.\right.\right.\right.$$

$$exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right]\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1-\lambda_2+\lambda_2\left\{1-\right.\right.$$

$$exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\right\}\right] \tag{7}$$

And the corresponding *pdf* is obtained by using (2) as

$$f_{XY}(x,y)$$

$$=\frac{\alpha_1\alpha_2}{\sigma_1\sigma_2}\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1-1}\left\{exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}\left(exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\left[1\right.$$

$$-\lambda_1$$

$$+2\lambda_1\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right]\left\{1\right.$$

$$-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\left[1-\lambda_2\right.$$

$$+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[1\right.$$

$$+\rho\left\{\left[\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\left[1-\lambda_1+\lambda_1\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right]\right.\right.$$

$$\left.-1\right]\left[\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1-\lambda_2+\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\right.$$

$$\left.\left.\left.-1\right]\right\}\right] \tag{8}$$

It can be shown that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dxdy = 1$$

4. Graphical Representation of Bivariate Transmuted Exponentiated Gumbel Distribution (BTEGD)

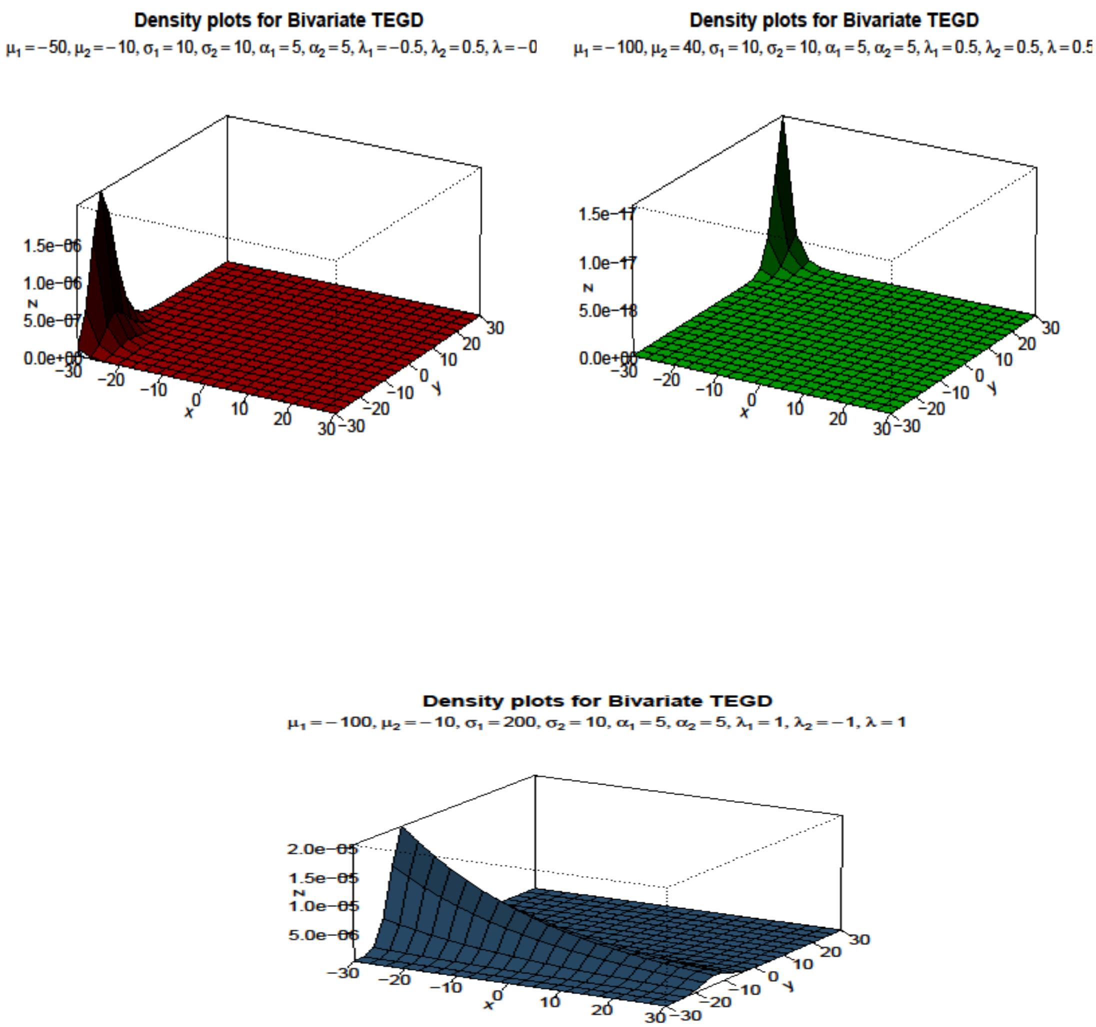

Fig1: *pdf* of Bivariate Transmuted Exponentiated Gumbel Distribution

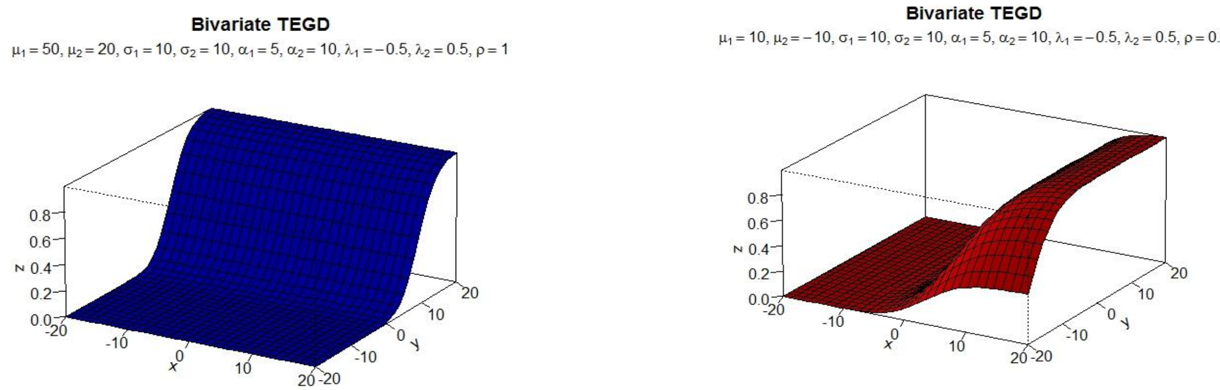

Fig 2: *cdf* of Bivariate Transmuted Exponentiated Gumbel Distribution

5. Distribution of Concomitants For Bivariate Transmuted Exponentiated Gumbel Distribution (BTEGD)

5.1. Distribution of r^{th} Concomitants for BTEGD

In this section, we obtain the distribution of the concomitants of the r^{th} order statistics for the Bivariate Transmuted Exponentiated Gumbel distribution, given in Eq. (7). To obtain the distribution of the concomitants of r^{th} order statistics we need the conditional distribution of Y given X and the distribution of r^{th} order statistics $X_{r:n}$. The conditional distribution of Y given X is obtained as follows:

We have,

$$f_{(Y|X)}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \tag{9}$$

Using equation (6) and (8), in equation (9) we get the conditional distribution of Y given X for BTEGD as

$$f_{(Y|X)}(y|x)$$

$$= \frac{\alpha_2}{\sigma_2}\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2 - 1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1 - \lambda_2 + 2\lambda_2\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[1 + \rho\left\{\left[\left\{1 - exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\left[1 - \lambda_1 + \lambda_1\left\{1 - exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right] - 1\right]\left[\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1 - \lambda_2 + \lambda_2\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right] - 1\right]\right\}\right] \quad (10)$$

Putting $r = 1$, in the equation (3), we get the distribution of the concomitants of 1st order statistics as

$$g_{[1:n]}(y) = \int f_{(Y|X)}(y|x) f_{1:n}(x)\, dx \quad (11)$$

Using series representation as

$$(1+z)^a = \sum_{j=0}^{a} \frac{\Gamma(a+1)}{\Gamma(a-j+1)} \frac{z^j}{j!} \quad (12)$$

Using Eq. (12) in Eq. (4), we get the expression for $f_{1:n}(x)$ as

$$f_{1:n}(x) = \sum_{j=0}^{\infty} (-1)^j \binom{n-1}{j} n[F(x)]^j f(x) \quad (13)$$

Using Eq. (5), (6), (12) and (13); in Eq. (11), we get the expression for $g_{[1:n]}(y)$ as

$$g_{[1:n]}(y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{n-1}{j}\binom{j}{k}\lambda_1{}^{j}\, n\frac{\alpha_1\alpha_2}{\sigma_1\sigma_2}\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1-\lambda_2 + 2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\int_{-\infty}^{+\infty}\left\{1 - exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1(n+k)-1}\left\{exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}\left\{exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right\}\left[1-\lambda_1 + 2\lambda_1\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right]\left[1 + \rho\left\{\left[\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\left[1-\lambda_1+\lambda_1\left\{1-exp\left(-exp\left(-\frac{x-\mu_1}{\sigma_1}\right)\right)\right\}^{\alpha_1}\right] - 1\right]\left[\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1-\lambda_2+\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right] - 1\right]\right\}\right]dx \quad (14)$$

Using Prudnikov *et al*. [1] after integration equation (14) becomes

$$g_{[1:n]}(y)$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j+k}\binom{n-1}{j}\binom{j}{k}\frac{n{\lambda_1}^{j}\alpha_2}{\sigma_2}\Bigg\{1$$
$$-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\Bigg[1-\lambda_2$$
$$+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg]\Bigg[\Bigg\{\frac{(1-\rho)(1-\lambda_1)}{n+k}-\frac{\rho+\lambda_1(\rho\lambda_1-3\rho+1)}{n+k+1}$$
$$-\frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2}-\frac{\rho\lambda_1}{n+k+3}\Bigg\}$$
$$+\Bigg\{\frac{\rho\lambda^2}{n+k+3}+\frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2}+\frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1}$$
$$-\frac{\rho(1-\lambda_1)}{n+k}\Bigg\}\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg[1-\lambda_2$$
$$+\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg]\Bigg] \qquad (15)$$

According to David [7] the *cdf* of the order statistics connected by the relation

$$F_{r:n}(x) = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}F_{1:i}(x);\;\; 1\le r\le n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*)$$

Therefore the relation (*) is also true for *pdf* of order statistics also. Thus we can obtained the *pdf* of $Y_{[r:n]}$ from the following relation

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}g_{[1:i]}(y) \qquad (16)$$

Now using equation (15) in equation (16) we get the *pdf* of the concomitants of rth order statistics as

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r} \binom{n}{i} \binom{i-1}{j} \binom{j}{k} \frac{i\lambda_1{}^j \alpha_2}{\sigma_2} \left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1} \left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\} \left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}$$
$$\left[1-\lambda_2+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[\left\{\frac{(1-\rho)(1-\lambda_1)}{n+k}-\frac{\rho+\lambda_1(\rho\lambda_1-3\rho+1)}{n+k+1} - \frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2}-\frac{\rho\lambda_1}{n+k+3}\right\}\right.$$
$$+\left\{\frac{\rho\lambda_1{}^2}{n+k+3}+\frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2}+\frac{\rho(1-2\lambda_2-\lambda_2+\rho\lambda_1)}{n+k+1} - \frac{\rho(1-\lambda_1)}{n+k}\right\}\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\left[1-\lambda_2\right.$$
$$\left.\left. + \lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\right] \tag{17}$$

5.2. Moment Generating Function of the Concomitants of Order Statistics for BTEGD

In this section, we have obtained the exact expression for the moment generating function (*mgf*) of the concomitants of r^{th} order statistics when random variables $(X_i, Y_i); (i = 1,2, \dots, n)$ are i.i.d. and follows TEGD. The *mgf* of the concomitants of r^{th} order statistics $Y_{[r:n]}$ is given by

$$M_{Y_{[r:n]}}(t) = E(e^{tY_{[r:n]}})$$
$$= \int_{-\infty}^{+\infty} e^{ty} g_{Y_{[r:n]}}(y) dy$$

$$\therefore\ M_{Y_{[r:n]}}(t)$$

$$= \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r} \binom{n}{i} \binom{i-1}{j} \binom{j}{k} \frac{i\lambda_1^j \alpha_2}{\sigma_2} \int_{-\infty}^{+\infty} e^{ty} \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2 - 1} \Bigg\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\} \Bigg\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\Bigg\} \Bigg[1 - \lambda_2 + 2\lambda_2 \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2}\Bigg] \Bigg[\Bigg\{\frac{(1-\rho)(1-\lambda_1)}{n+k} - \frac{\rho + \lambda_1(\rho\lambda_1 - 3\rho + 1)}{n+k+1} - \frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2} - \frac{\rho\lambda_1}{n+k+3}\Bigg\} + \Bigg\{\frac{\rho\lambda_1^2}{n+k+3} + \frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2} + \frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1} - \frac{\rho(1-\lambda_1)}{n+k}\Bigg\} \left(1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right)^{\alpha_2} \Bigg[1 - \lambda_2 + \lambda_2 \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2}\Bigg]\Bigg] dy$$

$$M_{Y_{[r:n]}}(t) = \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r} \binom{n}{i} \binom{i-1}{j} \binom{j}{k} \frac{i\lambda_1^j \alpha_2}{\sigma_2} \Bigg[\Bigg\{\frac{(1+\rho)(1-\lambda_1)}{n+k} - \frac{\rho + \lambda_1(\rho\lambda_1 - 3\rho + 1)}{n+k+1} - \frac{\rho\lambda_1}{n+k+3} - \frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2}\Bigg\} \int_{-\infty}^{+\infty} e^{ty} \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2 - 1} \left(exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right) \left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right) \Bigg[1 - \lambda_2 + 2\lambda_2 \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2}\Bigg] dy + \Bigg\{\frac{\rho\lambda_1^2}{n+k+3} + \frac{\rho\lambda_1(2-\lambda_1-\lambda_1)}{n+k+2} + \frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1} - \frac{\rho(1-\lambda_1)}{n+k}\Bigg\} \int_{-\infty}^{+\infty} e^{ty} \Bigg\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{2\alpha_2 - 1}$$

$$\left(exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right)\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\left[1-\lambda_2 + 2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]$$

$$\left[1-\lambda_2+\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy\Bigg] \tag{18}$$

$$Let \quad A = \int_{-\infty}^{+\infty} e^{ty}\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1}\left(exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right)\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)$$

$$\left[1-\lambda_2+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy$$

Putting $\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right) = u$ and using series representation for $(1+z)^a$; in A we get

$$A = (1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_1=0}^{\infty}(-1)^{l_1}\binom{\alpha_2-1}{l_1}\int_0^{+\infty} u^{(1-t\sigma_2)-1}\big(exp(-(l_1+1)u)\big)\,du +$$
$$2\lambda_2\sigma_2 e^{t\mu_2}\sum_{l_2=0}^{+\infty}(-1)^{l_2}\binom{2\alpha_2-1}{l_2}\int_0^{+\infty} u^{(1-t\sigma_2)-1}\big(exp(-(l_2+1)u)\big)du$$

After integration we get,

$$A = (1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_1=0}^{\infty}(-1)^{l_1}\binom{\alpha_2-1}{l_1}\frac{\Gamma(1-t\sigma_2)}{(l_1+1)^{1-t\sigma_2}}$$
$$+ 2\lambda_2\sigma_2 e^{t\mu_2}\sum_{l_2=0}^{\infty}(-1)^{l_2}\binom{2\alpha_2-1}{l_2}\frac{\Gamma(1-t\sigma_2)}{(l_2+1)^{1-t\sigma_2}} \tag{19}$$

Also let

$$B = \int_{-\infty}^{+\infty} e^{ty}\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{2\alpha_2-1}\left(exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right)\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)$$
$$\left[1-\lambda_2+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[1-\lambda_2 + \lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy$$

Putting $\left(exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)=u,$ and using series representation for $(1+z)^a$; in B we get

$$B = (1-\lambda_2)^2\sigma_2 e^{t\mu_2}\sum_{l_3=0}^{\infty}(-1)^{l_3}\binom{2\alpha_2-1}{l_3}\int_0^{+\infty} u^{(1-t\sigma_2)-1}\left(exp(-(l_3+1)u)\right)du +$$
$$3\lambda_2(1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_4=0}^{\infty}(-1)^{l_4}\binom{3\alpha_2-1}{l_4}\int_0^{+\infty} u^{(1-t\sigma_2)-1}\left(exp(-(l_4+1)u)\right)du +$$
$$2\lambda_2^2\sigma_2 e^{t\mu_2}\sum_{l_5=0}^{+\infty}(-1)^{l_5}\binom{4\alpha_2-1}{l_5}\int_0^{+\infty} u^{(1-t\sigma_2)-1}\,(exp(-(l_5+1)u)du)$$

After integration we get,

$$B = (1-\lambda_2)^2\sigma_2 e^{t\sigma_2}\sum_{l_3=0}^{\infty}(-1)^{l_3}\binom{2\alpha_2-1}{l_3}\frac{\Gamma(1-t\sigma_2)}{(l_3+1)^{1-t\sigma_2}}$$
$$+3\lambda_2(1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_4=0}^{\infty}(-1)^{l_4}\binom{3\alpha_2-1}{l_4}\frac{\Gamma(1-t\sigma_2)}{(l_4+1)^{1-t\sigma_2}}$$
$$+2\lambda_2^2\sigma_2 e^{t\mu_2}\sum_{l_5=0}^{\infty}(-1)^{l_5}\binom{4\alpha_2-1}{l_5}\frac{\Gamma(1-t\sigma_2)}{(l_5+1)^{1-t\sigma_2}} \tag{20}$$

Using (19) and (20), in equation (18) we get the expression for the moment generating function of the concomitants of r^{th} order statistics as

$$M_{Y_{[r:n]}}(t)=\sum_{i=n-r+1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i-n+r+j+k-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{j}\binom{j}{k}\frac{i\lambda_1^j\alpha_2}{\sigma_2}$$

$$\left[\left\{\frac{(1-\rho)(1-\lambda_1)}{n+k}-\frac{\rho+\lambda_1(\rho\lambda_1-3\lambda_1+1)}{n+k+1}-\frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2}-\frac{\rho\lambda_1}{n+k+3}\right\}\right.$$

$$\left\{(1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_1=0}^{\infty}(-1)^{l_1}\binom{\alpha_2-1}{l_1}\frac{\Gamma(1-t\sigma_2)}{(l_1+1)^{1-t\sigma_2}}\right.$$

$$\left.+2\lambda_2\sigma_2 e^{t\mu_2}\sum_{l_2=0}^{\infty}(-1)^{l_2}\binom{2\alpha_2-1}{l_2}\frac{\Gamma(1-t\sigma_2)}{(l_2+1)^{1-t\sigma_2}}\right\}+$$

$$\left\{\frac{\rho\lambda_1^2}{n+k+3}+\frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2}+\frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1}-\frac{\rho(1-\lambda_1)}{n+k}\right\}$$

$$\left\{(1-\lambda_2)^2\sigma_2 e^{t\mu_2}\sum_{l_3=0}^{\infty}(-1)^{l_3}\binom{2\alpha_2-1}{l_3}\frac{\Gamma(1-t\sigma_2)}{(l_3+1)^{1-t\sigma_2}}\right.$$

$$+3\lambda_2(1-\lambda_2)\sigma_2 e^{t\mu_2}\sum_{l_4=0}^{\infty}(-1)^{l_4}\binom{3\alpha_2-1}{l_4}\frac{\Gamma(1-t\sigma_2)}{(l_4+1)^{1-t\sigma_2}}+$$

$$\left.\left.2\lambda_2^2\sigma_2 e^{t\mu_2}\sum_{l_5=0}^{\infty}(-1)^{l_5}\binom{4\alpha_2-1}{l_5}\frac{\Gamma(1-t\sigma_2)}{(l_5+1)^{1-t\sigma_2}}\right\}\right] \tag{21}$$

5.3. Moments of Concomitants of Order Statistics

In this section, we have deduced the expression for the moment of concomitants of r^{th} order statistics when random variables $(X_i, Y_i); (i=1,2,3,\dots,n)$ are i.i.d. and follows TEGD. Utilizing these results, we can compute means and variates of concomitants of r^{th} order statistics. According to David [6], the j^{th} moment about origin of concomitants of r^{th} order statistics $Y_{[r:n]}$ is given by

$$\mu^{j/}_{Y_{[r:n]}}=\int_{-\infty}^{+\infty} y^j g_{[r:n]}(y)\,dy$$

Thus we get the p^{th} moment about origin for the concomitants of r^{th} order statistics $Y_{[r:n]}$ for TEGD as

$$\therefore \mu^{p/}_{Y_{[r:n]}} = \int_{-\infty}^{+\infty} y^p g_{[r:n]}(y)\,dy$$

$$\therefore \mu^{p/}_{Y_{[r:n]}}$$

$$= \sum_{i=n-r+1}^{n} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} (-1)^{i-n+r+j+k-1} \binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{j}\binom{j}{k} \frac{i\lambda_1^j \alpha_2}{\sigma_2} \int_{-\infty}^{+\infty} y^p \Bigg\{1$$

$$- exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\Bigg\}^{\alpha_2 - 1} \left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\Bigg[1-\lambda_2$$

$$+ 2\lambda_2\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg]\Bigg[\Bigg\{\frac{(1-\rho)(1-\lambda_1)}{n+k} - \frac{\rho+\lambda_1(\rho\lambda_1 - 3\rho + 1)}{n+k+1}$$

$$- \frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2} - \frac{\rho\lambda_1}{n+k+3}\Bigg\}$$

$$+ \Bigg\{\frac{\rho\lambda_1^2}{n+k+3} + \frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2} + \frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1}$$

$$- \frac{\rho(1-\lambda_1)}{n+k}\Bigg\}\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg[1-\lambda_2$$

$$+ \lambda_2\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\Bigg]\Bigg]dy$$

$$\therefore \mu^{p/}_{Y_{[r:n]}} = \sum_{i=n-r+1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i-n+r+j+k-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{j}\binom{j}{k}\frac{i\lambda_1^j\alpha_2}{\sigma_2}\left[\left\{\frac{(1-\rho)(1-\lambda_1)}{n+k} - \frac{\rho+\lambda_1(\rho\lambda_1-3\rho+1)}{n+k+1} - \frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2} - \frac{\rho\lambda_1}{n+k+3}\right\}\int_{-\infty}^{+\infty} y^p\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1-\lambda_2 + 2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy + \left\{\frac{\rho\lambda_1^2}{n+k+3}+\frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2}+\frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1} - \frac{\rho(1-\lambda_1)}{n+k}\right\}\int_{-\infty}^{+\infty} y^p\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{2\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1-\lambda_2 + 2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[1-\lambda_2 + \lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy\right] \tag{22}$$

Let

$$A = \int_{-\infty}^{+\infty} y^p\left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1-\lambda_2+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy$$

$$= (1-\lambda_2)\int_{-\infty}^{+\infty} y^p \left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2-1} \left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy$$

$$+ 2\lambda_2 \int_{-\infty}^{+\infty} y^p \left\{1 - exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{2\alpha_2-1} \left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy$$

Let

$$u = exp\left(-\frac{y-\mu_2}{\sigma_2}\right)$$

$$=> -\frac{y-\mu_2}{\sigma_2} = logu$$

$$=> y = \mu_2 - \sigma_2 logu$$

Also

$$\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left(-\frac{1}{\sigma_2}\right)dy = du$$

$$=> exp\left(-\frac{y-\mu_2}{\sigma_2}\right)dy = -\sigma_2 du$$

When

$$y = -\infty, u = \infty$$

$$y = \infty, u = 0$$

Therefore we can write

$$A = (1-\lambda_2)\sigma_2 \int_0^{+\infty} (\mu_2 - \sigma_2 logu)^p \{1 - \exp(-u)\}^{\alpha_2-1} \exp(-u)\, du + 2\lambda_2\sigma_2 \int_0^{+\infty} (\mu_2 - \sigma_2 logu)^p \{1 - \exp(-u)\}^{2\alpha_2-1} \exp(-u)\, du$$

$$=> A = (1-\lambda_2)\sigma_2 \sum_{k_1=0}^{p}\sum_{l_1=0}^{\infty} (-1)^{k_1+l_1} \binom{p}{k_1}\binom{\alpha_2-1}{l_1} \mu_2^{p-k_1}\sigma_2^{k_1} \int_0^{+\infty} (logu)^{k_1}\{exp(-(l_1+1)u)\}\, du$$

$$+ 2\lambda_2\sigma_2 \sum_{k_2=0}^{p}\sum_{l_2=0}^{\infty} (-1)^{k_2+l_2} \binom{p}{k_2}\binom{2\alpha_2-1}{l_2} \mu_2^{p-k_2}\sigma_2^{k_2} \int_0^{+\infty} (logu)^{k_2}\{exp(-(l_2+1)u)\}\, du$$

After integration we get;

$$A=(1-\lambda_2)\sigma_2\sum_{k_1=0}^{p}\sum_{l_1=0}^{\infty}(-1)^{k_1+l_1}\binom{p}{k_1}\binom{\alpha_2-1}{l_1}\mu_2^{p-k_1}\sigma_2^{k_1}\left(\frac{\partial}{\partial a}\right)^{k_1}\left[(l_1+1)^{-a}\Gamma[a]\right]\Big|_{a=1}+$$

$$2\lambda_2\sigma_2\sum_{k_2=0}^{p}\sum_{l_2=0}^{\infty}(-1)^{k_2+l_2}\binom{p}{k_2}\binom{2\alpha_2-1}{l_2}\mu_2^{p-k_2}\sigma_2^{k_2}\left(\frac{\partial}{\partial a}\right)^{k_2}\left[(l_2+1)^{-a}\Gamma[a]\right]\Big|_{a=1} \qquad (23)$$

Equation (23) is obtained by using Prudnikov, Brychkov and Marichev [1]

Let

$$B = \int_{-\infty}^{+\infty} y^p\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{2\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}\left[1-\lambda_2+2\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]\left[1-\lambda_2+\lambda_2\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{\alpha_2}\right]dy$$

$$=> B=(1-\lambda_2)^2\int_{-\infty}^{+\infty} y^p\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{2\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy+\lambda_2(1-\lambda_2)\int_{-\infty}^{+\infty} y^p\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{3\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy+2\lambda_2(1-\lambda_2)\int_{-\infty}^{+\infty} y^p\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{3\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy+2{\lambda_2}^2\int_{-\infty}^{+\infty} y^p\left\{1-exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}^{4\alpha_2-1}\left\{exp\left(-exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right)\right\}\left\{exp\left(-\frac{y-\mu_2}{\sigma_2}\right)\right\}dy$$

Let $exp\left(-\frac{y-\mu_2}{\sigma_2}\right)=u$

$$=> y=\mu_2-\sigma_2 logu$$

When $y=-\infty, u=\infty$

$y = \infty, u = 0$

and $exp\left(-\frac{y-\mu_2}{\sigma_2}\right) dy = -\sigma_2 du$

Thus we get

$$B = (1-\lambda_2)^2\sigma_2 \int_0^\infty (\mu_2 - \sigma_2 logu)^p \, (1-\exp(-u))^{2\alpha_2-1} \exp(-u)\, du + \lambda_2(1-\lambda_2)\sigma_2 \int_0^\infty (\mu_2 - \sigma_2 logu)^p (1-\exp(-u))^{3\alpha_2-1} \exp(-u)\, du + 2\lambda_2(1-\lambda_2)\sigma_2 \int_0^\infty (\mu_2 - \sigma_2 logu)^p (1-\exp(-u))^{3\alpha_2-1} \exp(-u)\, du + 2{\lambda_2}^2\sigma_2 \int_0^\infty (\mu_2 - \sigma_2 logu)^p (1-\exp(-u))^{4\alpha_2-1} \exp(-u)\, du$$

Using binomial expression for $(a+bz)^n$ and series representation for $(1+z)^a$ above expression can be written as

$$B = (1-\lambda_2)^2\sigma_2 \sum_{k_3=0}^{p}\sum_{l_3=0}^{\infty} (-1)^{k_3+l_3} \binom{p}{k_3}\binom{2\alpha_2-1}{l_3} \mu^{p-k_4} {\sigma_2}^{k_3} \int_0^\infty (logu)^{k_3} \left(exp\left(-(l_3+1)\right)u\right) du + \lambda_2(1-\lambda_2)\sigma_2 \sum_{k_4=0}^{p}\sum_{l_4=0}^{\infty} (-1)^{l_4+k_4} \binom{p}{k_4}\binom{3\alpha_2-1}{l_4} \mu^{p-k_4} {\sigma_2}^{k_4} \int_0^\infty (logu)^{k_4} \left(exp(-(l_4+1)u)\right) du + 2\lambda_2\,(1-\lambda_2)\sigma_2 \sum_{k_5=0}^{p}\sum_{l_5=0}^{\infty} (-1)^{l_5+k_5} \binom{p}{k_5}\binom{3\alpha_2-1}{l_5} {\mu_2}^{p-k_5}{\sigma_2}^{k_5} \int_0^\infty (logu)^{k_5}\left(exp(-(l_5+1)u)\right)du + 2{\lambda_2}^2\sigma_2 \sum_{k_6=0}^{p}\sum_{l_6=0}^{\infty} (-1)^{l_6+k_6} \binom{p}{k_6}\binom{4\alpha_2-1}{l_6} {\mu_2}^{p-k_6}{\sigma_2}^{k_6} \quad (24)$$

Finally, by using (2.6.21.1) from Prudnikov *et al*. [1] in equation (24), and after integration we get

$$B = (1-\lambda_2)^2\sigma_2 \sum_{k_3=0}^{p}\sum_{l_3=0}^{\infty} (-1)^{l_3+k_3} \binom{p}{k_3}\binom{2\alpha_2-1}{l_3} {\mu_2}^{p-k_3}\, {\sigma_2}^{k_3} \left(\frac{\partial}{\partial a}\right)^{k_3} \left[(l_3+1)^{-a}\, \Gamma a\right]\Big|_{a=1} + \lambda_2(1-\lambda_2)\sigma_2 \sum_{k_4=0}^{p}\sum_{l_4=0}^{\infty} (-1)^{l_4+k_4} \binom{p}{k_4}\binom{3\alpha_2-1}{l_4} {\mu_2}^{p-k_4}{\sigma_2}^{k_4} \left(\frac{\partial}{\partial a}\right)^{k_4} \left[(l_4+1)^{-a}\, \Gamma a\right]\Big|_{a=1} +$$

$$2\lambda_2(1-\lambda_2)\sigma_2\sum_{k_5=0}^{p}\sum_{l_5=0}^{\infty}(-1)^{l_5+k_5}\binom{p}{k_5}\binom{3\alpha_2-1}{l_5}\mu_2{}^{p-k_5}\sigma_2{}^{k_5}\left(\frac{\partial}{\partial a}\right)^{k_5}[(l_5+1)^{-a}\,\Gamma a]|_{a=1}+$$
$$2\lambda_2{}^2\sigma_2\sum_{k_6=0}^{p}\sum_{l_6=0}^{\infty}(-1)^{l_6+k_6}\binom{p}{k_6}\binom{4\alpha_2-1}{l_6}\mu_2{}^{p-k_6}\sigma_2{}^{k_6}\left(\frac{\partial}{\partial a}\right)^{k_6}[(l_6+1)^{-a}\,\Gamma a]|_{a=1} \tag{25}$$

Using (23) and (25) in (22), we get the expression for the p^{th} moment of concomitants of r^{th} order statistics as

$$\mu_{Y_{[r:n]}}{}^{p/}=\sum_{i=n-r+1}^{n}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i-n+r+j+k-1}\binom{i-1}{n-r}\binom{n}{i}\binom{i-1}{j}\binom{j}{k}i\lambda_1^j\,\alpha_2\Bigg[\Bigg\{\frac{(1-\rho)(1-\lambda_1)}{n+k}-\frac{\rho+\lambda_1(\rho\lambda_1-3\rho+1)}{n+k+1}-\frac{2\rho\lambda_1(1-\lambda_1)}{n+k+2}-\frac{\rho\lambda_1}{n+k+3}\Bigg\}\Bigg\{(1-\lambda_2)\sum_{k_1=0}^{p}\sum_{l_1=0}^{\infty}(-1)^{k_1+l_1}\binom{p}{k_1}\binom{\alpha_2-1}{l_1}\mu_2{}^{p-k_1}\sigma_2{}^{k_1}\left(\frac{\partial}{\partial a}\right)^{k_1}[(l_1+1)^{-a}\,\Gamma a]|_{a=1}$$
$$+2\lambda_2\sum_{k_2=0}^{p}\sum_{l_2=0}^{\infty}(-1)^{k_2+l_2}\binom{p}{k_2}\binom{2\alpha_2-1}{l_2}\mu_2{}^{p-k_2}\sigma_2{}^{k_2}\left(\frac{\partial}{\partial a}\right)^{k_2}[(l_2+1)^{-a}\,\Gamma a]|_{a=1}\Bigg\}$$
$$+\Bigg\{\frac{\rho\lambda_1^2}{n+k+3}+\frac{\rho\lambda_1(2-\lambda_1-\lambda_2)}{n+k+2}+\frac{\rho(1-2\lambda_1-\lambda_2+\rho\lambda_1)}{n+k+1}-\frac{\rho(1-\lambda_1)}{n+k}\Bigg\}\Bigg\{(1-\lambda_2)^2\sum_{k_3=0}^{p}\sum_{l_3=0}^{\infty}(-1)^{l_3+k_3}\binom{p}{k_3}\binom{2\alpha_2-1}{l_3}\mu_2{}^{p-k_3}\sigma_2{}^{k_3}\left(\frac{\partial}{\partial a}\right)^{k_3}[(l_3+1)^{-a}\,\Gamma a]|_{a=1}$$
$$+\lambda_2(1-\lambda_2)\sum_{k_4=0}^{p}\sum_{l_4=0}^{\infty}(-1)^{l_4+k_4}\binom{p}{k_4}\binom{3\alpha_2-1}{l_4}\mu_2{}^{p-k_4}\sigma_2{}^{k_4}\left(\frac{\partial}{\partial a}\right)^{k_4}[(l_4+1)^{-a}\,\Gamma a]|_{a=1}$$
$$+2\lambda_2(1-\lambda_2)\sum_{k_5=0}^{p}\sum_{l_5=0}^{\infty}(-1)^{l_5+k_5}\binom{p}{k_5}\binom{3\alpha_2-1}{l_5}\mu_2^{p-k_5}\sigma_2^{k_5}\left(\frac{\partial}{\partial a}\right)^{k_5}[(l_5+1)^{-a}\,\Gamma a]|_{a=1}$$
$$+2\lambda_2^2\sum_{k_6=0}^{p}\sum_{l_6=0}^{\infty}(-1)^{l_6+k_6}\binom{p}{k_6}\binom{4\alpha_2-1}{l_6}\mu_2^{p-k_6}\sigma_2^{k_6}\left(\frac{\partial}{\partial a}\right)^{k_6}[(l_6+1)^{-a}\,\Gamma a]|_{a=1}\Bigg\}\Bigg] \tag{26}$$

Here we compute some numerical values of the first four moments for some selected values of the parameters which are given in the following tables:

Table 1: 1st moment of the concomitants of Order Statistics from BTEGD for selected values of the parameters

n	r	$\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5,$ $\sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2$ $\rho = -0.5, \lambda_1 = -0.5,$ $\lambda_2 = -0.5$	$\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3,$ $\sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5$ $\rho = -1, \lambda_1 = 1,$ $\lambda_2 = 1$
5	1	2.653007	2.335468
	2	2.792302	3.096762
	3	2.931492	3.858057
	4	3.070682	4.619358
	5	3.209878	5.380615
6	1	2.633099	2.226722
	2	2.752538	2.879249
	3	2.871844	3.531787
	4	2.991145	4.184326
	5	3.110454	4.836864
	6	3.229755	5.489441
7	1	2.618163	2.145143
	2	2.722713	2.716116
	3	2.827102	3.287086
	4	2.931492	3.858081
	5	3.035889	4.429027
	6	3.140276	5.000003
	7	3.244664	5.570966

8	1	2.606542	2.081703
	2	2.699518	2.589232
	3	2.792308	3.096761
	4	2.865098	3.604294
	5	2.977891	4.111822
	6	3.070682	4.619348
	7	3.162295	5.126883
	8	3.257058	5.228389
9	1	2.597241	2.030949
	2	2.680958	2.847733
	3	2.764468	2.944504
	4	2.847981	3.401272
	5	2.931498	3.858065
	6	3.015001	4.314848
	7	3.098516	4.771633
	8	3.182033	5.228389
	9	3.265541	5.685161
10	1	2.589626	1.989424
	2	2.665775	2.404677
	3	2.741691	2.819927
	4	2.817613	3.235175
	5	2.893536	3.650433
	6	2.969456	4.065658
	7	3.045377	4.480935
	8	3.121293	4.896191
	9	3.197217	5.626683
	10	3.273134	5.716632

Table 2: 2nd moment of the Concomitants of Order Statistics from BTEGD for selected values of the parameters

n	r	$\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5,$ $\sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2$ $\rho = -0.5, \lambda_1 = -0.5,$ $\lambda_2 = -0.5$	$\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3,$ $\sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5$ $\rho = -1, \lambda_1 = 1,$ $\lambda_2 = 1$
5	1	16.74247	26.44601
	2	17.93063	34.17772
	3	19.11811	41.9094
	4	20.3056	49.6411
	5	21.49317	57.37285
6	1	16.57269	25.34148
	2	17.59136	31.96866
	3	18.6092	38.59582
	4	19.62704	45.22538
	5	20.64487	51.85032
	6	21.66273	58.47742
7	1	16.44532	24.5131
	2	17.33691	30.31188
	3	18.22751	36.11063
	4	19.11811	41.90942
	5	20.00877	47.70822
	6	20.89933	53.50699
	7	21.78997	59.30575
8	1	16.34621	23.86878
	2	16.72082	29.02325
	3	17.13901	34.17772
	4	17.93062	39.33222

	5	18.72229	44.48665
	6	19.51398	49.64116
	7	20.30566	54.79555
	8	21.88893	59.95018
9	1	16.26691	23.35333
	2	16.98067	27.99243
	3	17.69313	32.63138
	4	18.40563	37.27003
	5	19.11814	41.90948
	6	19.83066	46.54841
	7	20.5431	51.18753
	8	21.25557	55.82651
	9	21.9681	60.46554
10	1	16.20199	22.93162
	2	16.85113	27.14891
	3	17.49882	31.36619
	4	18.14655	35.58346
	5	18.79427	39.80078
	6	19.44201	44.01819
	7	20.08974	48.23534
	8	20.73741	52.45268
	9	21.38508	56.66997
	10	22.03288	60.88725

Table 3: 3rd moment of the Concomitants of Order Statistics from BTEGD for selected values of the parameters

n	r	$\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5,$ $\sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2$ $\rho = -0.5, \lambda_1 = -0.5,$ $\lambda_2 = -0.5$	$\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3,$ $\sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5$ $\rho = -1, \lambda_1 = 1,$ $\lambda_2 = 1$
	1	46.21034	304.2023
	2	46.36356	384.429
5	3	46.51494	464.5551
	4	46.66646	540.791
	5	46.81796	542.4345
	1	46.18826	292.7411
	2	46.3203	361.5084
	3	46.45004	430.2701
6	4	46.5799	498.8401
	5	46.70976	561.7638
	6	46.83957	538.5677
	1	46.17162	284.1452
	2	46.28786	344.3163
	3	46.40136	404.4554
7	4	46.51493	464.6494
	5	46.62895	524.4826
	6	46.74226	576.6806
	7	46.8558	532.2156
	1	46.15857	277.4594
	2	46.26278	330.9449
	3	46.3635	384.4287
8	4	46.46446	437.915

	5	46.56582	491.384
	6	46.66684	544.3416
	7	46.75886	587.4596
	8	46.86842	524.3235
9	1	46.14808	272.1111
	2	46.24251	320.2479
	3	46.3332	368.3824
	4	46.42407	416.5217
	5	46.51513	464.6573
	6	46.6062	512.7646
	7	46.69968	560.13
	8	46.78758	595.2691
	9	46.87883	515.4633
10	1	46.13938	267.735
	2	46.22629	311.4938
	3	46.30853	355.2529
	4	46.39105	399.0171
	5	46.47393	442.7771
	6	46.55676	486.537
	7	46.63924	530.2497
	8	46.72145	572.9357
	9	46.80406	600.8519
	10	46.88709	505.9661

Table 4: 4^{th} moment of the Concomitants of Order Statistics from BTEGD for selected values of the parameters

n	r	$\mu_1 = 5, \mu_2 = -5, \sigma_1 = 2.5,$ $\sigma_2 = 2.5, \alpha_1 = 2, \alpha_2 = 2$ $\rho = -0.5, \lambda_1 = -0.5,$ $\lambda_2 = -0.5$	$\mu_1 = -10, \mu_2 = -10, \sigma_1 = 3,$ $\sigma_2 = 3, \alpha_1 = 2.5, \alpha_2 = 2.5$ $\rho = -1, \lambda_1 = 1,$ $\lambda_2 = 1$
	1	120.7733	355.6867
	2	139.8303	440.4934
5	3	157.3197	525.1884
	4	174.7976	605.5808
	5	191.3078	603.488
	1	117.9617	343.571
	2	134.8312	416.265
6	3	149.8283	488.9521
	4	164.811	561.427
	5	179.791	627.6618
	6	193.6115	598.6534
	1	115.7753	334.4848
	2	131.0799	398.0905
7	3	144.2097	461.6975
	4	157.3208	525.2939
	5	170.4295	588.5272
	6	183.5355	643.3157
	7	195.2905	591.2097
	1	114.0061	327.4168
	2	128.1599	383.9558
	3	139.8398	440.4935
8	4	151.4931	497.0336

	5	163.1463	553.5519
	6	174.7995	609.5107
	7	186.4476	654.5829
	8	196.5538	582.1563
9	1	112.5292	321.763
	2	125.8216	372.648
	3	136.3437	423.5304
	4	146.8319	474.4184
	5	157.3202	525.3023
	6	167.8075	576.1526
	7	178.2954	626.1872
	8	188.7768	662.6969
	9	197.5259	572.0886
10	1	111.2651	317.1369
	2	123.9062	363.3961
	3	133.4833	409.6512
	4	143.0181	455.9151
	5	152.5525	502.174
	6	162.0869	548.4318
	7	171.6213	594.6379
	8	181.1557	639.7085
	9	190.682	668.4436
	10	198.2868	561.3813

From the above tables we can see that the numerical values of moments of concomitants of order statistics of BTEGD shown an increasing trend. For fixed values of n and for the fixed values of the parameters, the values of moments increase for increasing values of r.

6. CONCLUSION

In this paper, we have studied the Morgenstern Type Bivariate Transmuted Exponentiated Gumbel Distribution and their concomitants of order statistics when a sample is available from a Bivariate Transmuted Exponentiated Gumbel Distribution. We have obtained the distribution of the concomitants of r^{th} order statistics, moment generating function and the distribution of moments of the concomitants from BTEGD. Numerical computations have been done for the first four moments of the concomitants of order statistics for some selected values of the parameters, which show an increasing trend.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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