Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 3347-3360
https://doi.org/10.28919/jmcs/5722
ISSN: 1927-5307

BOUNDS ON HANKEL DETERMINANT FOR STARLIKE FUNCTIONS WITH RESPECT TO CONJUGATE POINTS

NUR HAZWANI AQILAH ABDUL WAHID, DAUD MOHAMAD*

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia

Abstract: The objective of this paper is to obtain the bounds of the Hankel determinants $|H_2(3)|$ and $|H_3(1)|$ for the subclass of starlike functions with respect to conjugate points in an open unit disk. We also give some consequences of our main results.

Keywords: coefficient bounds; univalent functions; starlike functions; Hankel determinant.

2010 AMS Subject Classification: 30C45, 30C50.

1. INTRODUCTION

Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

in the unit disk $E = \{z : |z| < 1\}$. Let S be the class of functions $f(z) \in A$ and univalent in E. The well-known subclass starlike of A is denoted by S^*. Analytically, this class is defined by

*Corresponding author
E-mail address: daud@tmsk.uitm.edu.my
Received April 19, 2021

(2) $$S^* = \left\{ f(z) \in \mathrm{A} : \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0,\ z \in E \right\}.$$

Let P denote the class of analytic functions $p(z)$ of the form

(3) $$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

in E whose real parts are positive in E. If $p(z) \in P$, then a Schwarz function $\omega(z)$ exists with $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in E$ such that [1]

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)}.$$

For two functions $F(z)$ and $G(z)$ analytic in E, we say that the function $F(z)$ is subordinate to $G(z)$ and we write it as $F(z) \prec G(z)$ if there exists a Schwarz function $\omega(z)$ such that $F(z) = G(\omega(z))$. Further, if $G(z)$ is univalent in E, then $F(z) \prec G(z) \Leftrightarrow F(0) = G(0)$ and $F(E) = G(E)$ (see Miller and Mocanu [2, 3] for details).

Let $S_C^*(\alpha,\delta,A,B)$ denote the class of analytic functions $f(z)$ of the form (1) and satisfying the condition

(4) $$\left[e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin\alpha \right] \frac{1}{t_{\alpha\delta}} \prec \frac{1+Az}{1+Bz},\ z \in E,$$

where $g(z) = \dfrac{f(z) + \overline{f(\bar{z})}}{2}$, $t_{\alpha\delta} = \cos\alpha - \delta$, $0 \le \delta < 1$, $|\alpha| < \dfrac{\pi}{2}$ and $-1 \le B < A \le 1$. Functions in the class $S_C^*(\alpha,\delta,A,B)$ are called tilted starlike functions with respect to conjugate points of order δ, introduced and studied by Wahid et al. [4].

In particular,

a) $S_C^*(0,0,1,-1) \equiv S_C^*$, the class of starlike functions with respect to conjugate points introduced by El-Ashwah and Thomas [5].

b) $S_C^*(0,\delta,1,-1) \equiv S_C^*(\delta)$, the class introduced by Halim [6].

c) $S_C^*(0,0,A,B) \equiv S_C^*(A,B)$, the class introduced by Dahhar and Janteng [7].

Obviously, $S_C^*(0) \equiv S_C^*$ and $S_C^*(1,-1) \equiv S_C^*$.

For given natural numbers n, q and $a_1 = 1$, the Hankel determinant $H_q(n)$ of a function $f(z) \in \mathrm{A}$ is defined by means of the following determinant [8,9]

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For some specific values of n and q, it is observed that $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2 a_4 - a_3^2$ are the Fekete-Szegö functional and second Hankel determinant respectively. Fekete and Szegö [10] then further generalized the estimate $a_3 - \mu a_2^2$ where μ is real and $f(z) \in S$. The sharp bound on $H_2(2)$ was investigated by Janteng et al. [11] for the class S^* and they proved that $|H_2(2)| \le 1$. The estimation of $H_3(1)$ is much more difficult than $H_2(2)$ and in 2009, Babalola [12] proved that $|H_3(1)| \le 16$ for the class S^*. Zaprawa [13] and Kwon et al. [14] improved the result of Babalola [12] by proving that $|H_3(1)| \le 1$ and $|H_3(1)| \le \frac{8}{9}$ respectively for the class S^*. However, they claimed that the bound is still not sharp.

The finding of the upper bounds of $H_q(n)$ over different subclasses of S is still an open problem and a subject of interest. In fact, the bounds of $|H_2(2)|$ and $|H_3(1)|$ for the whole class S are still unknown and the estimation of $|H_3(1)|$ is more difficult. In addition, there are very few papers on investigating the bounds of $|H_2(3)|$ and $|H_3(1)|$ for functions belonging to the class of starlike functions with respect to conjugate points. For more works on $H_2(2)$, $H_2(3)$, and $H_3(1)$ especially on the subclasses of S related to other points, i.e., symmetric points, conjugate points and symmetric conjugate points, see the references [15-27].

In this paper, we consider the Hankel determinant in the case of $n=3$, $q=2$ and $n=1$, $q=3$ given by

$$H_2(3)=\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = a_3a_5-a_4^2 \tag{5}$$

and

$$H_3(1)=\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_5\left(a_3-a_2^2\right)+a_3\left(a_2a_4-a_3^2\right)+a_4\left(a_2a_3-a_4\right) \tag{6}$$

respectively for functions in the class $S_C^*(\alpha,\delta,A,B)$ which is defined in (4).

The following lemmas will be required to derive our main results.

2. PRELIMINARIES

Lemma 2.1. *[28] For a function* $p(z)\in P$ *of the form (3), the sharp inequality* $|p_n|\le 2$ *holds for each* $n\ge 1$. *Equality holds for the function* $p(z)=\dfrac{1+z}{1-z}$.

Lemma 2.2. *[29] Let* $p(z)\in P$ *of the form (3) and* $\mu\in\mathbb{C}$. *Then*

$$\left|p_n-\mu p_k p_{n-k}\right|\le 2\max\left\{1,\left|2\mu-1\right|\right\},\ 1\le k\le n-1.$$

If $|2\mu-1|\ge 1$, *then the inequality is sharp for the function* $p(z)=\dfrac{1+z}{1-z}$ *or its rotations. If* $|2\mu-1|<1$, *then the inequality is sharp for the function* $p(z)=\dfrac{1+z^n}{1-z^n}$ *or its rotations.*

3. Main Results

Theorem 3.1. *If the function* $f(z)$ *given by (1) belongs to the class* $S_C^*(\alpha,\delta,A,B)$, *then*

$$\begin{aligned}|H_2(3)| \le \frac{T^2}{9216}\Big\{&1024+4|288\Upsilon-144\xi-288|+8|72\xi^2+(-24\Upsilon+192)\xi-64\Upsilon^2-224\Upsilon| \\ &+32|(-24\Upsilon+2)\xi^4+(88\Upsilon^2+42\Upsilon+6)\xi^3+(-96\Upsilon^3+58\Upsilon^2+68\Upsilon)\xi^2 \\ &+(32\Upsilon^4+38\Upsilon^3-162\Upsilon^2)\xi-36\Upsilon^4+80\Upsilon^3| \\ &+8|(-192\Upsilon-64)\xi^2+(128\Upsilon^2+288\Upsilon-144)\xi-160\Upsilon^2+144\Upsilon|\Big\}\end{aligned}$$

where $\xi = Te^{-i\alpha}$, $T=(A-B)t_{\alpha\delta}$, $t_{\alpha\delta}=\cos\alpha-\delta$ *and* $\Upsilon = 1+B.$

Proof. If $f(z)\in S_C^*(\alpha,\delta,A,B)$ of the form (1), then according to subordination relationship, there exists a Schwarz function $\omega(z)$ such that

(7)
$$\left[e^{i\alpha}\frac{zf'(z)}{g(z)}-\delta-i\sin\alpha\right]\frac{1}{t_{\alpha\delta}}=\frac{1+A\omega(z)}{1+B\omega(z)},$$

where $g(z)=\dfrac{f(z)+\overline{f(\bar{z})}}{2}$ and $t_{\alpha\delta}=\cos\alpha-\delta.$

Define a function

$$h(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+\sum_{n=1}^{\infty}k_n z^n.$$

We have $h(z)\in P$ and

(8)
$$\omega(z)=\frac{h(z)-1}{h(z)+1}.$$

Using (8), from (7), we have

(9)
$$e^{i\alpha}\frac{zf'(z)}{g(z)}=\frac{\left[e^{i\alpha}(1-B)-T\right]+h(z)\left[e^{i\alpha}(1+B)+T\right]}{1-B+h(z)(1+B)}$$

where $T=(A-B)t_{\alpha\delta}$.

Using the series representation for $f(z)$ and $h(z)$ in (9), we get

$$e^{i\alpha}(1-B)\left(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+6a_6z^6+\cdots\right)$$
$$+e^{i\alpha}(1+B)\left(z+2a_2z^2+3a_3z^3+4a_4z^4+5a_5z^5+6a_6z^6+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+k_4z^4+k_5z^5+\cdots\right)$$
$$=\left[e^{i\alpha}(1-B)-T\right]\left(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+a_6z^6+\cdots\right)$$
$$+\left[e^{i\alpha}(1+B)+T\right]\left(z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+a_6z^6+\cdots\right)\left(1+k_1z+k_2z^2+k_3z^3+k_4z^4+k_5z^5+\cdots\right).$$
(10)

Equating the coefficients of like powers of z^2, z^3, z^4 and z^5 respectively in the expansion of (10) and for simplicity, we denote $\xi = Te^{-i\alpha}$ and $\Upsilon = 1+B$ give us

(11) $$a_2=\frac{k_1\xi}{2},$$

(12) $$a_3=\frac{1}{8}\left[2k_2\xi+k_1^2\xi^2-k_1^2\Upsilon\xi\right],$$

(13) $$a_4=\frac{1}{48}\left[8k_3\xi+6k_1k_2\xi^2-8k_1k_2\Upsilon\xi+k_1^3\xi^3-3k_1^3\Upsilon\xi^2+2k_1^3\Upsilon^2\xi\right]$$

and

(14) $$a_5=\frac{1}{384}\Big[48k_4\xi+32k_1k_3\xi^2-48k_1k_3\Upsilon\xi+12k_2^2\xi^2-24k_2^2\Upsilon\xi+k_1^4\xi^4-6k_1^4\Upsilon\xi^3+11k_1^4\Upsilon^2\xi^2-6k_1^4\Upsilon^3\xi$$
$$+12k_1^2k_2\xi^3-44k_1^2k_2\Upsilon\xi^2+36k_1^2k_2\Upsilon^2\xi\Big].$$

In view of (5) and (12)-(14), we have

$$H_2(3)=\frac{\xi^2}{9216}\Big\{288k_4k_2+k_1k_2k_3(192\xi-288\Upsilon)+k_2^3(72\xi-144\Upsilon)$$
$$+k_1^4k_2\left(42\xi^3-204\Upsilon\xi^2+306\Upsilon^2\xi-144\Upsilon^3\right)+k_1^2k_2^2\left(108\xi^2-372\Upsilon\xi+288\Upsilon^2\right)$$
$$+k_1^2k_4(144\xi-144\Upsilon)+k_1^3k_3\left(96\xi^2-240\Upsilon\xi+144\Upsilon^2\right)$$
$$+k_1^6\left(3\xi^4-21\Upsilon\xi^3+51\Upsilon^2\xi^2-51\Upsilon^3\xi+18\Upsilon^4\right)+k_1^2k_2^2\left(-144\xi^2+384\Upsilon\xi-256\Upsilon^2\right)$$
$$+k_1^6\left[(12\Upsilon-4)\xi^4+\left(-44\Upsilon^2+12\Upsilon\right)\xi^3+\left(-8\Upsilon^2+48\Upsilon^3\right)\xi^2-16\Upsilon^4\xi\right]$$
$$-256k_3^2+k_1k_2k_3(512\Upsilon-384\xi)+k_1^3k_3\left[(96\Upsilon-64)\xi^2+\left(-64\Upsilon^2+96\Upsilon\right)\xi-64\Upsilon^2\right]$$
$$+k_1^4k_2\left[(-24\Upsilon-48)\xi^3+\left(-144\Upsilon^2+136\Upsilon\right)\xi^2+\left(-144\Upsilon^2+64\Upsilon^3\right)\xi+64\Upsilon^3\right]\Big\}$$
$$=\frac{\xi^2}{9216}\Big\{k_1^6\left[(12\Upsilon-1)\xi^4+\left(-44\Upsilon^2-9\Upsilon\right)\xi^3+\left(48\Upsilon^3+43\Upsilon^2\right)\xi^2+\left(-16\Upsilon^4-51\Upsilon^3\right)\xi+18\Upsilon^4\right]$$
$$+288k_2k_4+k_1^2k_4(144\xi-144\Upsilon)-256k_3^2+k_1k_2k_3(224\Upsilon-192\xi)$$
$$+k_2^3(72\xi-144\Upsilon)+k_1^2k_2^2\left(-36\xi^2+12\Upsilon\xi+32\Upsilon^2\right)$$
(15) $$+k_1^3k_3\left[(96\Upsilon+32)\xi^2+\left(-64\Upsilon^2-144\Upsilon\right)\xi+80\Upsilon^2\right]$$
$$+k_1^4k_2\left[(-24\Upsilon-6)\xi^3+\left(-144\Upsilon^2-68\Upsilon\right)\xi^2+\left(64\Upsilon^3+162\Upsilon^2\right)\xi-80\Upsilon^3\right]\Big\}.$$

Further, by suitably arranging the terms and taking modulus on both sides of (15) gives us

$$
\begin{aligned}
\left|H_2(3)\right| = \frac{\left|\xi^2\right|}{9216}\Bigg| & -256k_3^{\,2} + 288k_2\left[k_4 - k_2^{\,2}\left(\frac{144\Upsilon - 72\xi}{288}\right)\right] \\
& + k_1k_2\left(224\Upsilon - 192\xi\right)\left[k_3 - k_1k_2\left(\frac{36\xi^2 - 12\Upsilon\xi - 32\Upsilon^2}{224\Upsilon - 192\xi}\right)\right] \\
& + k_1^{\,2}\left(-144\Upsilon + 144\xi\right)\left[k_4 - k_1k_3\left(\frac{\left(-96\Upsilon - 32\right)\xi^2 + \left(64\Upsilon^2 + 144\Upsilon\right)\xi - 80\Upsilon^2}{-144\Upsilon + 144\xi}\right)\right] \\
& + k_1^{\,4}\left(\left(-24\Upsilon - 6\right)\xi^3 + \left(-144\Upsilon^2 - 68\Upsilon\right)\xi^2 + \left(64\Upsilon^3 + 162\Upsilon^2\right)\xi - 80\Upsilon^3\right)\Bigg[k_2 \\
& - k_1^{\,2}\left(\frac{\left(-12\Upsilon + 1\right)\xi^4 + \left(44\Upsilon^2 + 9\Upsilon\right)\xi^3 + \left(-48\Upsilon^3 - 43\Upsilon^2\right)\xi^2 + \left(16\Upsilon^4 + 51\Upsilon^3\right)\xi - 18\Upsilon^4}{\left(-24\Upsilon - 6\right)\xi^3 + \left(-144\Upsilon^2 - 68\Upsilon\right)\xi^2 + \left(64\Upsilon^3 + 162\Upsilon^2\right)\xi - 80\Upsilon^3}\right)\Bigg]\Bigg| \\
= \frac{\left|\xi^2\right|}{9216}\Big| & -256k_3^{\,2} + 288k_2\left[k_4 - \eta^* k_2^{\,2}\right] + k_1k_2\left(224\Upsilon - 192\xi\right)\left[k_3 - \lambda^* k_1k_2\right] \\
& + k_1^{\,4}\left(\left(-24\Upsilon - 6\right)\xi^3 + \left(-144\Upsilon^2 - 68\Upsilon\right)\xi^2 + \left(64\Upsilon^3 + 162\Upsilon^2\right)\xi - 80\Upsilon^3\right)\left[k_2 - \mu^* k_1^{\,2}\right] \\
& + k_1^{\,2}\left(-144\Upsilon + 144\xi\right)\left[k_4 - \nu^* k_1k_3\right]\Big|
\end{aligned}
$$

(16)

where

$$\eta^* = \frac{144\Upsilon - 72\xi}{288},$$

$$\lambda^* = \frac{36\xi^2 - 12\Upsilon\xi - 32\Upsilon^2}{224\Upsilon - 192\xi},$$

$$\nu^* = \frac{\left(-96\Upsilon - 32\right)\xi^2 + \left(64\Upsilon^2 + 144\Upsilon\right)\xi - 80\Upsilon^2}{-144\Upsilon + 144\xi}$$

and

$$\mu^* = \frac{\left(-12\Upsilon + 1\right)\xi^4 + \left(44\Upsilon^2 + 9\Upsilon\right)\xi^3 + \left(-48\Upsilon^3 - 43\Upsilon^2\right)\xi^2 + \left(16\Upsilon^4 + 51\Upsilon^3\right)\xi - 18\Upsilon^4}{\left(-24\Upsilon - 6\right)\xi^3 + \left(-144\Upsilon^2 - 68\Upsilon\right)\xi^2 + \left(64\Upsilon^3 + 162\Upsilon^2\right)\xi - 80\Upsilon^3}.$$

Hence, by the triangle inequality, from (16), we get

$$
\begin{aligned}
\left|H_2(3)\right| \le \frac{\left|\xi^2\right|}{9216}\Big\{ & 256\left|k_3\right|^2 + 288\left|k_2\right|\left|k_4 - \eta^* k_2^{\,2}\right| + \left|k_1\right|\left|k_2\right|\left|224\Upsilon - 192\xi\right|\left|k_3 - \lambda^* k_1k_2\right| \\
& + \left|k_1\right|^4\left|\left(-24\Upsilon - 6\right)\xi^3 + \left(-144\Upsilon^2 - 68\Upsilon\right)\xi^2 + \left(64\Upsilon^3 + 162\Upsilon^2\right)\xi - 80\Upsilon^3\right|\left|k_2 - \mu^* k_1^{\,2}\right| \\
& + \left|k_1\right|^2\left|-144\Upsilon + 144\xi\right|\left|k_4 - \nu^* k_1k_3\right|\Big\}.
\end{aligned}
$$

(17)

Now, applying Lemmas 2.1 and 2.2, we see that

(18) $$256|k_3|^2 \le 1024,$$

(19) $$288|k_2|\left|k_4 - \eta^* k_2^{\,2}\right| \le 4|288\Upsilon - 144\xi - 288|,$$

(20) $$|k_1||k_2||224\Upsilon - 192\xi|\left|k_3 - \lambda^* k_1 k_2\right| \le 8\left|72\xi^2 + (-24\Upsilon + 192)\xi - 64\Upsilon^2 - 224\Upsilon\right|,$$

$$|k_1|^4\left|(-24\Upsilon - 6)\xi^3 + (-144\Upsilon^2 - 68\Upsilon)\xi^2 + (64\Upsilon^3 + 162\Upsilon^2)\xi - 80\Upsilon^3\right|\left|k_2 - \mu^* k_1^{\,2}\right|$$
$$\le 32\left|(-24\Upsilon + 2)\xi^4 + (88\Upsilon^2 + 42\Upsilon + 6)\xi^3 + (-96\Upsilon^3 + 58\Upsilon^2 + 68\Upsilon)\xi^2\right.$$
$$\left. + (32\Upsilon^4 + 38\Upsilon^3 - 162\Upsilon^2)\xi - 36\Upsilon^4 + 80\Upsilon^3\right|$$

(21)

and

$$|k_1|^2|-144\Upsilon + 144\xi|\left|k_4 - \nu^* k_1 k_3\right| \le 8\left|(-192\Upsilon - 64)\xi^2 + (128\Upsilon^2 + 288\Upsilon - 144)\xi - 160\Upsilon^2 + 144\Upsilon\right|.$$

(22)

Hence, (17) yields

$$|H_2(3)| \le \frac{T^2}{9216}\Big\{1024 + 4|288\Upsilon - 144\xi - 288| + 8\left|72\xi^2 + (-24\Upsilon + 192)\xi - 64\Upsilon^2 - 224\Upsilon\right|$$
$$+32\left|(-24\Upsilon + 2)\xi^4 + (88\Upsilon^2 + 42\Upsilon + 6)\xi^3 + (-96\Upsilon^3 + 58\Upsilon^2 + 68\Upsilon)\xi^2\right.$$
$$\left. + (32\Upsilon^4 + 38\Upsilon^3 - 162\Upsilon^2)\xi - 36\Upsilon^4 + 80\Upsilon^3\right|$$
$$+8\left|(-192\Upsilon - 64)\xi^2 + (128\Upsilon^2 + 288\Upsilon - 144)\xi - 160\Upsilon^2 + 144\Upsilon\right|\Big\}.$$

This completes the proof of Theorem 3.1.

Theorem 3.2. *If the function* $f(z)$ *given by (1) belongs to the class* $S_C^*(\alpha, \delta, A, B)$, *then*

$$|H_3(1)| \le \frac{T^2}{9216}\Big\{4|288\Upsilon + 288\xi - 288| + 4|448\Upsilon + 384\xi - 256|$$
$$+8\left|72\xi^2 + (24\Upsilon + 72)\xi - 64\Upsilon^2 + 144\Upsilon\right| + 64\left|-\xi^4 - 3\Upsilon\xi^3 - \Upsilon^2\xi^2 + 3\Upsilon^3\xi + 2\Upsilon^4\right|$$
$$+16\left|-12\xi^3 + (-8\Upsilon - 32)\xi^2 + (36\Upsilon^2 - 48\Upsilon)\xi + 32\Upsilon^3 - 16\Upsilon^2\right|\Big\}$$

where $\xi = Te^{-i\alpha}$, $T = (A - B)t_{\alpha\delta}$, $t_{\alpha\delta} = \cos\alpha - \delta$ *and* $\Upsilon = 1 + B$.

Proof. Making use of (11)-(14) and (6) give us

$$H_3(1)=\frac{\xi^2}{9216}\Big\{288k_2k_4+k_1k_2k_3(192\xi-288\Upsilon)+k_2^3(72\xi-144\Upsilon)+k_1^2k_2^2(36\xi^2-228\Upsilon\xi+288\Upsilon^2)$$
$$+k_1^2k_4(-144\xi-144\Upsilon)+k_1^3k_3(-96\xi^2+48\Upsilon\xi+144\Upsilon^2)+k_1^2k_2^2(-72\xi^2+24\Upsilon\xi)$$
$$+k_1^4k_2(-30\xi^3+60\Upsilon\xi^2+90\Upsilon^2\xi-144\Upsilon^3)+k_1^6(-3\xi^4+15\Upsilon\xi^3-15\Upsilon^2\xi^2-15\Upsilon^3\xi+18\Upsilon^4)$$
$$+k_1^4k_2(-12\xi^3-24\Upsilon\xi^2+36\Upsilon^2\xi)+k_1^3k_3(96\xi^2-96\Upsilon\xi)+192k_1k_2k_3\xi-144k_2^3\xi-256k_3^2$$
$$+k_1^6(-6\xi^4+6\Upsilon\xi^3+6\Upsilon^2\xi^2-6\Upsilon^3\xi)+k_1k_2k_3(512\Upsilon-192\xi)+k_1^3k_3(32\xi^2+96\Upsilon\xi-128\Upsilon^2)$$
$$+k_1^2k_2^2(192\Upsilon\xi-256\Upsilon^2)+k_1^4k_2(48\xi^3-32\Upsilon\xi^2-144\Upsilon^2\xi+128\Upsilon^3)$$
$$+k_1^6(8\xi^4-24\Upsilon\xi^3+8\Upsilon^2\xi^2+24\Upsilon^3\xi-16\Upsilon^4)\Big\}$$
$$=\frac{\xi^2}{9216}\Big\{288k_2k_4+k_1^2k_4(-144\Upsilon-144\xi)-256k_3^2+k_1k_2k_3(224\Upsilon+192\xi)$$
$$+k_2^3(-144\Upsilon-72\xi)+k_1^2k_2^2(-36\xi^2-12\Upsilon\xi+32\Upsilon^2)+k_1^3k_3(32\xi^2+48\Upsilon\xi+16\Upsilon^2)$$
$$+k_1^4k_2(6\xi^3+4\Upsilon\xi^2-18\Upsilon^2\xi-16\Upsilon^3)+k_1^6(-\xi^4-3\Upsilon\xi^3-\Upsilon^2\xi^2+3\Upsilon^3\xi+2\Upsilon^4)\Big\}.$$

(23)

Further, by suitably arranging the terms and taking modulus on both sides of (23), we obtain

$$|H_3(1)|=\Bigg|\frac{\xi^2}{9216}\Bigg\{288k_4\left[k_2-k_1^2\left(\frac{144\Upsilon+144\xi}{288}\right)\right]-256k_3\left[k_3-k_1k_2\left(\frac{224\Upsilon+192\xi}{256}\right)\right]$$
$$+k_2^2(-144\Upsilon-72\xi)\left[k_2-k_1^2\left(\frac{36\xi^2+12\Upsilon\xi-32\Upsilon^2}{-144\Upsilon-72\xi}\right)\right]$$
$$+k_1^3(32\xi^2+48\Upsilon\xi+16\Upsilon^2)\left[k_3-k_1k_2\left(\frac{-6\xi^3-4\Upsilon\xi^2+18\Upsilon^2\xi+16\Upsilon^3}{32\xi^2+48\Upsilon\xi+16\Upsilon^2}\right)\right]$$
$$+k_1^6(-\xi^4-3\Upsilon\xi^3-\Upsilon^2\xi^2+3\Upsilon^3\xi+2\Upsilon^4)\Bigg\}\Bigg|$$
$$=\Bigg|\frac{\xi^2}{9216}\Big\{288k_4\left[k_2-\mu k_1^2\right]-256k_3\left[k_3-\eta k_1k_2\right]+k_2^2(-144\Upsilon-72\xi)\left[k_2-\lambda k_1^2\right]$$
$$+k_1^3(32\xi^2+48\Upsilon\xi+16\Upsilon^2)\left[k_3-\nu k_1k_2\right]+k_1^6(-\xi^4-3\Upsilon\xi^3-\Upsilon^2\xi^2+3\Upsilon^3\xi+2\Upsilon^4)\Big\}\Bigg|$$

(24)

where

$$\mu=\frac{144\Upsilon+144\xi}{288},$$
$$\eta=\frac{224\Upsilon+192\xi}{256},$$

$$\lambda = \frac{36\xi^2 + 12\Upsilon\xi - 32\Upsilon^2}{-144\Upsilon - 72\xi}$$

and

$$\nu = \frac{-6\xi^3 - 4\Upsilon\xi^2 + 18\Upsilon^2\xi + 16\Upsilon^3}{32\xi^2 + 48\Upsilon\xi + 16\Upsilon^2}.$$

Hence, by the triangle inequality, from (24), we get

$$|H_3(1)| \le \frac{|\xi^2|}{9216}\Big\{288|k_4||k_2 - \mu k_1^2| + 256|k_3||k_3 - \eta k_1 k_2| + |k_2|^2|-144\Upsilon - 72\xi||k_2 - \lambda k_1^2|$$
$$+|k_1|^6|-\xi^4 - 3\Upsilon\xi^3 - \Upsilon^2\xi^2 + 3\Upsilon^3\xi + 2\Upsilon^4| + |k_1|^3|32\xi^2 + 48\Upsilon\xi + 16\Upsilon^2||k_3 - \nu k_1 k_2|\Big\}.$$

(25)

Now, applying Lemmas 2.1 and 2.2 in (25), we see that

(26) $$288|k_4||k_2 - \mu k_1^2| \le 4|288\Upsilon + 288\xi - 288|,$$

(27) $$256|k_3||k_3 - \eta k_1 k_2| \le 4|448\Upsilon + 384\xi - 256|,$$

(28) $$|k_2|^2|-144\Upsilon - 72\xi||k_2 - \lambda k_1^2| \le 8|72\xi^2 + (24\Upsilon + 72)\xi - 64\Upsilon^2 + 144\Upsilon|,$$

(29) $$|k_1|^6|-\xi^4 - 3\Upsilon\xi^3 - \Upsilon^2\xi^2 + 3\Upsilon^3\xi + 2\Upsilon^4| \le 64|-\xi^4 - 3\Upsilon\xi^3 - \Upsilon^2\xi^2 + 3\Upsilon^3\xi + 2\Upsilon^4|$$

and

(30) $$|k_1|^3|32\xi^2 + 48\Upsilon\xi + 16\Upsilon^2||k_3 - \nu k_1 k_2| \le 16|-12\xi^3 + (-8\Upsilon - 32)\xi^2 + (36\Upsilon^2 - 48\Upsilon)\xi + 32\Upsilon^3 - 16\Upsilon^2|.$$

Thus, using (26)-(30), we obtain

$$|H_3(1)| \le \frac{T^2}{9216}\Big\{4|288\Upsilon + 288\xi - 288| + 4|448\Upsilon + 384\xi - 256|$$
$$+8|72\xi^2 + (24\Upsilon + 72)\xi - 64\Upsilon^2 + 144\Upsilon| + 64|-\xi^4 - 3\Upsilon\xi^3 - \Upsilon^2\xi^2 + 3\Upsilon^3\xi + 2\Upsilon^4|$$
$$+16|-12\xi^3 + (-8\Upsilon - 32)\xi^2 + (36\Upsilon^2 - 48\Upsilon)\xi + 32\Upsilon^3 - 16\Upsilon^2|\Big\}.$$

This completes the proof of Theorem 3.2.

Setting $\alpha = 0$, $\delta = 0$, $A = 1$ and $B = -1$ in Theorems 3.1 and 3.2, we get the bounds of the Hankel determinants for the class introduced by El-Ashwah and Thomas [5] as in Corollary 3.1.

Corollary 3.1. *Let* $f(z) \in S_C^*$. *Then*

$$|H_2(3)| \le \frac{61}{9}$$

and

$$\left|H_3(1)\right| \le \frac{44}{9}.$$

Setting $\alpha = 0$, $A = 1$ and $B = -1$ in Theorems 3.1 and 3.2, we get the bounds of the Hankel determinants for the class introduced by Halim [6] as in Corollary 3.2.

Corollary 3.2. *Let* $f(z) \in S_C^*(\delta)$. *Then*

$$\left|H_2(3)\right| \le \frac{(1-\delta)^2}{576}\Big\{256 + \left|-288(1-\delta) - 288\right| + 2\left|288(1-\delta)^2 + 384(1-\delta)\right|$$
$$+8\left|32(1-\delta)^4 + 48(1-\delta)^3\right| + 2\left|-256(1-\delta)^2 - 288(1-\delta)\right|\Big\}$$

and

$$\left|H_3(1)\right| \le \frac{(1-\delta)^2}{576}\Big\{\left|576(1-\delta) - 288\right| + \left|768(1-\delta) - 256\right| + 2\left|288(1-\delta)^2 + 144(1-\delta)\right|$$
$$+16\left|-16(1-\delta)^4\right| + 4\left|-96(1-\delta)^3 - 128(1-\delta)^2\right|\Big\}.$$

Setting $\alpha = 0$ and $\delta = 0$ in Theorems 3.1 and 3.2, we get the bounds of the Hankel determinants for the class introduced by Dahhar and Janteng [7] as in Corollary 3.3.

Corollary 3.3. *Let* $f(z) \in S_C^*(A,B)$. *Then*

$$\left|H_2(3)\right| \le \frac{(A-B)^2}{2304}\Big\{256 + \left|288\Upsilon - 144(A-B) - 288\right|$$
$$+2\left|72(A-B)^2 + (-24\Upsilon + 192)(A-B) - 64\Upsilon^2 - 224\Upsilon\right|$$
$$+8\Big|(-24\Upsilon + 2)(A-B)^4 + \left(88\Upsilon^2 + 42\Upsilon + 6\right)(A-B)^3 + \left(-96\Upsilon^3 + 58\Upsilon^2 + 68\Upsilon\right)(A-B)^2$$
$$+\left(32\Upsilon^4 + 38\Upsilon^3 - 162\Upsilon^2\right)(A-B) - 36\Upsilon^4 + 80\Upsilon^3\Big|$$
$$+2\left|(-192\Upsilon - 64)(A-B)^2 + \left(128\Upsilon^2 + 288\Upsilon - 144\right)(A-B) - 160\Upsilon^2 + 144\Upsilon\right|\Big\}$$

and

$$\left|H_3(1)\right| \le \frac{(A-B)^2}{2304}\Big\{\left|288\Upsilon + 288(A-B) - 288\right| + \left|448\Upsilon + 384(A-B) - 256\right|$$
$$+2\left|72(A-B)^2 + (24\Upsilon + 72)(A-B) - 64\Upsilon^2 + 144\Upsilon\right|$$
$$+16\left|-(A-B)^4 - 3\Upsilon(A-B)^3 - \Upsilon^2(A-B)^2 + 3\Upsilon^3(A-B) + 2\Upsilon^4\right|$$
$$+4\left|-12(A-B)^3 + (-8\Upsilon - 32)(A-B)^2 + \left(36\Upsilon^2 - 48\Upsilon\right)(A-B) + 32\Upsilon^3 - 16\Upsilon^2\right|\Big\}.$$

4. CONCLUSION

Motivated significantly by a number of recent works, we have obtained the bounds of $|H_2(3)|$ and $|H_3(1)|$ for $S_C^*(\alpha,\delta,A,B)$. We also gave some consequences of our main results which are stated in the corollaries.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their careful reading.

CONFLICT OF INTERESTS

The author(s) declares that there is no conflict of interests.

REFERENCES

[1] I. Graham, Geometric function theory in one and higher dimensions, CRC Press, New York, (2003).

[2] S.S. Miller, P.T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl. 65(2) (1978), 289-305.

[3] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(2) (1981), 157-172.

[4] N.H.A.A. Wahid, D. Mohamad, S.C. Soh, On a subclass of tilted starlike functions with respect to conjugate points, Menemui Mat. (Discover. Math.) 37(1) (2015), 1-6.

[5] R.M. El-Ashwah, D.K. Thomas, Some subclasses of close-to-convex functions, J. Ramanujan Math. Soc. 2(1) (1987), 85-100.

[6] S. Halim, Functions starlike with respect to other points, Int. J. Math. Math. Sci. 14(3) (1991), 451-456.

[7] S.A.F.M. Dahhar, A. Janteng, A subclass of starlike functions with respect to conjugate points, Int. Math. Forum. 4(28) (2009), 1373-1377.

[8] C. Pommerenke, On the Hankel determinants of univalent functions, Mathematika 14(1) (1967), 108-112.

[9] C. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. Lond. Math. Soc. 1(1) (1966), 111-122.

[10] M. Fekete, G. Szegö, Eine Bemerkung uber ungerade schlichte Funktionen, J. Lond. Math. Soc. 8 (1933), 85-89.

[11] A. Janteng, S.A. Halim, M. Darus, Hankel determinant for starlike and convex functions, Int. J. Math. Anal. 1(13) (2007), 619-625.

[12] K.O. Babalola, On $\left|H_3(1)\right|$ Hankel determinant for some classes of univalent functions, ArXiv:0910.3779 [Math]. (2009).

[13] P. Zaprawa, Third Hankel determinants for subclasses of univalent functions, Mediterr. J. Math. 14 (2017), 19.

[14] O.S. Kwon, A. Lecko, Y.J Sim, The bound of the Hankel determinant of the third kind for starlike functions, Bull. Malays. Math. Sci. Soc. 42(2) (2019), 767-780.

[15] A. Janteng, S.A. Halim, M. Darus, Hankel determinant for functions starlike and convex with respect to symmetric points, J. Qual. Measure. Anal. 2(1) (2006), 37-43.

[16] G. Singh, Hankel determinant for analytic functions with respect to other point, Eng. Math. Lett. 2(1) (2013), 115-123.

[17] G. Singh, Hankel determinant for new subclasses of functions with respect to symmetric points, Int. J. Mod. Math. Sci. 5(2) (2013), 67-76.

[18] A. Yahya, S.C. Soh, D. Mohamad, Second Hankel determinant for a class of a generalised Sakaguchi class of analytic functions, Int. J. Math. Anal. 7(33) (2013), 1601-1608.

[19] G. Singh, G. Singh, Upper bound of the second Hankel determinant for a subclass of analytic functions, New Trends Math. Sci. 2(1) (2014), 53-58.

[20] R. Bucur, D. Breaz, L. Georgescu, Third Hankel determinant for a class of analytic functions with respect to symmetric points, Acta Univ. Apulensis, 42 (2015), 79-86.

[21] C. Selvaraj, T.R.K. Kumar, Hankel determinant for analytic functions with respect to other points, Int. Electron. J. Pure Appl. Math. 9(2) (2015), 45-54.

[22] N.H.A.A. Wahid, Second Hankel determinant for a subclass of tilted starlike functions with respect to conjugate points, MATEMATIKA: Malays. J. Ind. Appl. Math. 31(2) (2015), 111-119.

[23] A.B. Akbarally, N.A.M. Isa, On new subclasses of analytic functions with respect to conjugate and symmetric conjugate points, Glob. J. Pure Appl. Math. 12(3) (2016), 2849-2865.

[24] S.P. Vijayalakshmi, T.V. Sudharsan, Upper bound of second Hankel determinant for a class of generalized Sakaguchi class of analytic functions, Int. J. Pure Appl. Math. 109(6) (2016), 49-55.

[25] V. Kumar, S. Kumar, V. Ravichandran, Third Hankel Determinant for Certain Classes of Analytic Functions, in: N. Deo, V. Gupta, A.M. Acu, P.N. Agrawal (Eds.), Mathematical Analysis I: Approximation Theory, Springer Singapore, Singapore, 2020: pp. 223–231.

[26] A.K. Wanas, S. Bulut, Upper bound of second Hankel determinant for bi-univalent functions with respect to symmetric conjugate, Gen. Math. 28(2) (2020), 67-80.

[27] A.L.P. Hern, A. Janteng, R. Omar, Hankel determinant for certain subclasses of univalent functions, Math. Stat.

8(5) (2020), 566-569.

[28] P.L. Duren, Univalent functions, Springer, New York–Berlin–Heidelberg–Tokyo, (1983).

[29] I. Efraimidis, A generalization of Livingston's coefficient inequalities for functions with positive real part, J. Math. Anal. Appl. 435(1) (2016), 369-379.