Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 3963-3983
https://doi.org/10.28919/jmcs/5728
ISSN: 1927-5307

NUMERICAL ANALYSIS OF PREDATOR-PREY MODEL IN PRESENCE OF TOXICANT BY A NOVEL APPROACH

P.C. JAYAPRAKASHA[1,2,*], CHANDRALI BAISHYA[2]

[1]Department of Mathematics, Government First Grade Women's College, Tumkur -572103, Karnataka, India

[2]Department of Studies and Research in Mathematics, Tumkur University, Tumkur -572103, Karnataka, India

Abstract. It is assumed in this paper that the environmental toxicant affects both prey and predator population. We consider the Holling Type-I predator-prey model and stage-structured predator-prey model in presence of environmental toxicant in the form of Volterra integro-differential equations. We solve the system of equations by a novel approach Trapezoidal Base Homotopy Perturbation Method. Using the dynamical behaviour of the systems as a validating tool, we validate the above mentioned method as an efficient method to solve Volterra integro-differential predator-prey model.

Keywords: predator-prey model; toxicant; trapezoidal based homotopy perturbation method; Volterra integro-differential equations.

2010 AMS Subject Classification: 34K20, 45D05, 45J05, 65D30.

1. INTRODUCTION

The modeling of predator-prey populations is an attractive research area for scientists from different fields like biologists, ecologists, mathematicians, economists, etc. Because there is the scope of proposing a wide variety of models to analyze different mechanisms such as control of populations, oscillations of population, delay maturation time effects and pattern formation

*Corresponding author
E-mail address: jayagagandeep@gmail.com

Received March 19, 2021

in predator-prey models with diffusion, migration, refuge, harvesting, etc. One of the important attributes in these models is the predator-prey coexistence, which is a key feature to obtain realistic models of predator-prey interactions. Many times eliminating one population from the environment and analyzing the other makes the models easy to handle. But coexistence makes the population model complicated to analyze. Mathematically, the first type of coexistence is established by a stable equilibrium which are the most basic solutions to the models. The second type of coexistence is given by stable periodic solutions and this is the most commonly accepted type in predator-prey systems[16]. Since Lotka (1925) and Volterra (1927) proposed the Lotka–Volterra population model, many mathematical models have been formed based on realistic explicit and implicit biological assumptions [1, 2, 3, 4, 5, 6].

One of the essential effects on population dynamics which is the need of the hour today to be studied in detail is the effect of environmental pollution. Many mathematical biologists have worked in this area. The problem of estimating qualitatively the effect of a toxicant on a population by mathematical models began in the early 1980s. For a general class of single population models with toxicant stress, Ma et al [7] obtained a survival threshold under the hypothesis that the capacity of the environment is large relative to the population biomass and that the exogenous input of toxicant into the environment is bounded. In 1987, Ma and Hallam [8] studied the two-dimensional non-autonomous Lotka–Volterra model by the average method and obtained sufficient conditions for persistence and for the extinction of the population. The threshold of the survival for a system of two species in a polluted environment was studied by Huaping and Ma [9]. Population toxicant coupling has been applied in several contexts including Lotka–Volterra and chemostat-like environments, resulting in ordinary, Integro-differential, and stochastic models [10, 12, 13, 15]. The Volterra model for a species population growth in a closed structure is as follows:

$$\frac{dp}{dt} = ap - bp^2 - cp\int_0^t p(x)dx \quad p(0) = p_0 \tag{1}$$

where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, $c > 0$ is the toxicity coefficient, p_0 is the initial population and $p = p(t)$ denotes the population at time t.

Considerable research work has been invested recently by Small[14], TeBeest[17] among others to the development of efficient strategies to determine numerical and analytic solutions of the population growth model 1. Although a closed-form solution has been achieved by them, but it was formally shown that the closed-form solution cannot lead to an insight into the behavior of the population evolution. Analytical approximations and Pade approximants for problem 1 are considered by Wazwaz[18]. In [19], a comparison of the Adomian decomposition method and Sinc-Galerkin method is given and it is shown that the Adomian decomposition method is more efficient for the solution of Volterra's population model 1. In [20] Syed et.al applied Homotopy perturbation method to solve model 1.

The effect of environmental pollutants in Integro-differential equations is investigated for a single population model in a wide range with various numerical strategies. The novelty of this work is that, in this paper, we have considered two predator-prey models viz. (i) Holling Type-I predator-prey model and (ii) Stage-Structured Predator-Prey(SSPP), where both predator and prey population are affected by environmental pollution. Trapezoidal Based Adams-Bashforth method, proposed by Deithlm et.al[23] is efficiently used to solve fractional differential equations. This is an attempt to apply a modified method in analogy to Deithlm et.al[23], named as Trapezoidal Based Homotopy Perturbation Method(TBHPM) to solve integro-differential equations. Instead of comparing the results obtained by TBHPM with other established methods, in this paper, we have considered the stability of the system as a tool to validate the efficiency of the method. Also, we expect that the presence of toxicant in the environment (i) affects both the population. (ii) But as the predator population is dependent on prey population for food, therefore due to the toxicant effect as well as lack of prey, the predator population reduces faster than the prey population, (iii) in the case of the SSPP model toxicant effects will be more on the immatured prey than the matured prey. We shall observe that the solutions obtained by TBHPM establish these expectations.

2. Trapezoidal Based Homotopy Perturbation Method for Volterra Integro-Differential Equation

Trapezoidal Based Adams-Bashforth Method applied by Diethelm et. al in [23]. In this paper, we have modified this method using the Homotopy perturbation method and call it as Trapezoidal Based Homotopy Perturbation Method(TBHPM) and we have applied this method for a class of integro-differential equation. Consider the nonlinear integrodifferential equation of the following form

(2)

$$D^q(x(t)) = f(t,x(t)) + \lambda x(t)\int_0^t K(x,\tau)x(\tau)d\tau, \quad x^k(0) = x_0^{(k)}, k = 0,1,....n-1, n<q<n+1$$

The initial value problem 1 is equivalent to a Volterra integral equation

$$x(t) = x_0^{(k)}\frac{t^k}{k!} + \frac{1}{\Gamma(q)}\int_0^t \big((t-\tau)^{n-1}(f(\tau,x(\tau)) + \lambda x(t)\int_0^\tau K(x,\tau)x(\tau)d\tau\big)d\tau \tag{3}$$

In the sense that a continuous function solves Eq. 3 if and only if it solves Eq.2.

Firstly the product trapezoidal quadrature formula is applied to replace the integrals of Eq. 3, setting $h = \frac{T}{N}, t_n = nh, n = 0,1,2.....N \in Z^+$. The integral part in the Eq. 3 is again simplified by using Trapezoidal rule. Then Eq. 3 can be discretized as follows:

$$x(t_{n+1}) = \sum_{k=0}^{q-1} x_0^{(k)}\frac{t^k}{k!} + \frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n+1} s_{j,n+1}\big(f(t_j,x(t_j)) + \lambda x(t_j)\sum_{i=0}^{j}\beta_i K(x_i,t_i)x(t_i)\big) \tag{4}$$

where

$$s_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q & j=0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1} & 1 \le j \le n \\ 1 & j = n+1 \end{cases} \tag{5}$$

and

$$\beta_i = \begin{cases} \frac{h}{2} & i=0 \\ h & 1 \le j \le n \\ \frac{h}{2} & i = n+1 \end{cases}$$

Eq. 3 may be rewritten as

$$\begin{aligned} x(t_{n+1}) = {} & \sum_{k=0}^{q-1} x_0^{(k)}\frac{t^k}{k!} + \frac{h^q}{\Gamma(q+2)}\big(f(t_{n+1},x(t_{n+1})) + \lambda x(t_{n+1})\sum_{i=0}^{n+1}\beta_i K(x_i,t_i)x(t_i)\big) \\ & + \frac{h^q}{\Gamma(q+2)}\sum_{j=0}^{n} s_{j,n+1}\big(f(t_j,x(t_j)) + \lambda x(t_j)\sum_{i=0}^{j}\beta_i K(x_i,t_i)x(t_i)\big) \end{aligned} \tag{6}$$

The right hand side of system 6 contains term $x(t_{n+1})$, so it is an implicit scheme. We shall evaluate the term $x(t_{n+1})$ in right hand side by HPM and the scheme is named as Trapezoidal rule based Homotopy Perturbation method(TBHPM).

3. Application of TBHPM

3.1. Holling Type-I Predator-Prey Model with Environmental Pollution: In this section, we investigate the perdator-prey model of the form

$$\frac{du}{dt} = ru(1-(u/k)) - auv - eu(t)\int_0^t u(s)ds$$
$$\frac{dv}{dt} = -dv + buv - ev(t)\int_0^t v(s)ds \tag{7}$$

where $u = u(t)$ and $v = v(t)$ are the population of identical individuals at a time 't' which exhibits crowding and sensitivity to the amount of toxic produced[14]. The meaning of the symbols used in system 7 are: r is the intrinsic growth rate and k is the environmental carrying capacity of prey. a and b represent respectively the predation rate of potential prey and conversion factor of predator after consuming potential prey. d is the death rate of predator and e is the toxic coefficient. If $e = 0$ we have the well-know logistic equation discussed in details in [15]. The integrals contained in the last terms of model 7 indicates the total amount of toxins produced since time zero. As stability is the key tool to validate the numerical strategy here, the bellow stated remark gives a short note on it.

Remark 3.1.1. The Jacobian matrix is corresponding to Eq. 7 is

$$\begin{pmatrix} \left(1-\frac{u}{k}\right)r - \frac{ur}{k} - av & -au \\ bv & bu-d \end{pmatrix}$$

In absence of toxic effect, the system 7 has three equilibrium states given by
$E_0 = (0,0)$, $E_1 = (k,0)$ and $E_2 = \left(\frac{d}{b}, \frac{(-d+bk)r}{abk}\right)$
E_0 and E_1 are always exist,but E_2 will exit only when $bk > d$ and the model 7 posses a co-existence equilibrium at the point E_2.

(1) Eigen values at E_0 are $\lambda_{0,1} = -d$, $\lambda_{0,2} = r$. So the equilibrium point E_0 is always a saddle point.
(2) Eigen values at E_1 are $\lambda_{1,1} = bk - d$, $\lambda_{1,2} = -r$. This point is a stable node if $bk < d$.
(3) Eigen values at E_2 are
$\lambda_{2,1} = \frac{-\sqrt{4a^2bdkr(d-bk)+a^2d^2r^2}+adr}{2abk}$

$$\lambda_{2,2} = \frac{\sqrt{4a^2bdkr(d-bk)+a^2d^2r^2}-adr}{2abk}$$

For stability of this point we must have $bk > d$. Again the point is a stable node if $-4b^2k^2 + 4bdk + dr > 0$ and stable spiral if $-4b^2k^2 + 4bdk + dr < 0$. In the following section, we have numerically analysed this behaviour by using TBHPM and shown that the method gives the expected results.

3.1.1. ***Solution of the Integro-differential Equation 7 by TBHPM***. In this section we shall apply the TBHPM described in section 2 to investigate the proposed integro-differential equation 7 numerically and establish the efficiency of the method in solving integro differential equation. Applying TBHPM method to model 7 yields

$$u(t_{n+1}) = \frac{h}{2}[ru(t_{n+1})(1-\frac{u(t_{n+1})}{k}) - au(t_{n+1})v(t_{n+1}) - eu(t_{n+1})\sum_{i=0}^{n+1}\beta_i u(t_i)] + u_0(t_0)$$
$$+\frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(ru(t_j)(1-\frac{u(t_j)}{k}) - au(t_j)v(t_j) - eu(t_j)\sum_{i=0}^{j}\beta_i u(t_i))]$$

$$v(t_{n+1}) = \frac{h}{2}[-dv(t_{n+1}) + bu(t_{n+1})v(t_{n+1}) - ev(t_{n+1})\sum_{i=0}^{n+1}\beta_i v(t_i)] + v_0(t_0)$$
$$+\frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(-dv(t_j)(1 + bu(t_j)v(t_j) - ev(t_j)\sum_{i=0}^{j}\beta_i v(t_i))]$$

Where, $s_{j,n+1}$ is defined by Eq. 5.

We represent the terms $u(t_{n+1})$ and $v(t_{n+1})$ in the right hand side by $u_p(t_{n+1})$ and $v_p(t_{n+1})$, which shall be predicted by using Homotopy Perturbation Method. We may constructed the homotopy as

$$u_p(t_{n+1}) = u_0(t_{n+1}) + p\frac{h}{2}[ru_p(t_{n+1})(1-\frac{u_p(t_{n+1})}{k}) - au_p(t_{n+1})v_p(t_{n+1}) - eu(t_{n+1})\sum_{i=0}^{n+1}\beta_i u(t_i)]$$

$$v_p(t_{n+1}) = v_0(t_{n+1}) + p\frac{h}{2}[-dv_p(t_{n+1}) + bu_p(t_{n+1})v_p(t_{n+1}) - ev(t_j)\sum_{i=0}^{n+1}\beta_i v(t_i))]$$

where

$$u_0(t_{n+1}) = u_0(t_0) + \frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(ru(t_j)(1-\frac{u(t_j)}{k}) - au(t_j)v(t_j) - e\sum_{i=0}^{j}\beta_i u(t_i))]$$

$$v_0(t_{n+1}) = v_0(t_0) + \frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(-dv(t_j)(1+bu(t_j)v(t_j) - ev(t_j)\sum_{i=0}^{j} v(t_j))]$$

According to HPM substitute in the above equations

$$u_p(t_{n+1}) = \sum_{l=0}^{\infty} p^l u_l(t_{n+1}), \quad v_p(t_{n+1}) = \sum_{l=0}^{\infty} p^l v_l(t_{n+1})$$

We obtain the following equations

$$u_1(t_{n+1}) = \frac{h}{2}[ru_0(t_{n+1})(1-\frac{u_0(t_{n+1})}{k}) - au_0(t_{n+1})v_0(t_{n+1}) - eu_0(t_{n+1})\sum_{i=0}^{n+1}\beta_i u_0(t_i)]$$

$$v_1(t_{n+1}) = \frac{h}{2}[-dv_0(t_{n+1} + bu_0(t_{n+1})v_0(t_{n+1}) - ev_0(t_{n+1})\sum_{i=0}^{n+1}\beta_i v_0(t_i)]$$

$$u_2(t_{n+1}) = \frac{h}{2}[ru_1(t_{n+1})(1-\frac{u_1(t_{n+1})}{k}) - au_1(t_{n+1})v_1(t_{n+1}) - eu_1(t_{n+1})\sum_{i=0}^{n+1}\beta_i u_1(t_i)]$$

$$v_2(t_{n+1}) = \frac{h}{2}[-dv_1(t_{n+1} + bu_1(t_{n+1})v_1(t_{n+1}) - ev_1(t_{n+1})\sum_{i=0}^{n+1}\beta_i v_1(t_i)]$$

$$u_3(t_{n+1}) = \frac{h}{2}[ru_2(t_{n+1})(1-\frac{u_2(t_{n+1})}{k}) - au_2(t_{n+1})v_2(t_{n+1}) - eu_2(t_{n+1})\sum_{i=0}^{n+1}\beta_i u_2(t_i)]$$

$$v_3(t_{n+1}) = \frac{h}{2}[-dv_2(t_{n+1} + bu_2(t_{n+1})v_2(t_{n+1}) - ev_2(t_{n+1})\sum_{i=0}^{n+1}\beta_i v_2(t_i)]$$

and so on. For different values of the parameters, below we have given the graphical representation of the solutions.

To establish the validity of the method, first, we consider the toxicant $e = 0$ so that we establish the convergence towards the point of equilibrium. Then we observe the behavior of the system by gradually increasing the effect of toxicity. Let us we consider following set of parameters $r = 0.4, k = 4, a = 0.1, d = 0.2, b = 0.5, e = 0$. The initial condition is $x_0 = 0.5, y_0 = 0.2$. For this set of parameters, we have the equilibrium point $E = (0.4, 3.6)$.

The condition for the stability $d - bk = 1.8 < 0$ is satisfied. Also $4bdk - 4b^2k^2 + dr = -14.32 < 0$. The eigen values at this point are $\lambda_{1,1} = -0.02 + 0.267582i$, $\lambda_{1,2} = -0.02 - 0.267582i$. Therefore the point E(0.4,3.6) is stable spiral. In Figure 1(A), we observed that when $e = 0$, the prey and predator populations tend to the point of equilibrium $E(0.4, 3.6)$. when the toxicity increased gradually in Figure 1(B), Figure 1(C), Figure 1(D), Figure 1(E) and Figure 1(F), we observe that prey and predator populations move towards extinction. Also, we observe that

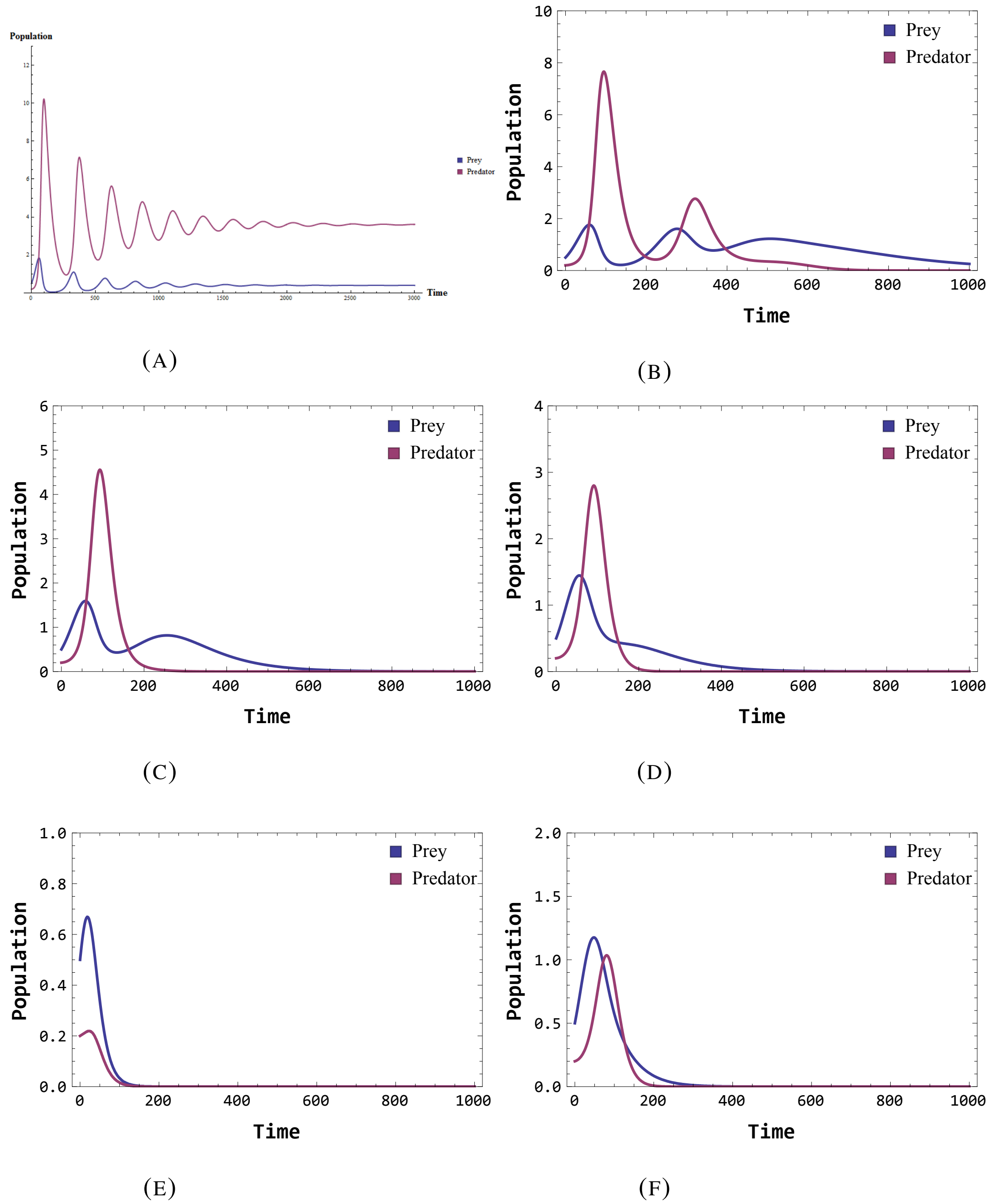

FIGURE 1. Behavior of the system 7 for (A) $e = 0$, (B) $e = 00.01$, (C) $e = 0.03$, (D) $e = 0.05$, (E) $e = 0.1$, (F) $e = 0.5$

extinction of predator population is faster than the prey populations. This is because predator are affected by toxicity as well as lack of food source(as prey population is affected due to toxicity). This analysis done by TBHPM for the predator-prey dynamics in presence of toxicant establishes that proposed method is reliable method to solve integro-differential equations.

3.2. Stage-Structured Predatpor-Prey(SSPP) Model with Environmental Pollution. Every species has two or more stages in its entire lifespan. Therefore mathematical biologists have proposed SSPP model incorporating various factors [24]-[34]. With reference to the model worked out in [34], in this section we consider a stage structured predator-prey model incorporating toxicant effect:

$$\begin{aligned}
\frac{du}{dt} &= rv-\delta_1u^2-d_1u-\alpha u-\gamma_1uw-eu(t)\int_0^t u(s)ds\\
\frac{dv}{dt} &= \alpha u-\delta_2v^2-d_2v-\gamma_2vw-ev(t)\int_0^t v(s)ds\\
\frac{dw}{dt} &= c_1\gamma_1uw+c_2\gamma_2vw-\delta_3w^2-d_3w-ew(t)\int_0^t w(s)ds
\end{aligned} \tag{8}$$

All the parameters $r,\delta_1,d_1,\beta,\gamma_1,m,\delta_2,d_2,\gamma_2,e_1,e_2,\delta_3,d_3$ are all positive and there biological meanings are as follows:

r is the intrinsic growth rate of immature prey depending on parents, δ_1, δ_2 and δ_3 represent Intraspecific competition rate between immature prey, matured prey population and predator population respectively. d_1, d_2 and d_3 are the natural death rate of immature prey, matured prey and predator population respectively. β is grownup rate of immature prey population. γ_1 and γ_2 represent the attack rate of predator to immature and matured prey population respectively. c_1 and c_2 are respectively the conversion factors of predator after consuming immature and matured prey with $0<c_1<1$, $0<c_2<1$. e represents the toxicant effect. Stability criteria of the system 8 in absence of toxicant effect and in presence prey refuge are discussed in details in [34].

Remark 3.2.1. : The system 8 has at most three biologically feasible equilibrium points viz. $E_i, i=0,1,2$.

(1) Trivial $E_0=(0,0,0)$ always exists.
(2) Predator extinct equilibrium point is $E_1=(u^*,v^*,0)$.
Here $u^*=\frac{\delta_2(v^*)^2+d_2v^*}{\beta}$ and v^* is the positive root od the equation

$$A(v^*)^3+B(v^*)^2+Cv^*+D=0$$

where $A=\delta_1\delta_2^2, \quad B=2d_2\delta_1\delta_2 y^2, \quad C=\beta d_1\delta_2+d_2^2\delta_1+\beta^2\delta_2, \quad D=\beta^2 d_2+\beta d_1 d_2-r\beta^2$. So there exists unique positive root only if $d_1 d_2<\beta(r-d_2)$

(3) Coexitence equilibrium point is (u^*,v^*,w^*).
Here, $u^*=\frac{v^*\left(-\gamma_2 d_3+\delta_3(d_2+\delta_2 v^*)+\gamma_2^2 c_2 v^*\right)}{-\gamma_1\gamma_2 c_1 v^*+\beta\delta_3}$
$z^*=\frac{-\beta d_3+v^*(\gamma_1 d_2 e_1+\beta\gamma_2 c_2+\gamma_1\delta_2 c_1 v^*)}{-\gamma_1\gamma_2 c_1 v^*+\beta\delta_3}$
and v^* is a positive root of the equation $A_1(v^*)^3+A_2(v^*)^2+A_3 v^*+A_4=0$
where,

$$A_1=(\delta_2\delta_3+\gamma_2^2 c_2)(\gamma_2^2\delta_1 c_2+\delta_2(\delta_1\delta_3+\gamma_1^2 c_1))$$

$$\begin{aligned}A_2=&\gamma_1 c_1(\gamma_1 d_2(2\delta_2\delta_3+\gamma_2^2 c_2)-\gamma_2(\gamma_2^2 c_2(\beta+d_1)+\delta_2(\delta_3(\beta+d_1)-\gamma_1 d_3)))\\&+(\delta_2\delta_3+\gamma_2^2 c_2)(2\delta_1(d_2\delta_3+\gamma_2 d_3)+\beta\gamma_1\gamma_2 c_2)-\gamma_1^2\gamma_2^2 c_1^2 r\end{aligned}$$

$$\begin{aligned}A_3=&\gamma_2 d_2(d_3(2\delta_1\delta_3+\gamma_1^2 c_1)-\gamma_1\delta_3(c_1(\beta+d_1)-\beta e_2))-\gamma_1 d_3(\gamma_2^2 c_1(\beta+d_1)\\&-\beta(\delta_2\delta_3+2\gamma_2^2 c_2))+\delta_3 d_2^2(\delta_1\delta_3+\gamma_1^2 c_1)+\beta\delta_3((\beta+d_1)(\delta_2\delta_3+\gamma_2^2 c_2)\\&+2\gamma_1\gamma_2 c_1 r)+\gamma_2^2 d_3^2\delta_1\end{aligned}$$

$$A_4=\beta(\delta_3 d_3(\gamma_2(\beta+d_1)+\gamma_1 d_2)+\gamma_1\gamma_2 d_3^2+\delta_3^2(d_2(\beta+d_1)-\beta r))$$

Interior equilibrium point exists if $A_4<0$ with $(A_2>0$ or $A_3<0)$
and $\left(d_3<\frac{d_2\delta_3+\gamma_2^2 e_2 v^*+\delta_2\delta_3 v^*}{\gamma_2}, \quad \frac{\beta d_3-\beta\gamma_2 e_2 v^*}{\gamma_1 d_2 v^*+\gamma_1\delta_2(v^*)^2}<e_1<\frac{\beta\delta_3}{\gamma_1\gamma_2 v^*}\right)$ or
$\left(d_3>\frac{d_2\delta_3+\gamma_2^2 e_2 v^*+\delta_2\delta_3 v^*}{\gamma_2}, \quad \frac{\beta\delta_3}{\gamma_1\gamma_2 v^*}<e_1<\frac{\beta d_3-\beta\gamma_2 e_2 v^*}{\gamma_1 d_2 v^*+\gamma_1\delta_2(v^*)^2}\right)$

Moreover the predator extinct equilibrium is stable if $d_3>\gamma_1 c_1 u^*+\gamma_2 c_2 v^*$ and $(d_2+2\delta_2 v^*)(\beta+d_1+2\delta_1 u^*)>\beta r$. And coexistence equilibrium is stable if (i)$\delta_1 e_1>\frac{re_1}{u}(\frac{1}{2}-\frac{v^*}{u^*})+\frac{\beta e_2}{2v}$ and (ii) $\delta_2 e_2>\frac{re_1}{2u}+\frac{\beta e_2}{v}(\frac{1}{2}-\frac{u^*}{v^*})$. Based on this information we shall verify the efficiency of the TBHPM in solving the SSPP model.

3.2.1. ***Solution of SSPP model by TBHPM***. Applying TRHP method to model 8 yields

$$u(t_{n+1}) = \frac{h}{2}[ru(t_{n+1})v(t_{n+1}) - \delta_1(u(t_{n+1}))^2 - \alpha u(t_{n+1}) - \gamma_1 u(t_{n+1})w(t_{n+1}) - eu(t_{n+1})\sum_{i=0}^{n+1}\beta_i u(t_i)] + u_0$$
$$+ \frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(ru(t_j)v(t_j) - \delta_1(u(t_j))^2 - \alpha u(t_j) - \gamma_1 u(t_j)w(t_j) - eu(t_j)\sum_{i=0}^{n+1}\beta_i u(t_i))]$$

$$v(t_{n+1}) = \frac{h}{2}[\alpha u(t_{n+1}) - \delta_2(v(t_{n+1}))^2 - d_2 v(t_{n+1}) - \gamma_2 v(t_{n+1})w(t_{n+1}) - ev(t_{n+1})\sum_{i=0}^{n+1}\beta_i v(t_i)] + v_0$$
$$+ \frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(\alpha u(t_j) - \delta_2(v(t_j))^2 - d_2 v(t_j) - \gamma_2 v(t_j)w(t_j) - ev(t_j)\sum_{i=0}^{n+1}\beta_i v(t_i))]$$

$$w(t_{n+1}) = \frac{h}{2}[c_1\gamma_1 u(t_{n+1})w(t_{n+1}) + c_2\gamma_2 v(t_{n+1})w(t_{n+1}) - \delta_3(w(t_{n+1}))^2 - d_3 w(t_{n+1})$$
$$- ew(t_{n+1})\sum_{i=0}^{n+1}\beta_i w(t_i)] + w_0 + \frac{h}{2}\sum_{j=0}^{n}[s_{j,n+1}(c_1\gamma_1 u(t_j)w(t_j) + c_2\gamma_2 v(t_j)w(t_j) - \delta_3(w(t_j))^2$$
$$- d_3 w(t_j) - ew(t_j)\sum_{i=0}^{n+1}\beta_i w(t_i))]$$

We represent the terms $u(t_{n+1})$,$v(t_{n+1})$ and $w(t_{n+1})$ in the right hand side by $u_p(t_{n+1})$,$v_p(t_{n+1})$ and $w_p(t_{n+1})$ and find them by applying the HPM. To establish the efficiency of the method we consider two types of equilibrium: (i) predator extinct equilibrium and (ii) co-exixtence or interior equilibrium.

Case (i): For the parameter values $r = 2; \delta_1 = 0.1; d_1 = 0.2; \beta = 1; \gamma_1 = 0.2; \delta_2 = 0.3; d_2 = 0.4; \gamma_2 = 0.1; c_1 = 0.2; c_2 = 0.4; \delta_3 = 0.3; d_3 = 0.7$; the stable point of equilibrum is the predator extinct point $(3.70681, 2.91111, 0)$.

TABLE 1. Predator extinct equilibrium in absence of toxicant

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
3.0	0.81351	0.731231	0.0249132
6.0	2.45486	2.03983	0.00431764
9.0	3.44267	2.73542	0.00102186
12.0	3.66363	2.88268	0.000270021
15.0	3.70005	2.90666	0.0000728647
18.0	3.70575	2.91041	0.0000197293
21.0	3.70664	2.911	5.34493×10^{-6}
24.0	3.70678	2.91109	1.44814×10^{-6}
25.5	3.7068	2.9111	7.53787×10^{-7}
27.0	3.70681	2.91111	3.92362×10^{-7}
28.5	3.70681	2.91111	2.04232×10^{-7}
30.0	3.70681	2.91111	1.06307×10^{-7}

TABLE 2. Predator extinct analysis when toxicant e=0.01

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
40.	1.19082	1.05868	1.678066×10^{-11}
80.	0.704336	0.642253	1.821661×10^{-22}
120.	0.500502	0.461318	7.713873×10^{-34}
160.	0.388254	0.359995	2.065200×10^{-45}
200.	0.317157	0.295187	4.210017×10^{-57}
240.	0.268079	0.250161	7.161895×10^{-69}
280.	0.232158	0.217058	1.070954×10^{-80}
320.	0.204729	0.191693	1.454012×10^{-92}
360.	0.183097	0.171637	$1.8313328 \times 10^{-104}$
400.	0.165601	0.155381	$2.172207 \times 10^{-116}$
440.	0.151157	0.141938	$2.453110 \times 10^{-128}$
480.	0.139031	0.130637	$2.659328 \times 10^{-140}$
520.	0.128706	0.121002	$2.784843 \times 10^{-152}$
560.	0.119809	0.112691	$2.831077 \times 10^{-164}$
600.	0.112062	0.105449	$2.805064 \times 10^{-176}$
640.	0.105257	0.0990808	$2.717501 \times 10^{-188}$
680.	0.0992305	0.0934384	2.58096×10^{-200}

TABLE 3. Predator extinct analysis when toxicant e=0.1

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
40.	0.171057	0.160508	4.254673×10^{-13}
80.	0.0860894	0.0811353	4.256882×10^{-25}
120.	0.0575198	0.0542899	3.657847×10^{-37}
160.	0.0431878	0.0407929	2.951801×10^{-49}
200.	0.0345733	0.0326707	2.301202×10^{-61}
240.	0.0288239	0.0272458	1.755175×10^{-73}
280.	0.0247141	0.023366	1.318623×10^{-85}
320.	0.02163	0.0204534	9.797412×10^{-98}
360.	0.0192303	0.0181864	$7.218153 \times 10^{-110}$
400.	0.0173098	0.0163718	$5.282503 \times 10^{-122}$
440.	0.0157381	0.0148865	$3.845081 \times 10^{-134}$
480.	0.0144281	0.0136483	$2.786324 \times 10^{-146}$
520.	0.0133194	0.0126002	$2.011541 \times 10^{-158}$
560.	0.0123689	0.0117017	$1.447566 \times 10^{-170}$
600.	0.011545	0.0109227	$1.038847 \times 10^{-182}$
640.	0.0108241	0.010241	$7.437437 \times 10^{-195}$
680.	0.0101879	0.00963938	$5.313481 \times 10^{-207}$

From Table 1, we noticed that when $e=0$, the population of immature prey, mature prey, and predators continues to reach a point of equilibrium at $(3.70681, 2.9111, 0)$. Table 2 and Table 3 shows that as the toxicity level increases, the population of immature prey decreases rapidly than mature prey. Furthermore, we conclude that in the face of toxicity and a decline in prey populations, predator species degrade quicker than other populations. Hence, TBHPM gives an expected numerical analysis at this point of equilibrium.

Case (ii): For parameter values $r=4; \delta_1=0.05; d_1=0.5; \beta=1; \gamma_1=0.5; \delta_2=0.1; d_2=0.6; \gamma_2=0.2; e_1=0.2; m=0.7; e_2=0.4; \delta_3=0.1; d_3=0.7$, interior equilibrium point is $(5.79227, 4.08069, 2.05682)$ and eigenvalues of the jacobian matrix is {-4.5312,-0.304813+0.754334 i,-0.304813-0.754334 i}. So the point of interior equilibrium is stable spiral. Solutions obtained by using the TBHPM are presented in the following tables:

TABLE 4. Interior equilibrium in absence of toxicant

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
4.0	5.85685	3.61676	0.033899
8.0	11.4266	7.70157	2.09451
12.0	4.84086	3.41742	1.94666
16.0	6.13746	4.31021	2.05862
20.0	5.6932	4.01552	2.05999
24.0	5.82209	4.09987	2.05425
28.0	5.78359	4.07522	2.058
32.0	5.79477	4.08223	2.05635
36.0	5.79156	4.08026	2.057
40.0	5.79247	4.08081	2.05676
44.0	5.79222	4.08066	2.05684
48.0	5.79228	4.0807	2.05682
52.0	5.79227	4.08069	2.05683
54.0	5.79227	4.08069	2.05682
56.0	5.79227	4.08069	2.05682
58.0	5.79227	4.08069	2.05682

TABLE 5. coexistence analysis when toxicant e=0.01

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
40.	2.69093	1.81111	0.00465775
80.	1.44546	0.981843	5.057885×10^{-10}
120.	0.987473	0.673067	5.366835×10^{-19}
160.	0.749893	0.512047	7.464032×10^{-29}
200.	0.604467	0.413199	3.296635×10^{-39}
240.	0.506285	0.346341	6.960366×10^{-50}
280.	0.435542	0.298106	8.780138×10^{-61}
320.	0.382145	0.261665	7.574234×10^{-72}
360.	0.340411	0.233162	4.879575×10^{-83}
400.	0.306895	0.210259	2.494298×10^{-94}
440.	0.279388	0.191453	$1.0566384 \times 10^{-105}$
480.	0.256406	0.175735	$3.831182 \times 10^{-117}$
520.	0.236917	0.162402	$1.218587 \times 10^{-128}$
560.	0.220182	0.150949	$3.4660501 \times 10^{-140}$
600.	0.205655	0.141006	$8.951514 \times 10^{-152}$
640.	0.192927	0.132291	$2.125196 \times 10^{-163}$
680.	0.181682	0.124591	$4.685264 \times 10^{-175}$

TABLE 6. coexitence analysis when toxicant e=0.1

Time	Immatured prey	Matured prey	Predator
0	0.2	0.1	0.2
40.	0.313606	0.215208	2.093739×10^{-11}
80.	0.156328	0.107315	5.554673×10^{-23}
120.	0.104113	0.0714793	8.421791×10^{-35}
160.	0.0780455	0.0535855	1.016076×10^{-46}
200.	0.0624175	0.0428569	1.081780×10^{-58}
240.	0.0520041	0.0357077	1.064197×10^{-70}
280.	0.0445685	0.0306027	9.913427×10^{-83}
320.	0.0389932	0.0267748	8.873552×10^{-95}
360.	0.0346577	0.023798	7.7044105×10^{-107}
400.	0.0311899	0.0214169	6.530486×10^{-119}
440.	0.0283529	0.019469	5.428875×10^{-131}
480.	0.0259889	0.0178459	4.4412406×10^{-143}
520.	0.0239889	0.0164725	3.584647×10^{-155}
560.	0.0222746	0.0152955	2.860257×10^{-167}
600.	0.0207891	0.0142754	2.259793×10^{-179}
640.	0.0194893	0.0133829	1.770078×10^{-191}
680.	0.0183424	0.0125954	1.376036×10^{-203}

Table 4 displays the results when $e = 0$, the population of immature prey, mature prey, and predators begins to achieve an interior equilibrium point at $(5.79227, 4.08069, 2.05682)$, as we saw earlier. Table 5 and Table 6 demonstrate that as the degree of toxicity rises, Immature prey populations decline rather than mature prey populations. Furthermore, we believe that predator species decline more rapidly than other populations in the face of toxicity and diminishing prey populations.

4. CONCLUSION

In this work, we have investigated two predator-prey models under the influence of environmental pollutants. Taking convergence towards the stable point of equilibrium point as a validity factor, we have shown the efficiency of TBHPM in solving the population model. We have solved Holling Type-I predator-prey model with toxic term and analyzed results graphically. Whereas in section 3.2 for the SSPP model results are presented in tabular form. Graphical analysis of the solution of the system 7 gives expected results. From the figures, it is seen that even though both predator and prey population are exposed to the same amount of toxicant, predator population decay faster than prey population. This indicates that the predator population faces death due to natural death, toxic effect, and lack of food. In the case of model 8, in absence of toxicity, we observe that the solution converges to predator extinct equilibrium and interior equilibria. Whereas, in presence of toxicity in the environment, immatured prey are affected by its effect more than the matured prey. Also, all the table show it clearly that in presence of toxicity predator population decays faster than other population. Moreover, the toxicity effect is directly proportional to the growth of the predator and prey population. These observations strongly support the proposed TBHPM.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] R.M. May, Stability and complexity in model ecosystems, 2nd edn. Princeton University Press, Princeton, (1974).

[2] R.H. Smith, R. Mead, Age structure and stability in models of predator-prey systems, Theor. Popul. Biol. 6 (1974), 308-322.

[3] M.E. Gurtin, D.S. Levine, On predator-prey interactions with predation dependent on the age of prey, Math. Biosci. 47 (1979), 207-219.

[4] D.S. Levine, On the stability of a predator-prey system with egg eating predators, Math. Biosci. 56 (1981), 27-56.

[5] R.W. Thompson, D. Dibiasio, M. Charles, Predator-prey interaction: egg eating predators, Math. Biosci. 60 (1982), 109-120.

[6] M. Saleem, Egg eating age structured predators in interaction with age structured prey, Math. Biosci 70 (1984), 91-104.

[7] Z. Ma, S.B. Jun, T.G. Hallam, The Threshold of Survival for System in a Fluctuating Environment, Bull. Math. Biol. 51 (1989), 311-323.

[8] Z. Ma, T.G. Hallam, Effects of parameter fluctuations on community survival, Math. Biosci. 86 (1987), 35-49.

[9] L. Huaping, Z. Ma, The threshold of survival for system of two species in a polluted environment, J. Math. Biol. 30 (1990), 49-61.

[10] Z. Li, Z. Shuai, K. Wang, Persistence and extinction of single population in a polluted environment, Electron. J. Differ. Equ. 2004 (2004), 18.

[11] H. Liu, Z. Ma, The threshold of survival for system of two species in a polluted environment, J. Math. Biol. 30 (1990), 49–61.

[12] J. Chattopadhyay, O. Arino, A predator-prey model with disease in the prey, Nonlinear Anal., Theory Meth. Appl. 36 (1999), 747–766.

[13] M.R. Arkoosh, E. Casillas, E. Clemons, Anna N. Kagley, R. Olson, P. Reno, John. Stein, Effect of pollution on fish diseases: potential impacts on Salmonid populations, J. Aquat. Anim. Health 10 (1998), 182–190.

[14] R.D. Small, Population growth in a closed model, In: Mathematical Modelling: Classroom Notes in Applied Mathematics, SIAM, Philadelphia, (1989).

[15] F.M. Scudo, Vito Volterra and theoretical ecology, Theor. Popul. Biol. 2 (1971), 1–23.

[16] F.J. Solis, R.A. Ku-Carrillo, Birth rate effects on an age-structured predator-prey model with cannibalism in the prey, Abstr. Appl. Anal. 2015 (2015), 241312.

[17] K.G. TeBeest, Numerical and analytical solutions of Volterra's population method, SIAM Rev. 39 (1997), 484–493.

[18] A.-M. Wazwaz, Analytical approximations and Pade approximants for Volterra's population model, Appl. Math. Comput. 100 (1999), 13–25.

[19] K. Al-Khaled, Numerical approximations for population growth models, Appl. Math. Comput. 160 (2005), 865–873.

[20] S.T. Mohyud-Din, A. Yıldırım, Y. Gülkanat, Analytical solution of Volterra's population model, J. King Saud Univ.-Sci. 22 (2010), 247–250.

[21] G. Adomian, Solving frontier problems of physics: the decomposition method, Kluwer Academic Publishers, Dordecht, 1994.

[22] A.H. Nayfeh, Introduction to perturbation technique, John Wiley and Sons, New York, 1981.

[23] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29 (2002), 3-22.

[24] W. Wang, L. Chen, A predator-prey system with stage structure for predator, Computers Math. Appl. 33(1997), 83–91.

[25] X. Zhang, L. Chen, A.U. Neumann, The stage-structured predator-prey model and optimal harvesting policy, Math. Biosci. 168 (2000), 201–210.

[26] J. Cui, L. Chen, W. Wang, The effect of dispersal on population growth with stage-structure, Computers Math. Appl. 39 (2000), 91–102.

[27] J. Cui, Y. Takeuchi, A predator-prey system with a stage structure for the prey, Math. Computer Model. 44 (2006), 1126–1132.

[28] S. Liu, E. Beretta, A stage-structured predator-prey model of Beddington-DeAngelis type, SIAM J. Appl. Math. 66 (2006), 1101–1129.

[29] T. Zhang, X. Meng, Y. Song, T. Zhang, A stage-structured predator-prey si model with disease in the prey and impulsive effects, Math. model Anal. 18 (2013), 505-528.

[30] S.A. Gourley, Y. Kuang, A stage structured predator-prey model and its dependence on maturation delay and death rate, J. Math. Biol. 49 (2004), 188–200.

[31] H. You, R. Yuan, A stage-structured predator-prey model with two delays due to juvenile maturation, Acta Math. Appl. Sin. Engl. Ser. (2011), https://doi.org/10.1007/s10255-011-0099-1.

[32] S.G. Mortoja, P. Panja, S.K. Mondal, Dynamics of a predator-prey model with stage-structure on both species and anti-predator behavior, Inform. Med. Unlocked. 10 (2018), 50–57.

[33] S. Xu, Dynamics of a general prey–predator model with prey-stage structure and diffusive effects, Computers Math. Appl. 68 (2014), 405–423.

[34] R.K. Naji, S.J. Majeed, The dynamical analysis of a prey-predator model with a refuge-stage structure prey population, Int. J. Differ. Equ. 2016 (2016), 2010464.