AN INTERACTIVE APPROACH TO MULTI-OBJECTIVE MEAN SEMI-VARIANCE MODEL WITH TRANSACTION COST FOR PORTFOLIO OPTIMIZATION

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Abstract: Portfolio selection may be considered as a complicated decision making under uncertain conditions. Regardless of an unknown future, a fund manager has to make critical financial choices based on the investors' preferences towards risk and return and perceptions. Since the pioneer work of Markowitz, a lot of studies have been published considering his Mean-Variance (MV) model as a basis. In our proposed model we had considered Semivariance as the risk measure in our proposed portfolio optimization model. We also extend our model to include the effect of transaction cost in portfolio optimization and then the concept of entropy is incorporated as an objective function in our model to obtain a well diversified portfolio within optimal asset allocation.

An interactive approach is used to solve the model. A numerical example is used to illustrate that the model can be efficiently used in practice. Finally the result obtained in interactive method has been compared with the result obtained by Fuzzy Non-Linear Programming (FNLP) and Fuzzy Additive Goal Programming (FAGP) techniques.

Keywords: portfolio; entropy; transaction cost.

2010 AMS Subject Classification: 91B05, 91B86, 91G10.

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1. INTRODUCTION

In financial management one of the most important topics is portfolio management. Basically portfolio management gives the direction to obtain a portfolio that will satisfy an investor concerning his risk and return. The main aim of portfolio management is to choose the best amalgamation of assets that will provide the highest expected return while maintaining acceptable level of risk simultaneously.

In portfolio optimization an investor always want to achieve maximum portfolio return, while ensuring an acceptable level of risk at the same time. Investor will have to manage the risk-return trade off for their investments since risk will reimburse the return. That is why single optimization portfolio is not appropriate. So while determining optimal portfolio one will have to consider the risk-return preferences of the investor.

The pioneer model in the field of portfolio management is the Mean-Variance (MV) model, proposed by Markowitz[1] in 1952. In the basic mean-variance model of portfolio framework, Markowitz trade-off between expected return and risk of the portfolio , each of them represented by the mean of historical performances i.e. mean of return of an asset and the dispersion of the return as risk respectively.

Over the last few decades the pioneer model proposed by Markowitz, mathematical programming techniques have become necessary tools to support financial decision making process and applied a lot in real life situation. There are several mathematical tools used in general to find the best solution in portfolio optimization. Such as Forecasting, Simulation, Statistical Model, Mathematical programming models. Among these models mathematical programming is the best option to the decision maker to find the optimal solution.

As far as we know from the existing literature that mathematical model for portfolio involving transaction cost usually trying to create a modified portfolio from cash i.e.in general trying to move to a new portfolio from a current portfolio. Most of the model introduces at least an extra binary variable and also bounds on the portfolio. Hence all these components for transaction cost increases the complexity of the problem. Now let us have a look on the literature

available on transaction cost. Angelelli et al [2] had considered a mixed integer linear programming model, which includes transaction cost and cardinality constraints with CVaR and MAD model. Chen and Cai [3] in his generalized MV Markowitz model had incorporated transaction cost. As per their assumption transaction cost is a V-shaped function was known at the beginning of the period and paid at the end of the period. Baule [4] had considered transaction cost as a non-convex function in his transaction cost model. Adcock and Meade [5] in their mixed quadratic portfolio optimization model formulation had incorporated variable transaction cost using a weighting factor. Also few more journals are there including the concept of transaction cost in portfolio optimization.

Entropy is an established concept of Information theory. The main aim of entropy is to quantify conveyed by a distribution including all higher order moments. Hence it is not surprising at all to notice that the concept of Shanon entropy has been well accepted in the field of finance also. However when entropy is used in construction of optimal portfolios the literature is limited so far. The effectiveness of entropy in optimal portfolio selection are illustrated in few well known books such as Fang et all [6], Kapur [7], Kapur and Kesavan [8]. Jana and Roy [9] had used entropy as a diversification measure in their multi-objective portfolio optimization model.

Also several exact method based techniques had been applied to solve the portfolio optimization models, such as integer programming method [10], goal programming method [11], lexiographic goal programming approach [12] etc. Some meta-heuristics based approaches are also used such as simulated annealing [13], genetic algorithm [14], particle swarm optimization [15], ant colony optimization [16]. In [17] authors had discussed about some advanced optimization techniques.

But in practice in order to take best portfolio decisions, decision maker will have to take help of some vaguely define financial parameters such as *return more than 20%, risk lower than 10%* etc. Under such vague expression it is difficult to construct satisfactory portfolios using crisp or interval numbers. In such a situation decision maker has to take help of fuzzy set theory in order to formulate the models of portfolio selection. Not only the uncertainty and the

vagueness are controlled by fuzzy set theory, but also help decision maker to take flexible decision by considering the investors' preferences.

The concept of fuzzy decision theory was developed by Bellmann and Zadeh [18] based on the idea of fuzzy sets introduced by Zadeh in 1965 [19]. Also some authors [20,21,22,23,24] had already used the fuzzy frame work to select the efficient portfolio using mean-variance model. Wang et all [25] had used fuzzy decision theory in single objective portfolio optimization model.

In [26] authors had used a fuzzy programming technique to solve his transportation model. Also M.Evangeline Jebaseeli and D.Paul Dhayabaran had presented a comperative study on fully time-cost trade off.

In this paper we had proposed a multi-objective portfolio optimization model. As a risk measure we had considered semi variance in order to overcome some drawback of using variance as a risk measure. The costs associated with trading equities were not included in the classical portfolio optimization model of Markowitz. But in current context it is important to integrate the transaction cost in a new portfolio as well as in revising an existing portfolio. So we had incorporated the concept of both fixed and variable transaction cost in our model. Again sometimes traditional MV approach make the portfolio highly concentrated on few assets. Also this method gives negative values for some portfolio weights which is termed as short sell, but in reality most of the investors are not allowed to sell short. Since maximizing entropy function (Shanon entropy) subject to the moment constraint implies estimating the probability which is closest to the uniform. So entropy may be considered as a well accepted measure for diversification. Therefore in order to achieve a well diversified portfolio we had considered maximization of entropy function in our model.

An interactive fuzzy multi-objective decision making method is used in this paper to obtain an optimal portfolio. Numerical examples are given to illustrate the problems and then the result will be compared.

1.1 Semi variance:

Though there are several limitations, till researchers prefers variance as a popular risk measures. The major drawback of using variance as a risk measure is that it penalizes extreme downside and upside deviations from the expected return. Therefore in the case of asymmetric probability distribution of asset return, the variance will be a less appropriate measure of portfolio risk. This is because of the fact that in terms of sacrificing higher expected return the obtained portfolio may have a potential danger. So it may be preferred to replace variance with a downside risk measure. This measure of risk only considers the negative deviations from a reference return level. Markowitz had introduced one of the best known downside risk measure Semi variance. Semi variance does not take values beyond the gain as risk; it is main advantage of using variance over Semi variance. If the investors are concerned about portfolio underperformance rather than over performance then semi variance will be a more appropriate measure of risk.

Semi variance is basically the expected value of the squared negative deviations of possible outcomes from the expected return. So the Semi variance which is a measure of portfolio risk denoted by $s(x_1, x_2, \ldots \ldots \ldots, x_n)$ is defined as: $s(x_1, x_2, \ldots \ldots \ldots, x_n) = E[[\sum_{i=1}^{n} R_i x_i - E[\sum_{i=1}^{n} R_i x_i]]^-]^2$

Where

$$\left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right]^- = \begin{cases} \left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right], if \left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right] < 0, \\ 0 \qquad\qquad\qquad , if \left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right] \geq 0. \end{cases}$$

So using semi variance if we want to select portfolio we need not to compute variance covariance matrix but we just need the joint distribution. Therefore if the dispersion of the portfolio return is below the expected return then only this risk measure will try to minimize the dispersion of portfolio return.

Since we shall approximate expected value of the random variable by the average derived from the past data, so we shall use $r_i = E[R_i] = \frac{\sum_{t=1}^{T} r_{it}}{T}$, the semi variance will be approximated as $s(x_1, x_2, \dots \dots \dots, x_n) = E[[\sum_{i=1}^{n} R_i x_i - E[\sum_{i=1}^{n} R_i x_i]]^{-}]^2$

$$= \frac{1}{T} \sum_{i=1}^{T} \{[\sum_{i=1}^{n} (r_{it} - r_i) x_i]^{-}\}^2,$$

Where,

$$\left[\sum_{i=1}^{n} (r_{it} - r_i) x_i\right]^{-} = \begin{cases} \sum_{i=1}^{n} (r_{it} - r_i) x_i, & if \sum_{i=1}^{n} (r_{it} - r_i) x_i < 0, \\ 0 & , if \sum_{i=1}^{n} (r_{it} - r_i) x_i \geq 0, \end{cases} \quad t = 1, 2, \dots \dots \dots, T.$$

Therefore in our model to obtain optimal portfolio we had used semi variance as the risk measure.

1.2 Entropy Theory and entropy model

In 1948 to find out the solution for problem of quantitative measurement information Claude Elwood Shanon had established the concept of Information entropy. Claude Elwood Shannon, the father of information theory had first explained the relationship between information redundancy and probability in mathematical terms.

A discrete probability distribution of a random variable $p = (p_1, p_2, \dots \dots \dots, p_N)^T$ taking N values gives us a measure of uncertainty concerning those random variables. This measure of disorder is termed as Entropy in the literature of information theory.

A portfolio allocation $\pi = (w_1, w_2, \dots \dots \dots, w_N)^T$ of N risky assets having the properties $w_i \geq 0$ for $i = 1, 2, \dots \dots \dots, N$ and $\sum_{i=1}^{N} w_i = 1$ has the form of a proper probability distribution.

In our paper as a measure of portfolio diversification we will use Shanon Entropy measure defined as $SE(\pi) = -\sum_{i=1}^{N} w_i \ln(w_i)$. When $w_i = \frac{1}{N}$ for all i then $SE(\pi)$ has its maximum value $\ln N$. And when $w_i = 1$ for one i and $=0$ for the rest, then in these extreme cases $SE(\pi) = 0$. So we can consider entropy as a measure of portfolio diversification since from above it is clear that entropy gives a good measure of disorder in a system or expected information in a probability distribution. After obtaining portfolio using different selection procedures, using the Shanon Entropy measure portfolios are generally evaluated in terms of their degree of

diversification. Therefore in our paper in order to obtain maximum diversified portfolio we had considered maximization of entropy as an objective function.

1.3 Transaction cost in Portfolio optimization:

If investors want to buy or sell an asset, each time an expense is incurred. The costs associated with trading equities were not included in the classical portfolio optimization model of Markowitz. But in current context it is important to integrate the transaction cost in a new portfolio as well as in revising an existing portfolio. A portfolio manager shall carefully maintain trading and its associated cost since transaction cost should be desirably low. We may classify Transaction cost into two types: Fixed and Variable transaction cost.

Fixed Cost: Fixed cost is the cost paid on all transaction but does not depend upon the volume of the transaction. Transfer fees and brokerage commission are costs included in this category.

Variable Cost: This type of cost depends upon the volume of the transaction. While buying or selling any cost variable costs are proportional to the volume traded. Execution costs and opportunity costs are part of this cost. Again execution cost can be dividing into market impact and market timing cost. Market impact is the result of a trade plus the market-maker's spread that is the movement in the price of an asset. Market timing cost is the movement in the prices of an asset at the time of transaction.

2. MATHEMATICAL FORMULATION OF THE PROPOSED PORTFOLIO OPTIMIZATION MODEL

We will now discuss about the proposed portfolio optimization model. The decision variables and the notations of the proposed model are discussed below.

2.1 Notation:

In this section the notation used for the formulation of the model is presented.

N= the number of assets which are available for investment.

μ_i= the expected return for i-th asset (i=1,2,………,N).

R=the desired level of expected return.

P_i = The current price of per unit of i-th asset (i=1,2,………,N).

X_i = The number of units of current portfolio holding for i-th asset (i=1,2,………,N).

f_i^b = The fixed transaction cost to be paid while buying i-th asset (i=1,2,………,N).

f_i^s = The fixed transaction cost to be paid while selling i-th asset (i=1,2,………,N).

c_i^b = The variable transaction cost to be paid while buying i-th asset (i=1,2,………,N).

c_i^s = The variable transaction cost to be paid while selling i-th asset (i=1,2,………,N).

D= the upper limit for the transaction cost.

L_i^b= The number of unit of assets to buy in minimum while buying i-th asset (i=1,2,………,N).

L_i^s= The number of unit of assets to sell in minimum while selling i-th asset (i=1,2,………,N).

U_i^b= The number of unit of assets to buy in maximum while buying i-th asset (i=1,2,………,N).

$[U_i^b \geq L_i^b]$

U_i^s= The number of unit of assets to sell in maximum while selling i-th asset (i=1,2,………,N).

$[U_i^s \geq L_i^s]$

2.2 Decision Variables

x_i = the number of units of i-th asset in the new portfolio.

e_i = the proportion of i-th asset in the new portfolio.

G_i = The total sum of fixed and variable cost i.e. total transaction cost acquired in trading of i-th asset.

y_i^s = The number of units of i-th asset that have been sold.

y_i^b = The number of units of i-th asset that have been bought.

$\beta_i^s = \begin{cases} 1 \text{ ; if we sell i} - \text{th asset} \\ \quad 0 \text{ ; otherwise} \end{cases}$.

$\beta_i^b = \begin{cases} 1 \text{ ; if we sell i} - \text{th asset} \\ \quad 0 \text{ ; otherwise} \end{cases}$.

2.3 The Proposed Model:

$$Minimize \ S_v(P) = \frac{1}{T}\sum_{i=1}^{T} p_t{}^2 \qquad (1.1)$$

$$Maximize \; E_n(P) = -\sum_{i=1}^{N} e_i \ln(e_i) \tag{1.2}$$

Subject to

$$p_t = \left[\sum_{i=1}^{N}(r_{it} - r_i)x_i\right]^{-} = \begin{cases} \sum_{i=1}^{N}(r_{it} - r_i)x_i, & if \; \sum_{i=1}^{N}(r_{it} - r_i)x_i < 0, \\ 0, & if \; \sum_{i=1}^{N}(r_{it} - r_i)x_i \geq 0, \end{cases} \quad t = 1,2,\ldots\ldots\ldots,T$$

$$p_t \geq -\sum_{i=1}^{N}(r_{it} - r_i)e_i \; ; \qquad t = 1,2,\ldots\ldots\ldots,T \tag{1.3}$$

$$\frac{\sum_{i=1}^{N}\mu_i P_i x_i}{\sum_{k=1}^{N}P_k x_k} = R \; ; \tag{1.4}$$

$$L_i^s \beta_i^s \leq y_i^s \leq U_i^s \beta_i^s \; ; \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.5}$$

$$L_i^b \beta_i^b \leq y_i^b \leq U_i^b \beta_i^b \; ; \qquad i = 1,2,\ldots\ldots\ldots, \tag{1.6}$$

$$\beta_i^s + \beta_i^b \leq 1 \; ; \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.7}$$

$$x_i = X_i + y_i^b - y_i^s \; ; \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.8}$$

$$G_i = c_i^s y_i^s + c_i^b y_i^b + f_i^s \beta_i^s + f_i^b \beta_i^b \; ; \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.9}$$

$$C_t(P) = \sum_{i=1}^{N} G_i \leq D \tag{1.10}$$

$$\sum_{i=1}^{N} P_i x_i = \sum_{i=1}^{N} P_i X_i - \sum_{i=1}^{N} G_i \tag{1.11}$$

$$e_i = \frac{P_i x_i}{\sum_{k=1}^{N} P_k x_k} \tag{1.12}$$

$$E_t(P) = \sum_{i=1}^{N} e_i < 1; \tag{1.13}$$

$$p_t \geq 0; t = 1,2,\ldots\ldots\ldots,T \tag{1.14}$$

$$e_i, x_i, y_i^s, y_i^b, G_i \geq 0; \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.15}$$

$$\beta_i^s, \beta_i^b \in [0,1], \qquad i = 1,2,\ldots\ldots\ldots,N \tag{1.16}$$

2.4 Descriptions of the objectives and the constraints

In this section the constraints of our proposed portfolio optimization model are given below. We had given a brief explanation of each of the constraints below.

We have a bid-ask spread also known as bid-offer spread if the price at which an asset i ($i = 1,2, ..., N$)can be bought differs from its present market price P_i .i.e the difference between the price for an immediate sale and immediate purchase. This situation is captured in the variable transaction cost c_i^b andc_i^s, which is example of a market timing cost.

In order to discuss about the constraint (1.3) we will discuss about it some different manner. If possible let for any given value of 't' the right hand side of the constraint is either negative or zero, .i.e. if $\sum_{i=1}^{n}(r_{it} - r_i)x_i \geq 0$, the constraint (1.3) and the constraint (1.14) leaves the variable p_t free to take any non-negative value. So as the variable p_t^2 appears with coefficient +1 in the objective function it will take value 0 in any optimal solution, since we are minimizing the sum of p_t^2 . If on the contrary, we consider the right hand side of (1.3) as positive, .i.e. $\sum_{i=1}^{n}(r_{it} - r_i)x_i < 0$, p_t will be equal to the right hand side of (1.3) in any optimal solution.

The constraint (1.4) is the return equation. Here the invested amount in i -th asset ($i = 1,2, ..., N$) is $P_i x_i$ for an expected return of μ_i. The numerator $\sum_{i=1}^{N} \mu_i P_i x_i$indicates the total interest income, while the denominator $\sum_{k=1}^{N} P_k x_k$ indicates the total investment made. The equation (1.3) is related to the expected return from the chosen portfolio under the assumption that this expected return will continue over time. R is the desired level of the expected return.

The constraints (1.5) and (1.6) give us idea about the appropriate limits to the number of units bought or sold. Here $\beta_i^s = 0$ indicates that none of asset i is sold while $\beta_i^s = 1$ indicates that the number of units sold would be between the upper and lower bound of selling, i.e. $L_i^s \leq y_i^s \leq U_i^s$. Similarly $\beta_i^b = 0$ indicates that none of asset i is bought while $\beta_i^b = 1$ indicates that the number of units sold would be between the upper and lower bound of selling, i.e. $L_i^b \leq y_i^b \leq U_i^b$.

The constraint (1.7) makes sure that it is not possible to buy and sell an asset i ($i = 1,2, ..., N$) simultaneously. When the binary decision variables are both zero for buying and selling of i-th asset i.e. $\beta_i^s = \beta_i^b = 0$ then this constraint does not allow us to trade the i-th asset.

The constraint (1.8) is the balance constraint on the number of shares in the portfolio. This constraint states that for of i-th asset sum of the number of units in old portfolio and number of units bought minus number of units sold be equal to number of units in the new portfolio.

In the constraint (1.9) our focus is on budget management. For a fund manager budget is possibly the most important tool. In order to make a game edge through difficult financial plans for operating a portfolio this budget constraint puts in perspective an investor's ideas about risk and the resources available. This budget related constraint limits the degree of total market exposure agreed by an investor, using the idea that the available wealth be greater than or equal to the total value of the portfolio. A fund manager can properly use the budget to achieve their goals, and also can create more profitable portfolios that may give them

The constraints related to budget is $G_i = c_i^s y_i^s + c_i^b y_i^b + f_i^s \beta_i^s + f_i^b \beta_i^b$; $\quad i = 1,2, \dots \dots \dots, N.$ from the equation it is clear that for each of the assets the sum of fixed transaction cost and variable transaction cost incurred in selling and buying an asset gives the total transaction cost.

The constraint (1.10) is limit constraint for transaction cost. It explains that sum of the transaction cost (variable cost and fixed cost) is less than or equal to pre specified transaction cost limit.

The constraint (1.11) is the monetary balance constraint. It guarantees that monetary value of the new portfolio equals to the monetary value of the current portfolio less the total transaction cost for all the assets.

In equation (1.12) it had been defined that the invested proportion in i-th asset is $e_i = \frac{P_i x_i}{\sum_{k=1}^N P_k x_k}$.

Here the term w_i is non-linear because of the effect of transaction cost the denominator is not constant.

From the Markowitz budget constraint the sum of the invested proportion is in-generally 1. But in transaction cost model the invested proportion w_i in real situation underestimates the actual proportion invested in i-th asset in the newly constructed portfolio. That's why in constraint (1.13) we have considered $\sum_{i=1}^N e_i < 1$.

The rest constraints are all non-negativity constraints.

Now we will discuss about the objectives. We have considered semi variance as an appropriate measure of risk in this paper. So, in the first objective minimization of risk has been considered i.e. minimization of semi-variance. And as second objective we have considered maximization of entropy in order to get a well diversified portfolio.

3. INTERACTIVE FUZZY MULTI-OBJECTIVE DECISION MAKING METHOD

Interactive fuzzy multi-objective decision making is one of the systematic and efficient methods of Multi-objective Decision Making (MODM) techniques. Either the decision maker (DM) will be satisfied with the solutions obtained in the MODM problem; otherwise he may want to change the actual model when he is not satisfied with the particular solutions or resources used. In such cases DM may proceed with the interactive process to design a high productivity system.

Definition : A solution $x \in X$ is said to be *fuzzy efficient* if there exist no $x^* \in X$ such that :

$Z_p(x^*) \geq Z_p(x)$ for p=1,2,………,n and

$f_i(x^*) \geq f_i(x)$ for i= 1,2,……….,m.

And also

$Z_p(x^*) > Z_p(x)$ for at least one $p \in \{1,2, … … … , n\}$ and

$f_i(x^*) > f_i(x)$ for at least one $i \in \{1,2, … … … , m\}$

3.1 Steps of solution of an optimization model using IFMODM

The solution procedure using Interactive fuzzy multi-objective decision making (IFMODM) approaches is summarized below in few steps.

Step 1: Consider a fuzzy MODM problem

$$Maximize\ (Z_1(x), Z_2(x), … … , Z_n(x))$$

Subject to $g_i(x) \leq \widetilde{b_i};\ i = 1,2, … … … , m, x \geq 0.$

Step 2: In order to construct the membership functions the best upper bound and worst lower bound will determined as follows

$$Z_p^- = Z_p(x_p^-) = max_{x\in X^-} Z_p(x), \forall p$$

where, $X^- = \{x: g_i(x) \leq \widetilde{b_i}, \forall i, x \geq 0\};$ and

$$Z_p^+ = Z_p(x_p^+) = max_{x \in X^+} Z_p(x), \forall p$$

where, $X^+ = \{x : g_i(x) \le \widetilde{b_i} + \delta_i, \forall i, x \ge 0\}$;

Here δ_i is the maximum tolerance corresponding to the fuzzy resource $\widetilde{b_i}$.

The pay-off matrix of efficient extreme solutions will be as follow

	Z_1	Z_2		Z_n
x_1^+	$Z_1^*(x_1^+)$	$Z_2(x_1^+)$	……….	$Z_n(x_1^+)$
x_2^+	$Z_1(x_2^+)$	$Z_2^*(x_2^+)$	……….	$Z_n(x_2^+)$
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
x_n^+	$Z_1(x_n^+)$	$Z_2(x_n^+)$	……….	$Z_n^*(x_n^+)$
x_1^-	$Z_1^{**}(x_1^-)$	$Z_2(x_1^-)$	……….	$Z_n(x_1^-)$
x_2^-	$Z_1(x_2^-)$	$Z_2^{**}(x_2^-)$	……….	$Z_n(x_2^-)$
⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮
x_n^-	$Z_1(x_n^-)$	$Z_2(x_n^-)$	……….	$Z_n^{**}(x_n^-)$

Step 3: The membership function of the objective functions and constraints are constructed as follow:

$$f_p^{w_p}\left(Z_p(x)\right) = \begin{cases} 0 & if\ Z_p(x) < Z_{p,min}; \\ w_p\left(\dfrac{Z_p(x) - Z_{p,min}}{Z_p^+ - Z_{p,min}}\right) & if\ Z_{p,min} \le Z_p(x) \le Z_p^+;\ \forall\, p = 1,2,\dots\dots,n \\ w_p & if\ Z_p^+ < Z_p(x); \end{cases}$$

and

$$f_i(g_i(x)) = \begin{cases} 0 & if\ b_i + \delta_i < g_i(x); \\ 1 - \left(\dfrac{g_i(x) - b_i}{\delta_i}\right) & if\ b_i \le g_i(x) \le b_i + \delta_i;\ \forall i = 1,2,\dots\dots,m \\ 1 & if\ g_i(x) < b_i \end{cases}$$

where $Z_{p,min} = min_p[Z_p(x_p^+), Z_p(x_p^-)]$ $(p = 1,2, \dots \dots \dots, n)$ yields the pessimistic values and the optimistic values is given by the diagonal in the upper half of the payoff matrix, which is the maximum achievable values of the corresponding objectives.

Step 4: Now fuzzy decision as per min-operator introduced by Bellman and Zadeh is defined as

$$fuzzy\ decisions\ (D) = fuzzy\ objective\ goals\ (G) \cap fuzzy\ constraints\ (C)$$

Hence the corresponding membership function is characterized by

$$f_D(x) = \min(f_G(x), f_C(x))$$

Now we introduce the variable λ as follow

$$\lambda = \min[f_1^{w_1}(Z_1(x)), \dots \dots, f_n^{w_n}(Z_n(x))]$$
$$= \min[f_1^w(Z_1(x)), \dots \dots, f_n^w(Z_n(x))]$$
$$= \min[wf_1(Z_1(x)), \dots \dots, wf_n(Z_n(x))]$$

Where, $w = \min(w_1, w_2, \dots \dots, w_n) = min_p(w_p), for\ p = 1,2, \dots \dots, n.$

Step 5: Now the following $\lambda - Fuzzy\ minimization\ problem$ will be solved to get a preffered solution.

$$Maximize\ \lambda$$

$$subject\ to\ , wf_p\left(Z_p(x)\right) \geq \lambda\ ; \forall\ p = 1,2, \dots \dots, n;$$

$$f_i\left(g_i(x)\right) \geq \lambda\ ; \forall\ i = 1,2, \dots \dots, m;$$

$$\lambda \in [0, w]\ and\ x \geq 0, w \in (0,1]$$

Now for the objective $Z_p(x)$, using positive weights W_p $(p = 1,2, \dots \dots, n)$ we have

$$Maximize\ \lambda$$

$$subject\ to\ , W_p wf_p\left(Z_p(x)\right) \geq \lambda\ ; \forall\ p = 1,2, \dots \dots, n;$$

$$f_i\left(g_i(x)\right) \geq \lambda\ ; \forall\ i = 1,2, \dots \dots, m;$$

$$\lambda \in [0, w]\ and\ x \geq 0, w \in (0,1], \sum_{p=1}^{n} W_p = 1;$$

Here the decision maker's preference regarding the relative importance of each objective goal is reflected by the positive weights W_p $(p = 1,2, \dots \dots, n)$.

Steps 6: If any of the fuzzy efficient extreme solution in the pay-off matrix or the preferred solution obtained from the $\lambda - Fuzzy\ minimization\ problem$ is satisfactory for the DM, then the process will be successfully terminated else need to go to the next step.

Step 7: Now to get preferred result of the DM, need to modify the membership function of the objective and the constraints, assuming the linear membership function.

3.2 Interactive method to solve the Multi-Objective Fuzzy Portfolio Optimization model

To solve the proposed model defined above using the already mentioned interactive method, first the table of extreme solutions is formulated as follow:

	$S_v(P)$	$E_n(P)$
P_1^+	$S_v^*(P_1^+)$	$E_n(P_1^+)$
P_2^+	$S_v(P_2^+)$	$E_n^*(P_2^+)$
P_1^-	$S_v(P_1^-)$	$E_n(P_1^-)$
P_2^-	$S_v(P_2^-)$	$E_n(P_2^-)$

Now the identified optimistic and pessimistic values are given by $S_v^*(P_1^+)$, $E_n^*(P_2^+)$ and S_v^{min}, E_n^{min} respectively, where $S_v^{min} = min[S_v(P_1^+), S_v(P_2^+), S_v(P_1^-), S_v(P_2^-)]$

and $E_n^{min} = min[E_n(P_1^+),\ E_n(P_2^+),\ E_n(P_1^-), E_n(P_2^-)]$.

Now the linear membership functions for the objectives $S_v(P)$, $E_n(P)$ and constraints $E_t(P)$, $C_t(P)$ are defined as follows:

$$f_{S_v}^{w_1}(S_v(P)) = \begin{cases} 0 & if\ S_v(P) < S_v^{min}; \\ w_1\left(\dfrac{S_v(P) - S_v^{min}}{S_v^*(P_1^+) - S_v^{min}}\right) & if\ \ S_v^{min} \leq \ \ S_v(P) \leq S_v^*(P_1^+); \\ w_1 & if\ S_v^*(P_1^+) < S_v(P); \end{cases}$$

$$f_{E_n}^{w_2}(E_n(P)) = \begin{cases} 0 & if\ E_n(P) < E_n^{min}; \\ w_2\left(\dfrac{E_n(P) - E_n^{min}}{E_n^*(P_2^+) - E_n^{min}}\right) & if\ \ E_n^{min} \leq \ \ E_n(P) \leq E_n^*(P_2^+); \\ w_2 & if\ E_n^*(P_2^+) < E_n(P); \end{cases}$$

$$f_{E_t}\big(E_t(P)\big) = \begin{cases} 0 & if\ E_{lim} + \delta_{E_t} < E_t(P); \\ 1 - \left(\dfrac{E_t(P) - E_{lim}}{\delta_{E_t}}\right) & if\ E_{lim} \le E_t(P) \le E_{lim} + \delta_{E_t}; \\ 1 & if\ E_t(P) < E_{lim}; \end{cases}$$

$$f_{C_t}\big(C_t(P)\big) = \begin{cases} 0 & if\ C_{lim} + \delta_{C_t} < C_t(P); \\ 1 - \left(\dfrac{C_t(P) - C_{lim}}{\delta_{C_t}}\right) & if\ C_{lim} \le C_t(P) \le C_{lim} + \delta_{C_t}; \\ 1 & if\ C_t(P) < C_{lim}; \end{cases}$$

Now after incorporating the membership functions defined above the crisp model is formulated as follows:

$$maximize\ \lambda$$

Subject to:

$$W_1 w \left(\frac{S_v(P) - S_v^{min}}{S_v^*(P_1^+) - S_v^{min}}\right) \ge \lambda$$

$$W_2 w \left(\frac{E_n(P) - E_n^{min}}{E_n^*(P_2^+) - E_n^{min}}\right) \ge \lambda$$

$$\delta_{E_t}(1 - \lambda) - \left(\sum_{i=1}^{N} e_i - I(E_{lim})\right) \ge 0;$$

$$\delta_{C_t}(1 - \lambda) - \left(\sum_{i=1}^{N} G_i - I(G_{lim})\right) \ge 0;$$

$$p_t = \left[\sum_{i=1}^{N}(r_{it} - r_i)x_i\right]^- = \begin{cases} \displaystyle\sum_{i=1}^{N}(r_{it} - r_i)x_i, & if\ \displaystyle\sum_{i=1}^{N}(r_{it} - r_i)x_i < 0, \\ 0, & if\ \displaystyle\sum_{i=1}^{N}(r_{it} - r_i)x_i \ge 0, \end{cases} \quad t = 1,2,\ldots\ldots\ldots,T$$

$$p_t \ge -\sum_{i=1}^{N}(r_{it} - r_i)e_i ; \qquad t = 1,2,\ldots\ldots\ldots,T$$

$$\frac{\sum_{i=1}^{N}\mu_i P_i x_i}{\sum_{k=1}^{N} P_k x_k} = R ;$$

$$L_i^s \beta_i^s \le y_i^s \le U_i^s \beta_i^s ; \qquad i = 1,2,\ldots\ldots\ldots,N$$

$$L_i^b \beta_i^b \leq y_i^b \leq U_i^b \beta_i^b ; \qquad i = 1,2, \ldots \ldots \ldots,$$

$$\beta_i^s + \beta_i^b \leq 1 ; \qquad i = 1,2, \ldots \ldots \ldots, N$$

$$x_i = X_i + y_i^b - y_i^s ; \qquad i = 1,2, \ldots \ldots \ldots, N$$

$$G_i = c_i^s y_i^s + c_i^b y_i^b + f_i^s \beta_i^s + f_i^b \beta_i^b ; \qquad i = 1,2, \ldots \ldots \ldots, N$$

$$\sum_{i=1}^{N} P_i x_i = \sum_{i=1}^{N} P_i X_i - \sum_{i=1}^{N} G_i$$

$$e_i = \frac{P_i x_i}{\sum_{k=1}^{N} P_k x_k}$$

$$p_t \geq 0; t = 1,2, \ldots \ldots \ldots, T$$

$$e_i, x_i, y_i^s, y_i^b, G_i \geq 0; \qquad i = 1,2, \ldots \ldots \ldots, N$$

$$\beta_i^s, \beta_i^b \in [0,1], \qquad i = 1,2, \ldots \ldots \ldots, N$$

$$\lambda \in [0, w];$$

$$W_1 + W_2 = 1;$$

Where $I(G_{lim})$ and $I(E_{lim})$ denote the integral value of the corresponding resources.

4. NUMERICAL ILLUSTRATION

The validity of the proposed portfolio optimization model is demonstrated using a data set of 10 randomly selected assets taken from [27] which was actually taken from National Stock Exchange (NSE).

Company	1	2	3	4	5	6	7	8	9	10	11	12
ABL	0.072	0.32032	0.2971	0.236	-0.05161	0.50633	-0.02516	0.90484	0.03214	0.45968	0.227	-0.87871
ALL	-0.14433	0.19032	0.75032	0.03433	-0.33581	0.247	0.49968	0.27032	-0.32786	0.31968	0.11933	-0.50903
BHL	0.08667	1.05613	0.05516	0.27567	-0.21839	0.49233	1.11516	0.57613	0.17143	0.92258	0.22367	-0.67903
CGL	-0.18567	0.76774	0.16194	0.48633	-0.2071	0.47833	0.2571	0.59484	-0.02321	0.55387	0.07333	-0.11871
HHM	0.18233	0.33	0.13677	0.46533	-0.12774	0.56067	0.10839	0	0.14321	0.00968	-0.15767	-0.27258
HCC	-0.157	0.61226	1.23548	0.56067	-0.71065	0.97333	0.32839	0.61581	0.03286	0.49935	-0.03733	-0.59452
KMB	0.18567	0.27806	0.55097	0.02733	-0.46613	0.73333	0.20581	0.17065	-0.05286	0.6671	0.373	-0.08355
MML	0.37533	0.65903	0.1929	0.16533	-0.15226	0.80867	0.39097	0.29	0.1975	0.21839	0.031	-0.06548
SIL	-0.10467	0.200552	0.31161	0.43333	-0.3171	1.104	0.37194	0.73097	0.03321	0.75903	0.09467	-0.44903
UNL	0.26367	0.41581	0.24484	0.12967	-0.0829	0.54	0.93258	0.61871	0.2275	0.68968	0.65433	0.65258

We have solved the above model in three fuzzy multi-objective program techniques namely Fuzzy multi-objective Non Linear Programming technique (FMONLP), Fuzzy multi-objective Additive Goal Programming technique (FMOAGP) and Interactive fuzzy multi-objective decision making method using the LINGO 12.0 and the corresponding computational result is given below.

Method	Weight	ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL
Fuzzy muti-objective non linear programming method (FMONLP)	$W_1 = 0.2$ $W_2 = 0.8$	0	0	0	0	0	0	0	0.1534	0	0.8466
	$W_1 = 0.5$ $W_2 = 0.5$	0	0	0	0	0.1371	0	0	0.0823	0	0.7806
	$W_1 = 0.8$ $W_2 = 0.2$	0.00217	0	0	0	0.3125	0	0	0	0	0.68533
Fuzzy muti-objective additive goal programming method (FMOAGP)	$W_1 = 0.2$ $W_2 = 0.8$	0	0	0	0	0.3275	0	0	0.1208	0	0.5517
	$W_1 = 0.5$ $W_2 = 0.5$	0.0128	0	0	0	0.295	0	0	0.1072	0	0.585
	$W_1 = 0.8$ $W_2 = 0.2$	0.1054	0	0	0	0.2744	0	0	0.1782	0	0.442
Interactive Fuzzy muti-objective decision making method (IFMODM)	$W_1 = 0.2$ $W_2 = 0.8$	0.1823	0	0	0.0231	0.243	0	0	0.1732	0	0.3784
	$W_1 = 0.5$ $W_2 = 0.5$	0.1285	0	0	0.1003	0.2812	0	0	0.2103	0	0.2797
	$W_1 = 0.8$ $W_2 = 0.2$	0.1032	0	0	0.0843	0.2932	0	0	0.2334	0	0.2859

Table 1: Summary result of portfolio selection for different weights

Method	Weight	$S_v(P)$	$E_n(P)$
Fuzzy muti-objective non linear programming method (FMONLP)	$W_1 = 0.2$ $W_2 = 0.8$	0.7143	0.6792
	$W_1 = 0.5$ $W_2 = 0.5$	0.8631	0.6913
	$W_1 = 0.8$ $W_2 = 0.2$	0.9138	0.7123
Fuzzy muti-objective additive goal programming method (FMOAGP)	$W_1 = 0.2$ $W_2 = 0.8$	0.9879	0.7214
	$W_1 = 0.5$ $W_2 = 0.5$	0.8485	0.6918
	$W_1 = 0.8$ $W_2 = 0.2$	0.7839	0.6823
Interactive Fuzzy muti-objective decision making method (IFMODM)	$W_1 = 0.2$ $W_2 = 0.8$	0.9234	0.6975
	$W_1 = 0.5$ $W_2 = 0.5$	0.9335	0.6927
	$W_1 = 0.8$ $W_2 = 0.2$	0.9392	0.7227

Table 2: Optimal solutions for different weightages of mean semi variance (W_1) and entropy functions (W_2) for our Portfolio optimization model

From our work it had been noticed that Entropy acts as a, measure of dispersal for portfolio allocation. So realistically it will be more potential if we would like to have maximum entropy.

It had also been noticed that in FMONLP weight effects directly on the objective functions but in FMOAGP weight effect inversely to the objective function.

But IFMODM is an efficient and modified technique for optimization and gives a highly reliable system in compare to other existing methods.

Hence Interactive method is more effective, efficient and powerful tool to solve the proposed multi-objective portfolio optimization model.

5. CONCLUSIONS

In this paper we had developed a mathematical model for efficient portfolio selection. In our model we had considered semi-variance as a risk measure in order to overcome the drawbacks of using variance as a risk measure. We had also integrated the transaction cost for trading equities in our proposed model. For this purpose we had considered both fixed and variable transaction cost. Also to get a well diversified portfolio we had incorporated the concept of Entropy in the form of maximization of entropy as an objective function in the proposed model.

Here we have validated the model by taking some real data from NSE. We have solved our proposed model by Interactive fuzzy multi-objective decision making method and finally compared the result with the results obtained by solving the model by FMONLP and FMOAGP approaches. The optimal solutions with different weight to different objectives had also been presented.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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