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ON $(m;M;\varphi_n)$ SCHUR h-CONVEX STOCHASTIC PROCESS

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Abstract. In this work, we introduce and examine $(m;M;\varphi)$ h-convex, $(m;M;\varphi)$ Jensen h-convex and $(m;M;\varphi_n)$-Schur h-convex stochastic processes. Also some characterizations of $(m;M;\varphi)$-Wright h-convex stochastic process are given, where $h:(0,1)\to\mathbb{R}$ is a positive function with $h(t)\leq t$ for any $t\in(0,1)$.

Keywords: stochastic process; $(m;M;\varphi_n)$-schur convex; majorization; convex stochastic process; Wright-convex; Jensen-convex.

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1. Introduction

Recently much attention has been given to the theory of convexity due to its great importance in other fields of pure and applied sciences.

In this paper we present some results concerning $(m;M;\varphi)$ h-convex,$(m;M;\varphi)$Jensen h-convex, $(m;M;\varphi_n)$-Schur h-convex and $(m;M;\varphi)$-Wright h-convex stochastic processes.

2. Preliminaries

Let $(\Omega,\mathscr{A},\mathrm{P})$ be a probability space and $I\subset\mathbb{R}$ is an interval.

A function $X:\Omega\to\mathbb{R}$ is a random variable if is $\mathscr{A}$-measurable.

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A stochastic processes is defined as function $X : I \times \Omega \to \mathbb{R}$ if for every $t \in I$, the function $X(t,.)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called:

(1) Continuous in probability in interval I, if for all $t_0 \in$ I :
$\mathrm{P}-\lim_{t\to t_0} X(t;.) = X(t_0;.)$. Where P − lim denotes the limit in probability.

(2) Mean square continuous in the interval I, if for all $t_0 \in$ I:
$\lim_{t\to t_0} \mathrm{E}\left[(X(t;.) - X(t_0;.))^2\right] = 0.$
Where E$[X(t;.)]$ denote the expectation value of the random variable $X(t,.)$.

Recall also that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is called:

- convex if for all $a,b \in I$ and $t \in [0,1]$, the following inequality holds:

$X(ta+(1-t)b,\cdot) \le tX(a,\cdot)+(1-t)X(b,\cdot)$

If the above inequality is assumed only for $t = \frac{1}{2}$, then the process X is Jensen-convex.

- h-convex if for all $\lambda \in [0,1]$ and $a,b \in I$ the inequality:

$X(\lambda a+(1-\lambda)b,.) \le h(\lambda)X(a,.)+h(1-\lambda)X(b,.)$ is satisfied.Where $h : (0,1) \to \mathbb{R}$ be a positive function ,$h \neq 0$.

- Wright-convex if for all $a,b \in I$ and $t \in [0,1]$, the following condition holds:

$X(ta+(1-t)b,\cdot)+X((1-t)a+tb,\cdot) \le X(a,\cdot)+X(b,\cdot)$

-Additive if for all $a,b \in I$, the following condition holds: $X(a+b,\cdot) = X(a,\cdot)+X(b,\cdot)$

Let $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in I^n$,(the integer $n \ge 2$). We say that x is majorized by y and write $x \preceq y$, if there exists a doubly stochastic $n \times n$ matrix P such that $x = y \cdot P$.

Note that the stochastic process $X : I^n \times \Omega \to \mathbb{R}$ is called Schur-convex if for all $x,y \in I^n$:
$x \preceq y \implies X(x,\cdot) \le X(y,\cdot)$.

The notion of $(m;\Psi)$-lower convex , $(M;\Psi)$-upper convex and (m,M,Ψ)-convex function was introduced by Dragomir [**1**].

For more informations we refer to [**2**, **3**, **5**, **6**, **10**, **12**, **13**, **14**].

3. Main Results

For the next results we consider $m,M \in \mathbb{R}$ and $I \subset \mathbb{R}$.

Definition 3.1. Let $h:(0,1)\to\mathbb{R}$ be a positive function $,h\neq 0$.

Let $\varphi: I\times\Omega\to\mathbb{R}$ is h-convex stochastic process.

We say that a stochastic process $X: I\times\Omega\to\mathbb{R}$ is $(m;M;\varphi)$ h-convex if:

$X-m\varphi$ *is convex* (1)

and

$M\varphi-X$ *is convex* (2)

If only condition (1) is satisfied, we say that X is $(m;\varphi)$ -$lower\, h-convex$, but if only condition (2) is satisfied, we say that X is $(M;\varphi)$ -upper h-convex.

Proposition 3.2. *Let $h:(0,1)\to\mathbb{R}$ be a positive function $,h\neq 0$ with $h(t)\leq t$ for any $t\in(0,1)$ Let $\varphi: I\times\Omega\to\mathbb{R}$ is non negative, h-convex stochastic process .*
Then the stochastic process $\varphi_n: I^n\times\Omega\to\mathbb{R}$ defined by:
$\varphi_n(x_1,\ldots,x_n,\omega)=\varphi(x_1,\omega)+\cdots+\varphi(x_n,\omega),\,(x_1,\ldots,x_n,\omega)\in I^n\times\Omega$ is schur convex for all integr $n\geq 2$.

Proof. Let $x=(x_1,\ldots,x_n)\,;y=(y_1,\ldots,y_n)\in I^n$ such that $x\preccurlyeq y$,then:

$$
\begin{aligned}
\textstyle\sum_{j=1}^{n}\varphi\left(x_j;.\right)=\sum_{j=1}^{n}\varphi\left(\sum_{i=1}^{n}t_{ij}y_i;.\right) &\leq \sum_{j=1}^{n}\sum_{i=1}^{n}h(t_{ij})\varphi\left(y_i;.\right)\\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n}t_{ij}\varphi\left(y_i;.\right)\\
&= \sum_{i=1}^{n}\varphi\left(y_i;.\right)\sum_{j=1}^{n}t_{ij}
\end{aligned}
$$
□

Definition 3.3. Let $h:(0,1)\to\mathbb{R}$ be a positive function $,h\neq 0$.

Let $\varphi: I\times\Omega\to\mathbb{R}$ is h-convex stochastic process.

Let $\varphi_n: I^n\times\Omega\to\mathbb{R}$ the stochastic process given by:

$\varphi_n(x_1,\ldots,x_n,\omega)=\varphi(x_1,\omega)+\cdots+\varphi(x_n,\omega)$

We say that a stochastic process $X: I^n\times\Omega\to\mathbb{R}$ is $(m;M;\varphi_n)$ -Schur $h-convex$ if for all $x,y\in I^n$:

$x\preceq y\Longrightarrow X(x;.)\leq X(y;.)-m\left(\varphi_n(y;.)-\varphi_n(x;.)\right)$ (3)

and

$$x \preceq y \quad \Longrightarrow \quad X(x;.) \geq X(y;.) - M(\varphi_n(y;.) - \varphi_n(x;.)) \qquad (4)$$

Remark 3.4. -If only condition (3) is satisfied, we say that X is $(m;\varphi_n)$ -lower Schur $h-convex$.
$-$ If only condition (4) is satisfied, we say that X *is* $(M;\varphi_n)$ -upper Schur h-convex.

Theorem 3.5.

Let $h:(0,1)\to\mathbb{R}$ be a positive function ,$h\neq 0$ with $h(t)\leq t$ for any $t\in(0,1)$

Let $\varphi : I\times\Omega\to\mathbb{R}$ is non negative, h-convex stochastic process .

Let $\varphi_n : I^n\times\Omega\to\mathbb{R}$ the stochastic process given by:

$\varphi_n(x_1,\ldots,x_n,\omega) = \varphi(x_1,\omega)+\cdots+\varphi(x_n,\omega)$, $(x_1,\ldots,x_n,\omega)\in I^n\times\Omega.$

Let $X : I\times\Omega\to\mathbb{R}$ be a stochastic process,and $S_n : I^n\times\Omega\to\mathbb{R}$ be a stochastic process such that: $S_n(x_1,\ldots,x_n,\omega) = X(x_1;\omega)+\cdots+X(x_n;\omega)$

(i) If X is (m,φ) -lower h-convex, then the stochastic process S_n is (m,φ_n) -lower Schur h-convex

(ii) If X is (M,φ) upper h-convex, then the stochastic process S_n is (M,φ_n) -upper Schur h-convex

(iii) If X is (m,M,φ)-h-convex, then the stochastic process S_n is (m,M,φ_n) Schur h-convex.

Proof.

(i) Let $x;y\in I^n$ with $x\preceq y$ There exists a doubly stochastic matrix $P=[t_{ij}]$ such that $x=y\cdot P$ then: X *is* (m,φ) -lower hconvex,so the stochastic process $Y = X - m\varphi$ is hconvex, by using the previous propostion,we have $\mathrm{Y}(x_1;.)+\cdots+Y(x_n;.)\, is\, schur\, convex.Then,$

$S_n(x;.) \leq Y(y_1;.)+\cdots+Y(y_n;.)+m(\varphi(x_1;.)+\cdots+\varphi(x_n;.))$

$=X(y_1;.)+\cdots+X(y_n;.)-m(\varphi(y_1;.)+\cdots+\varphi(y_n;.))$

$+m(\varphi(x_1;.)+\cdots+\varphi(x_n;.))$

$(ii).The\, proof\, is\, analogous.$

$(iii).From(i)\, and(ii)\, we\, obtain\, the\, desired\, result.$ □

In the following result we use the above theorem, to give a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem $(See[\mathbf{4},\mathbf{7},\mathbf{8}])$.

Corollary 3.6. *Let $h:(0,1)\to\mathbb{R}$ be a positive function ,$h\neq 0$ with $h(t)\leq t$ for any $t\in(0,1)$*

Let $\varphi : I\times\Omega\to\mathbb{R}$ is non negative, h-convex stochastic process.

$\varphi_n : I^n \times \Omega \to \mathbb{R}$*the stochastic process given by:*

$\varphi_n(x_1,\ldots,x_n,\omega) = \varphi(x_1,\omega) + \cdots + \varphi(x_n,\omega)$, $(x_1,\ldots,x_n,\omega) \in I^n \times \Omega$.

Let $X : I \times \Omega \to \mathbb{R}$ *be a stochastic process.*

Assume that $x = (x_1,\ldots,x_n)$ *;* $y = (y_1,\ldots,y_n) \in I^n$ $(n \geq 2)$ *satisfy:*

(a) $x_1 \leq \cdots \leq x_n, y_1 \leq \cdots \leq y_n$

(b) $y_1 + \cdots + y_k \leq x_1 + \cdots + x_k, \quad k = 1,\ldots,n-1$

(c) $y_1 + \cdots + y_n = x_1 + \cdots + x_n$

(i) If X is (m,φ) *-lower h-convex, then:*

$X(x_1;.) + \cdots + X(x_n;.) \leq X(y_1;.) + \cdots + X(y_n;.) - m(\varphi_n(y;.) - \varphi_n(x;.))$

(ii) If X is (M,φ) *-upper h-convex then:*

$X(x_1;.) + \cdots + X(x_n;.) \geq X(y_1;.) + \cdots + X(y_n;.) - M(\varphi_n(y;.) - \varphi_n(x;.))$

(iii) If X is (m,M,φ) *h-convex then:*

$X(y_1;.) + \cdots + X(y_n;.) - M(\varphi_n(y;.) - \varphi_n(x;.))$

$\leq X(x_1;.) + \cdots + X(x_n;.)$

$\leq X(y_1;.) + \cdots + X(y_n;.) - m(\varphi_n(y;.) - \varphi_n(x;.))$

Proof. Recall that assumptions (a)-(c) imply $x \preceq y$ (see [**7**]) and apply theorem 3.5. □

Corollary 3.7.

Let $h : (0,1) \to \mathbb{R}$ *be a positive function* $, h \neq 0$ *with* $h(t) \leq t$ *for any* $t \in (0,1)$

Let $\varphi : I \times \Omega \to \mathbb{R}$ *is non negative, h-convex stochastic process.*

Let $X : I \times \Omega \to \mathbb{R}$ *be a stochastic process.*

(i) If X is (m,φ) *-lower h-convex, then:*

$$(X - m\varphi)\left(\frac{x_1 + x_2 + \cdots x_n}{n},.\right) \leq \frac{\sum_{i=1}^{n}(X - m\varphi)(x_i,.)}{n}$$

(ii) If X is (M,φ)*-upper h-convex, then:*

$$(X - M\varphi)\left(\frac{x_1 + x_2 + \cdots x_n}{n},.\right) \geq \frac{\sum_{i=1}^{n}(X - M\varphi)(x_i,.)}{n}$$

Proof. Let $\bar{x} = \frac{1}{n}(x_1 + \cdots + x_n)$. Then $(\bar{x},\ldots,\bar{x}) \preceq (x_1,\ldots,x_n)$

By using theorem 3.5,we obtain the desired result. □

Definition 3.8. Let $h:(0,1)\to\mathbb{R}$ be a positive function ,$h\neq 0$.
Let $\varphi: I\times\Omega\to\mathbb{R}$ is h-convex stochastic process.
We say that a stochastic process $X: I\times\Omega\to\mathbb{R}$ is (m,φ) -lower Jensen h-convex if the stochastic process $X-m\varphi$ *is Jensen* $-$ *convex*.
Also the stochastic process X: I$\times\Omega\to\mathbb{R}$ is called $(M,\varphi)-$ upper Jensen h-convex if the stochastic process $M\varphi-X$ is Jensen-convex.
We say that $X: I\times\Omega\to\mathbb{R}$ is $(m,M,\varphi)-$ Jensen h-convex if it is (m,φ) -lower Jensen h-convex and (M,φ) -upper Jensen h-convex.

Now,We will proves that stochastic processes generating (m,M,φ_n) -Schur h-convex sums must be (m,M,φ) -Jensen h-convex.

Theorem 3.9. *Let $h:(0,1)\to\mathbb{R}$ be a positive function ,$h\neq 0$ with $h(t)\leq t$ for any $t\in(0,1)$*
Let $\varphi: I\times\Omega\to\mathbb{R}$ is non negative, h-convex stochastic process.
$\varphi_n: I^n\times\Omega\to\mathbb{R}$*the stochastic process given by:*
$\varphi_n(x_1,\dots,x_n,\omega)=\varphi(x_1,\omega)+\cdots+\varphi(x_n,\omega),\ (x_1,\dots,x_n,\omega)\in I^n\times\Omega.$
Let $X: I\times\Omega\to\mathbb{R}$ be a stochastic process,and $S_n: I^n\times\Omega\to\mathbb{R}$ be a stochastic process such that: $S_n(x_1,\dots,x_n,\omega)=X(x_1;\omega)+\cdots+X(x_n;\omega)$
For some $n\geq 2$ we have:
(i) If the stochastic process S_n is (m,φ_n) -lower Schur h-convex, then X is (m,φ) -lower Jensen h-convex.
(ii) If the stochastic process S_n is (M,φ_n) -upper Schur h-convex, then X is (M,φ) -upper Jensen h-convex.
(iii) If the stochastic process S_n is (m,M,φ_n) -Schur h-convex, then X is (m,M,φ) - Jensen h-convex.

Proof. (i) Take $y_1,y_2\in I$ and put $x_1=x_2=\frac{1}{2}(y_1+y_2)$.
Let $y=(y_1,y_2,y_2,\dots,y_2),\ x=(x_1,x_2,y_2,\dots,y_2)$ (if $n=2$, we take $y=(y_1,y_2),x=(x_1,x_2)$).
then $x\preceq y$, we get: $S_n(x;.)\leq S_n(y;.)-m(\varphi_n(y;.)-\varphi_n(x;.))$. Then

$$2X\left(\frac{y_1+y_2}{2};.\right)\leq X(y_1;.)+X(y_2;.)-m\left(\varphi(y_1;.)+\varphi(y_2;.)-2\varphi\left(\frac{y_1+y_2}{2};.\right)\right)$$

So, for $Y = X - m\varphi$ we have

$$2Y\left(\frac{y_1+y_2}{2};.\right) = 2X\left(\frac{y_1+y_2}{2};.\right) - 2m\varphi\left(\frac{y_1+y_2}{2};.\right)$$

$$\leq X(y_1;.) + X(y_2;.) - m\left(\left(\varphi(y_1;.) + \varphi(y_2;.)\right)\right)$$

$$= Y(y_1;.) + Y(y_2;.)$$

Then X is (m,φ) -lower Jensen h-convex.

(ii) The proof is analogous.

(iii) From (i) and (ii) we obtain the desired result.

□

Definition 3.10. Let $h : (0,1) \to \mathbb{R}$ be a positive function , $h \neq 0$.

Let $\varphi : I \times \Omega \to \mathbb{R}$ is h-convex stochastic process.

We say that a stochastic process $X : I \times \mathbb{R} \to \mathbb{R}$ is (m,φ) -lower Wright h-convex ((M,φ) upper Wright-h-convex) if the stochastic process $X - m\varphi$ (the stochastic process $M\varphi - X$) is Wright convex.

We say that X: I×$\mathbb{R} \to \mathbb{R}$ is $(m,M,\varphi)-$ Wright-h-convex if it is (m,φ) -lower Wright-h-convex and (M,φ) -upper Wright h-convex.

The next theorem is a counterpart of the result of Ng [**9**] on functions generating Schur-convex sums.

Theorem 3.11. *Let $h : (0,1) \to \mathbb{R}$ be a positive function , $h \neq 0$ with $h(t) \leq t$ for any $t \in (0,1)$*

Let $\varphi : I \times \Omega \to \mathbb{R}$ is non negative, h-convex stochastic process .

$\varphi_n : I^n \times \Omega \to \mathbb{R}$ the stochastic process given by:

$\varphi_n(x_1,\ldots,x_n,\omega) = \varphi(x_1,\omega) + \cdots + \varphi(x_n,\omega)$, $(x_1,\ldots,x_n,\omega) \in I^n \times \Omega$.

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process,and $S_n : I^n \times \Omega \to \mathbb{R}$ be a stochastic process such that: $S_n(x_1,\ldots,x_n,\omega) = X(x_1;\omega) + \cdots + X(x_n;\omega)$:

(i) If X is (m,φ) -lower Wright h-convex,then for every $n \geq 2$, S_n is (m,φ_n) -lower Schur-h-convex.

Conversely if for some $n \geq 2$, S_n is (m,φ_n) -lower Schur h-convex then X is (m,φ) -lower Wright h-convex.

(ii) If X is (M,φ) -upper Wright h-convex,then for every $n \geq 2$, S_n is (M,φ_n) -upper Schur h-convex.

Conversely if for some $n \geq 2$, S_n is (M,φ_n) -upper Schur h-convex then X is (M,φ) -upper Wright h-convex.

(iii)If X is (m,M,φ) - Wright h-convex,then for every $n \geq 2$, S_n is (m,M,φ_n) - Schur h-convex. Conversely if for some $n \geq 2$, S_n is (m,M,φ_n) - Schur h-convex then X is (m,M,φ) - Wright h-convex.

Proof.

(i)$\Rightarrow$)

Suppose X is (m,φ) -lower Wright h-convex .

Then the stochastic process $Y = X - m\varphi$ is Wright convex, it is of the form $Y = Y_1 + A$, where Y_1 is convex and A is additive $(See[\mathbf{11}])$.

For $x = (x_1,\ldots,x_n) \preceq y = (y_1,\ldots,y_n)$, we have

$$Y(x_1;.) + \cdots + Y(x_n;.) \leq Y(y_1;.) + \cdots + Y(y_n;.)$$

So,

$$\begin{aligned}&X(x_1;.) + \cdots + X(x_n;.) - m(\varphi(x_1;.) + \cdots + \varphi(x_n;.))\\&\quad \leq X(y_1;.) + \cdots + X(y_n;.) - m(\varphi(y_1;.) + \cdots + \varphi(y_n;.))\end{aligned}$$

Then:

$$S_n(x;.) \leq S_n(y;.) - m(\varphi_n(y;.) - \varphi_n(x;.))$$

Then: S_n is (m,φ_n) -lower Schur h-convex.

$\Leftarrow$)

Suppose S_n is (m,φ_n) -lower Schur h-convex. Take $y_1, y_2 \in I$ and $t \in (0,1)$

Put $x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$ and $x_i = y_i = z \in I$ for $i = 3,\ldots,n$. (for $n > 2$)

Then: $x = (x_1,\ldots,x_n) \preceq y = (y_1,\ldots,y_n)$.

Using (3), we get $S_n(x;.) \leq S_n(y;.) - m(\varphi_n(y;.) - \varphi_n(x;.))$

Then:

$$X(ty_1+(1-t)y_2;.)+X((1-t)y_1+ty_2;.)$$
$$\leq X(y_1;.)+X(y_2;.)-m(\varphi(y_1;.)+\varphi(y_2;.)-\varphi(x_1;.)-\varphi(x_2;.))$$

Then, for $Y=X-m\varphi$ we obtain:

$$\begin{aligned}&Y(ty_1+(1-t)y_2;.)+Y((1-t)y_1+ty_2;.)\\&=X(ty_1+(1-t)y_2;.)+X((1-t)y_1+ty_2;.)-m\varphi(ty_1+(1-t)y_2;.)\\&\quad-m\varphi((1-t)y_1+ty_2;.)\\&\leq X(y_1;.)+X(y_2;.)-m\varphi(y_1;.)-m\varphi(y_2;.)=Y(y_1;.)+Y(y_2;.)\end{aligned}$$

This shows that X is (m,φ) -lower Wright h-convex.

(ii).The proof is analogous of (i).

(iii) from (i) and (ii) we obtain the desired result. □

Now,we establish a representation for (m,M,φ) -Wright h-convex sthocastic process.

Theorem 3.12. *Let $h:(0,1)\to\mathbb{R}$ be a positive function ,$h\neq 0$ with $h(t)\leq t$ for any $t\in(0,1)$*
Let $\varphi: I\times\Omega\to\mathbb{R}$ is non negative, h-convex stochastic process .
Let $X: I\times\Omega\to\mathbb{R}$ be a stochastic process :

(i) X is (m,φ) -lower Wright h-convex if and only if $X=Y_1+H_1$.
Where Y_1 is (m,φ)-lower h-convex and $H_1: I\times\Omega\to\mathbb{R}$ is additive

(ii) X is (M,φ) -upper Wright h-convex if and only if $X=Y_2+H_2$.
Where Y_2 is (M,φ)-upper h-convex and $H_2: I\times\Omega\to\mathbb{R}$ is additive.

(iii) X is (m,M,φ) - Wright h-convex if and only if $X=Y_3+H_3$.
Where Y_3 is (m,M,φ)-h-convex and $H_3: I\times\Omega\to\mathbb{R}$ is additive.

Proof. i)$\Rightarrow$)

Suppose X is (m,φ) -lower Wright h-convex, then $Z=X-m\varphi$ is Wright convex. Then, there exist a convex stochastic process $Z_1: I\times\Omega\to\mathbb{R}$ and an additive stochastic process $H_1: I\times\Omega\to\mathbb{R}$ such that $Z=Z_1+H_1$.

Then $Y_1=Z_1+m\varphi$ is (m,φ)-lower h-convex and $X=Z+m\varphi=Z_1+H_1+m\varphi=Y_1+H_1$

Then we obtain the desired result.

$\Leftarrow$)

If $X = Y_1 + H_1$ where Y_1 is (m,φ)-lower h-convex and H_1 additive, then $X - m\varphi = Y_1 - m\varphi + H_1$ is Wright-convex as a sum of a convex stochastic process and an additive stochastic process.

Then X is (m,φ) -lower Wright h-convex.

ii) The proof is similar.

iii):$\Rightarrow$)

If X is (m,M,φ) Wright-h-convex, then $X - m\varphi$ and $M\varphi - X$ are Wright-convex. Then $X - m\varphi = Z_1 + G_1$ and $M\varphi - X = Z_2 + G_2$ with some convex stochastic processes Z_1, Z_2 and additive stochastic process G_1, G_2. Hence $G_1 + G_2 = (M-m)\varphi - (Z_1 + Z_2)$.

We have: $G = G_1 + G_2$ is additive. Let $H_3 = G_1$ and $Y_3 = X - H_3$. Then:

$Y_3 - m\varphi = X - H_3 - m\varphi = Z_1$

Since Z_1 is convex,then Y_3 is (m,φ)-lower convex .

Also: $M\varphi - Y_3 = M\varphi - X + H_3 = Z_2 + G_2 + H_3 = Z_2 + G$ is convex, then Y_3 is (M,φ)-upper h-convex.

Consequently, Y_3 is (m,M,φ)- h-convex and $X = Y_3 + H_3$.

$\Leftarrow$)

If $X = Y_3 + H_3$, where Y_3 is (m,M,φ)-h-convex and $H_3 : I \times \Omega \to \mathbb{R}$ is additive, then, by (i) and(ii) X is (m,φ) -lower Wright h-convex and (M,ψ) -upper Wright h-convex. Then X is (m,M,φ) -Wright h-convex.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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