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ON THE GENERALIZED k-FIBONACCI AND k-LUCAS NUMBERS

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Abstract. In this paper, we give some new identities for a family of Fibonacci and Lucas numbers. We investigate the relationships among Fibonacci, Lucas, generalized k-Fibonacci, and generalized k-Lucas numbers and we also prove some properties of those numbers.

Keywords: Fibonacci number; Lucas number; generalized k-Fibonacci number; generalized k-Lucas number.

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1. Introduction

Fibonacci sequences are one of the most well-known sequences in mathematics which is applied in other fields such as computer science, physics, biology, and statistics. The application of Fibonacci sequences in group theories was studied by C. M. Campbell, P. P. Campbell, and Ö. Deveci, see in ([2], [3], [4]). Fibonacci and Lucas numbers have many applications to various fields of mathematics and science, see in [8]. Because the structure of the Lucas sequence

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is similar to that of the Fibonacci sequences, so it can be seen that there are several studies looking at both of them in the same manners. The generalized Fibonacci and Lucas sequences were investigated by many researchers, see in ([1], [5], [6], [7], [9], [12], [13], [14]). One of the interesting research of generalized Fibonacci and Lucas numbers is generalized k-Fibonacci and generalized k-Lucas numbers which can be seen in [10], [11], and [15].

In this paper, we give some new identities of generalized k-Fibonacci and generalized k-Lucas numbers. We investigate the relationships among Fibonacci, Lucas, generalized k-Fibonacci, and generalized k-Lucas numbers, especially in the case of $k = 3$.

2. Preliminaries

It is well known that the Fibonacci number F_n is a term of the sequence 1,1,2,3,5,8,13,.... The Fibonacci number F_n satisfies the second order linear recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n = 2,3,4,...$$

with the initial values $F_0 = 1$, $F_1 = 1$. It is well known that the Fibonacci numbers are defined by Binet's formula

$$F_n = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \quad n = 0,1,2,...$$

where α and β are the root of the quadratic equation $x^2 - x - 1 = 0$, so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The Lucas number L_n satisfies the second order linear recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \quad n = 2,3,4,...$$

with the initial values $L_0 = 2$, $L_1 = 1$. Then the Lucas number L_n is a term of the sequence $2,1,3,4,7,11,18,....$ The Binet's formula for the Lucas numbers is

$$L_n = \alpha^n + \beta^n, \quad n = 0,1,2,...$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The Fibonacci numbers and the Lucas numbers are related by

$$L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}$$

In 2010, Mikkawy and Sogabe [16] gave the definition of generalized k-Fibonacci numbers as follows.

Definition 2.1. [16] Let n and $k(\neq 0)$ be natural numbers, then there exist unique numbers m and n such that $n = mk + r(0 \leq r < k)$. The generalized k-Fibonacci number $F_n^{(k)}$ is defined by

$$F_n^{(k)} = \frac{1}{(\sqrt{5})^k}(\alpha^{m+2} - \beta^{m+2})^r(\alpha^{m+1} - \beta^{m+1})^{k-r}, \quad n = mk + r$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

The generalized k-Fibonacci and Fibonacci numbers are related by

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r \text{ , } n = mk + r.$$

From Definition 2.1, we have that the generalized 1-Fibonacci number $F_n^{(1)}$ is just the ordinary Fibonacci number F_n because the sequence $\{F_n^{(1)}\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}$. The first few numbers of the generalized k-Fibonacci numbers for $k = 2, 3$ are as follows:

$$\{F_n^{(2)}\}_{n=0}^{10} = \{1, 1, 1, 2, 4, 6, 9, 15, 25, 40, 64\}$$
$$\{F_n^{(3)}\}_{n=0}^{10} = \{1, 1, 1, 1, 2, 4, 8, 12, 18, 27, 45\}.$$

In 2017, Özkan, Altun, and Göcer [10] introduced the definition of the generalized k-Lucas numbers as follows.

Definition 2.2. [10] Let n and $k(\neq 0)$ be natural numbers, then there exist unique numbers m and n such that $n = mk + r\,(0 \leq r < k)$. The generalized k-Lucas number $L_n^{(k)}$ is defined by

$$L_n^{(k)} = (\alpha^{m+1} + \beta^{m+1})^r(\alpha^m + \beta^m)^{k-r}, \;\; n = mk + r$$

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

From Definition 2.2, the generalized k-Lucas and Lucas numbers are related by

$$L_n^{(k)} = (L_m)^{k-r}(L_{m+1})^r, \quad n = mk + r.$$

In the same manner as a case of the generalized k-Fibonacci numbers, the generalized 1-Lucas number $L_n^{(1)}$ is just the ordinary Lucas number L_n because $\{L_n^{(1)}\} = \{2, 1, 3, 4, 7, 11, 18, 29, 47, 76, ...\}$. The first few numbers of generalized k-Lucas numbers $L_n^{(k)}$ for $k = 2, 3$ are as follows:

$$\{L_n^{(2)}\}_{n=0}^{10} = \{4, 2, 1, 3, 9, 12, 16, 28, 49, 77, 121\}$$
$$\{L_n^{(3)}\}_{n=0}^{10} = \{8, 4, 2, 1, 3, 9, 27, 36, 48, 64, 112\}.$$

3. MAIN RESULTS

In this section, we investigate the relationships among Fibonacci, Lucas, generalized k-Fibonacci, and generalized k-Lucas numbers, especially in the case of $k = 3$.

We start with the relations on the generalized k-Fibonacci numbers

Theorem 3.1. Let n be a positive integer. The generalized 3-Fibonacci numbers satisfy

$$F_{3n}^{(3)} = F_{3n+1}^{(3)} - F_{3n-1}^{(3)}.$$

Proof. We have $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r(0 \leq r < k)$. Consider

$$\begin{aligned} F_{3n}^{(3)} &= (F_n)^3 \\ &= (F_n)^2(F_{n+1} - F_{n-1}) \\ &= (F_n)^2 F_{n+1} - F_{n-1}(F_n)^2 \\ &= F_{3n+1}^{(3)} - F_{3n-1}^{(3)}. \end{aligned}$$

□

Theorem 3.2. Let n be a positive integer. The generalized 3-Fibonacci numbers satisfy

$$F_{3n+3}^{(3)} = F_{3n+2}^{(3)} + F_{3n-1}^{(3)} + 2F_{3n-2}^{(3)} + F_{3n-3}^{(3)}.$$

Proof. Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r(0 \leq r < k)$. We obtain $F_{3n+3}^{(3)} - F_{3n-3}^{(3)} = F_{3(n+1)}^{(3)} - F_{3(n-1)}^{(3)} = (F_{n+1})^3 - (F_{n-1})^3$. This implies that $F_{3n+3}^{(3)} = (F_{n+1})^3 - (F_{n-1})^3 + F_{3n-3}^{(3)}$. Consider

$$\begin{aligned} F_{3n+3}^{(3)} &= (F_{n+1})^3 - (F_{n-1})^3 + F_{3n-3}^{(3)} \\ &= (F_{n+1} - F_{n-1})((F_{n+1})^2 + F_{n+1}F_{n-1} + (F_{n-1})^2) + F_{3n-3}^{(3)} \\ &= F_n((F_{n+1})^2 + F_{n+1}F_{n-1} + (F_{n-1})^2) + F_{3n-3}^{(3)} \\ &= F_n(F_{n+1})^2 + F_nF_{n-1}F_{n+1} + (F_{n-1})^2F_n + F_{3n-3}^{(3)} \\ &= F_{3n+2}^{(3)} + F_nF_{n-1}(F_n + F_{n-1}) + F_{3(n-1)+1}^{(3)} + F_{3n-3}^{(3)} \\ &= F_{3n+2}^{(3)} + F_{n-1}(F_n)^2 + (F_{n-1})^2F_n + F_{3n-2}^{(3)} + F_{3n-3}^{(3)} \\ &= F_{3n+2}^{(3)} + F_{3n-1}^{(3)} + F_{3n-2}^{(3)} + F_{3n-2}^{(3)} + F_{3n-3}^{(3)} \\ &= F_{3n+2}^{(3)} + F_{3n-1}^{(3)} + 2F_{3n-2}^{(3)} + F_{3n-3}^{(3)}. \end{aligned}$$

□

Theorem 3.3. Let n be a positive integer. The generalized 3-Fibonacci numbers satisfy

$$F_{3n+3}^{(3)} = F_{3n}^{(3)} + 3F_{3n-1}^{(3)} + 3F_{3n-2}^{(3)} + F_{3n-3}^{(3)}.$$

Proof. Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r(0 \leq r < k)$, we obtain

$$\begin{aligned}
F_{3n+3}^{(3)} &= (F_{n+1})^3 \\
&= (F_n + F_{n-1})^3 \\
&= (F_n)^3 + 3F_{n-1}(F_n)^2 + 3(F_{n-1})^2F_n + (F_{n-1})^3 \\
&= F_{3n}^{(3)} + 3F_{3n-1}^{(3)} + 3F_{3n-2}^{(3)} + F_{3n-3}^{(3)}.
\end{aligned}$$

□

In the next results, we are focused on the generalized 3-Lucas numbers.

Theorem 3.4. Let n be a positive integer. The generalized 3-Lucas numbers satisfy

$$L_{3n}^{(3)} = L_{3n+1}^{(3)} - L_{3n-1}^{(3)}.$$

Proof. We have $L_n^{(k)} = (L_m)^{k-r}(L_{m+1})^r$ for $n = mk + r(0 \leq r < k)$. Then

$$\begin{aligned}
L_{3n}^{(3)} &= (L_n)^3 \\
&= (L_n)^2(L_{n+1} - L_{n-1}) \\
&= (L_n)^2L_{n+1} - L_{n-1}(L_n)^2 \\
&= L_{3n+1}^{(3)} - L_{3n-1}^{(3)}.
\end{aligned}$$

□

Theorem 3.5. Let n be a positive integer. The generalized 3-Lucas numbers satisfy

$$L_{3n+3}^{(3)} = L_{3n+2}^{(3)} + L_{3n-1}^{(3)} + 2L_{3n-2}^{(3)} + L_{3n-3}^{(3)}.$$

Proof. Since $L_n^{(k)} = (L_m)^{k-r}(L_{m+1})^r$ for $n = mk + r(0 \leq r < k)$, we have $L_{3n+3}^{(3)} - L_{3n-3}^{(3)} = L_{3(n+1)}^{(3)} - L_{3(n-1)}^{(3)} = (L_{n+1})^3 - (L_{n-1})^3$. Therefore, $L_{3n+3}^{(3)} = (L_{n+1})^3 - (L_{n-1})^3 + L_{3n-3}^{(3)}$.

Consider

$$\begin{aligned}
L^{(3)}_{3n+3} &= (L_{n+1})^3-(L_{n-1})^3+L^{(3)}_{3n-3}\\
&= (L_{n+1}-L_{n-1})((L_{n+1})^2+L_{n+1}L_{n-1}+(L_{n-1})^2)+L^{(3)}_{3n-3}\\
&= L_n((L_{n+1})^2+L_{n-1}L_{n+1}+(L_{n-1})^2)+L^{(3)}_{3n-3}\\
&= (L_n)(L_{n+1})^2+L_nL_{n-1}L_{n+1}+(L_{n-1})^2(L_n)+L^{(3)}_{3n-3}\\
&= L^{(3)}_{3n+2}+L_nL_{n-1}(L_n+L_{n-1})+L^{(3)}_{3(n-1)+1}+L^{(3)}_{3n-3}\\
&= L^{(3)}_{3n+2}+L_{n-1}(L_n)^2+(L_{n-1})^2L_n+L^{(3)}_{3n-2}+L^{(3)}_{3n-3}\\
&= L^{(3)}_{3n+2}+L^{(3)}_{3n-1}+L^{(3)}_{3n-2}+L^{(3)}_{3n-2}+L^{(3)}_{3n-3}\\
&= L^{(3)}_{3n+2}+L^{(3)}_{3n-1}+2L^{(3)}_{3n-2}+L^{(3)}_{3n-3}.
\end{aligned}$$

□

Theorem 3.6. Let n be a positive integer. The generalized 3-Lucas numbers satisfy

$$L^{(3)}_{3n+3}=L^{(3)}_{3n}+3L^{(3)}_{3n-1}+3L^{(3)}_{3n-2}+L^{(3)}_{3n-3}.$$

Proof. Since $L^{(k)}_n=(L_m)^{k-r}(L_{m+1})^r$ for $n=mk+r(0\le r<k)$, we obtain

$$\begin{aligned}
L^{(3)}_{3n+3} &= (L_{n+1})^3\\
&= (L_n+L_{n-1})^3\\
&= (L_n)^3+3L_{n-1}(L_n)^2+3(L_{n-1})^2L_n+(L_{n-1})^3\\
&= L^{(3)}_{3n}+3L^{(3)}_{3n-1}+3L^{(3)}_{3n-2}+L^{(3)}_{3n-3}.
\end{aligned}$$

□

Finally, we provide some relations among Fibonacci, Lucas, generalized 3-Fibonacci, and generalized 3-Lucas numbers.

Theorem 3.7. Let n be a positive integer. Then

$$(L_n+F_n)^3=27F^{(3)}_{3n}-27F^{(3)}_{3n-1}+9F^{(3)}_{3n-2}-F^{(3)}_{3n-3}.$$

Proof. We have $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r (0 \le r < k)$. By the identity $L_n = F_n + F_{n-2}$, we obtain

$$\begin{aligned}
(L_n + F_n)^3 &= (F_n + F_{n-2} + F_n)^3 \\
&= (F_n + F_n - F_{n-1} + F_n)^3 \\
&= (3F_n - F_{n-1})^3 \\
&= (3F_n)^3 - 3(3F_n)^2 F_{n-1} + 3(3F_n)(F_{n-1})^2 - (F_{n-1})^3 \\
&= 27(F_n)^3 - 27F_{n-1}(F_n)^2 + 9(F_{n-1})^2 F_n - (F_{n-1})^3 \\
&= 27F_{3n}^{(3)} - 27F_{3n-1}^{(3)} + 9F_{3n-2}^{(3)} - F_{3n-3}^{(3)}.
\end{aligned}$$

□

Theorem 3.8. Let n be a positive integer. Then

$$L_{3n}^{(3)} - F_{3n}^{(3)} = F_{3n-6}^{(3)} + 3F_{n-2}F_n L_n.$$

Proof. Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r (0 \le r < k)$, by the identity $L_n = F_n + F_{n-2}$, we obtain

$$\begin{aligned}
L_{3n}^{(3)} - F_{3n}^{(3)} &= (L_n)^3 - (F_n)^3 \\
&= (F_n + F_{n-2})^3 - (F_n)^3 \\
&= (F_n)^3 + 3(F_n)^2 F_{n-2} + 3F_n (F_{n-2})^2 + (F_{n-2})^3 - (F_n)^3 \\
&= 3F_{n-2}(F_n)^2 + 3(F_{n-2})^2 F_n + (F_{n-2})^3 \\
&= 3F_{n-2}F_n(F_n + F_{n-2}) + F_{3n-6}^{(3)} \\
&= F_{3n-6}^{(3)} + 3F_{n-2}F_n L_n.
\end{aligned}$$

□

Theorem 3.9. Let n be a positive integer. Then

$$L_{3n}^{(3)} = 7F_{3n}^{(3)} - 9F_{3n-1}^{(3)} + 3F_{3n-2}^{(3)} + F_{3n-6}^{(3)}.$$

Proof. Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r (0 \leq r < k)$, by the identity $L_n = F_n + F_{n-2}$, we have

$$\begin{aligned}
L_{3n}^{(3)} &= (L_n)^3 \\
&= (F_n + F_{n-2})^3 \\
&= (F_n)^3 + 3(F_n)^2 F_{n-2} + 3F_n(F_{n-2})^2 + (F_{n-2})^3 \\
&= (F_n)^3 + 3(F_n)^2(F_n - F_{n-1}) + 3F_n(F_n - F_{n-1})^2 + (F_{n-2})^3 \\
&= F_{3n}^{(3)} + 3(F_n)^3 - 3(F_n)^2 F_{n-1} + 3F_n((F_n)^2 - 2F_n F_{n-1} + (F_{n-1})^2) + F_{3n-6}^{(3)} \\
&= F_{3n}^{(3)} + 3(F_n)^3 - 3F_{n-1}(F_n)^2 + 3(F_n)^3 - 6F_{n-1}(F_n)^2 + 3(F_{n-1})^2 F_n + F_{3n-6}^{(3)} \\
&= F_{3n}^{(3)} + 6(F_n)^3 - 9F_{n-1}(F_n)^2 + 3(F_{n-1})^2 F_n + F_{3n-6}^{(3)} \\
&= F_{3n}^{(3)} + 6F_{3n}^{(3)} - 9F_{3n-1}^{(3)} + 3F_{3n-2}^{(3)} + F_{3n-6}^{(3)} \\
&= 7F_{3n}^{(3)} - 9F_{3n-1}^{(3)} + 3F_{3n-2}^{(3)} + F_{3n-6}^{(3)}.
\end{aligned}$$

□

Theorem 3.10. Let n be a positive integer. Then

$$F_n(L_n)^2 = 4F_{3n}^{(3)} - 4F_{3n-1}^{(3)} + F_{3n-2}^{(3)}.$$

Proof. Since $F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r$ for $n = mk + r (0 \leq r < k)$, by the identity $L_n = F_n + F_{n-2}$, we obtain

$$\begin{aligned}
F_n(L_n)^2 &= F_n(F_n + F_{n-2})^2 \\
&= F_n[(F_n)^2 + 2F_{n-2}F_n + (F_{n-2})^2] \\
&= (F_n)^3 + 2F_{n-2}(F_n)^2 + (F_{n-2})^2 F_n \\
&= F_{3n}^{(3)} + 2(F_n - F_{n-1})(F_n)^2 + (F_n - F_{n-1})^2 F_n \\
&= F_{3n}^{(3)} + 2(F_n)^3 - 2F_{n-1}(F_n)^2 + (F_n)^3 - 2F_{n-1}(F_n)^2 + (F_{n-1})^2 F_n \\
&= F_{3n}^3 + 3(F_n)^3 - 4F_{n-1}(F_n)^2 + (F_{n-1})^2 F_n \\
&= 4F_{3n}^{(3)} - 4F_{3n-1}^{(3)} + F_{3n-2}^{(3)}.
\end{aligned}$$

□

Theorem 3.11. Let n be a positive integer. Then

$$L_n(F_n)^2 = 2F_{3n}^{(3)} - F_{3n-1}^{(3)}.$$

Proof. By the identity $L_n = F_n + F_{n-2}$. Consider

$$\begin{aligned} L_n(F_n)^2 &= (F_n + F_{n-2})(F_n)^2 \\ &= (F_n)^3 + F_{n-2}(F_n)^2 \\ &= (F_n)^3 + (F_n - F_{n-1})(F_n)^2 \\ &= 2(F_n)^3 - F_{n-1}(F_n)^2 \\ &= 2F_{3n}^{(3)} - F_{3n-1}^{(3)} \end{aligned}$$

□

4. Conclusion

In this paper, we have been proved some theorems concerning the generalized k-Fibonacci numbers and generalized k-Lucas numbers, especially in the case $k = 3$, and given some relationships among generalized 3-Fibonacci numbers, relationships among generalized 3-Lucas numbers, and relationships among the ordinary Fibonacci, ordinary Lucas, generalized 3-Fibonacci and generalized 3-Lucas numbers.

Conflict of interests

The authors declare that there is no conflict of interests.

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