ZERO DIVISOR GRAPH ASSOCIATED TO $\Gamma-$SEMIGROUP OF $\mathbb{Z}_n$

PINKU CHANDRA DEY*, KUNTALA PATRA

Department of Mathematics, Gauhati University, Guwahati, India

Abstract. In this paper we have studied the zero divisor graph $G_1(\mathbb{Z}_n(\Gamma))$ associated with a $\Gamma-$semigroup. Here we have taken two non empty sets $\mathbb{Z}_n$ and set Γ of non zero zero divisors of $\mathbb{Z}_n$ then $\mathbb{Z}_n(\Gamma)$ is a $\Gamma-$semi group. In this paper we have taken all the elements of $\mathbb{Z}_n$ as the vertices of the graph $G_1(\mathbb{Z}_n(\Gamma))$ and two distinct vertices a and b are adjacent if and only if for any $\alpha \in \Gamma$ we have $a\alpha b = 0$. In this paper we have studied the degree of vertices of $G_1(\mathbb{Z}_n(\Gamma))$, number of edges, girth, diameter, planarity, and traversibility of $G_1(\mathbb{Z}_n(\Gamma))$.

Keywords: graph theory; zero divisor graph; $\Gamma-$semigroup.

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1. INTRODUCTION

From the very beginning of the introduction of graph theory researchers were interested in representing the various algebraic structure in the form of graph structures. One such Mathematician I. Beck[6] introduced us to a beautiful graph structure and named it as zero divisor graph. This graph structure establishes a connection between ring theory and graph theory. Beck [5] presented the idea of coloring a commutative ring with the help of this notion of graph. Motivated by this Definition D.F. Anderson and M. Naseer[3] continued the work done by I. Beck[6] in which they have cosiodered all the elements of a commutative ring as vertices

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*Corresponding author

E-mail address: pikulmg@gmail.com

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and any two distinct elements of the ring R are adjacent if and only if their Product is zero. Later this definition was modified by D.F Anderson and P.S Livingston [4].In their notion they have considered a commutative ring R with unity and $Z(R) - 0$ as the set of zero divisors of R. In their graph structure they have considered all the non zero zero divisors as the vertices and any two distinct vertices x and y of $Z(R) - 0$ are adjacent if and only if their product $xy = 0$. After the introduction of these graphs many researchers introduced several graph structures associated to different algebraic structures.

In our work we have obtained a graph structure for an algebraic structure known as $\Gamma-$semigroup by taking two non empty sets $\mathbb{Z}_n$ and $\Gamma = $ set of non zero zero divisors of $\mathbb{Z}_n$.

In the year 1964, N. Nobusawa[8] in his paper "on a generalization of ring theory" introduced $\Gamma-$ algebraic system known as $\Gamma-$ ring . The concept of $\Gamma-$ semigroup was first introduced by M. K. Sen[9] in 1981 by considering two non empty sets S and Γ. S is called a $\Gamma-$semi group if there exist mappings $S \times \Gamma \times S \rightarrow S$, written as $(a, \alpha, b) \rightarrow a\alpha b$ and $\Gamma \times S \times \Gamma \rightarrow \Gamma$ by $(\alpha, a, \beta) \rightarrow \alpha a \beta$ which satisfies the identities $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$. In 1986 M.K. Sen and N.K. Saha[10] improved the definition of Γ-semigroup by taking $S = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ where S is a $\Gamma-$semigroup if the following relations are satisfied $a\alpha b \in S$ and $(a\alpha b)c = a\alpha(b\beta c)$ for all $\alpha, \beta \in \Gamma$ and $a, b, c \in S$. Motivated by the above concepts we have introduced the graph structure zero divisor graph associated with a Γ-semigroup as follows: $Z(\mathbb{Z}_n)$ be the set of zero divisors of a commutative ring $\mathbb{Z}_n$.We introduce a set $\Gamma = Z(\mathbb{Z}_n) - \{0\}$ to form a $\Gamma-$Semi group $\mathbb{Z}_n(\Gamma)$. In our graph structure we have considered all the elements of $\mathbb{Z}_n$ as vertices and any two distinct vertices a and, b of $\mathbb{Z}_n$ are adjacent if and only if there exist an α in Γ such that aαb=$a\alpha b = 0$. Various properties of this graph structure are studdied in our work which will be given in the main results section.

2. PRELIMINARIES

A graph $G(V, E)$ with vertex set V and edge set E is connected if there exist a path between any two vertices a and b 0f G. The degree of any vertex x of a graph is defined as the number of vertices of $G(V, E)$ which are adjacent to it. In Algebraic Graph theory the study of degree of vertices is very important as the study opens up many characteristics of the graph which further can be studied to various aspects. A graph $G(V, E)$ is regular if the degree of each vertex in the

graph $G(V,E)$ is equal. If the vertex set can be partitioned into two sets in such a way that each vertex of the same set has same degree but the degrees of the two vertex sets are not equal then the graph is said to be semi-regular. The diameter of the graph is defined to be the supremum of the shortest distance between any two distinct vertices of the graph $G(V,E)$. A cycle of a graph $G(V,E)$ is a closed walk $(v_0, v_1, v_2, ..., v_n, v_0)$ in which degree of each of the vertices has degree 2. The smallest such cycles is known as girth. A graph $G(V,E)$ is said to be Eulerian if there exist a path between each vertices and the path is obtained by crossing each edge of the Graph $G(V,E)$ exactly once. And if we can get a cycle in which it contains all the vertices of the graph $G(V,E)$ and the path of the cycle is obtained by crossing each vertex exactly once. This type of cycle is termed as hamiltonian cycle. Any graph $G(V,E)$ containing an Hamiltonian cycle is known as hamiltonian graph. If a graph has no edges between any two vertices of the graph then the graph is termed as empty graph. A graph $G(V,E)$ is said to be Planar if the graph can be drawn in a plane in such a way that no two edges of the graph $G(V,E)$ intersects in the plane except at the vertices. For some basic definition and terminology we will refer to C. Godsil and G. Royle[1], F. Haray[5] and N. Biggs[7]. A plane graph with a fixed embedding forms a planar graph and the regions forms by the edges dividing the plane termed as faces. If adding an edge between any two non adjacent vertices make the graph non planar then the graph so formed is termed as maximal planar graph.

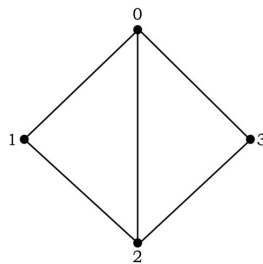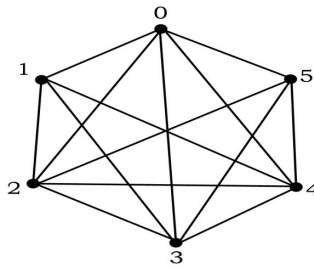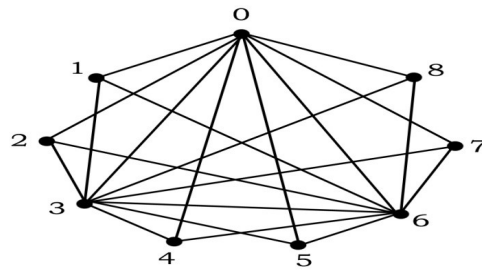Some of the examples of our Graph structure for some n is given as follows:

FIGURE 1. Graph of $(\mathbb{Z}_4(\Gamma))$

FIGURE 2. Graph of $(\mathbb{Z}_6(\Gamma))$

FIGURE 3. Graph of $(\mathbb{Z}_9(\Gamma))$

3. MAIN RESULTS

MThe symbol $\phi(n)$ represents the Euler $\phi-$function which is defined as the number of elements which are less than n and relatively prime to n. We will use this definition more frequently as we move forward.

We will use the following two 'results' in our proof of finding out the degrees of the vertices and the number of edges of our graph:

Result 1: Product of two unit elements in $\mathbb{Z}_n$ is again a unit element in $\mathbb{Z}_n$.

Result 2:. Sum of the degree of the vertices of a graph $G(V,E)$ is twice the total number of edges in the graph $G(V,E)$.

Theorem 1: The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is a totally disconnected graph if n is prime.

Proof: since n is prime every vertices of the of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is an unit element. Thus non adjacent. Hence the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is a disconnected graph for n prime.

Theorem 2: The degree of each vertices of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is $n-1$ if the vertex is a zero divisor or the zero element of $\mathbb{Z}_n$.

Proof: Let a be an arbitrary element(vertex) of $\mathbb{Z}_n$.

Case 1: if $a = 0$

Since 0 is adjacent to all the other vertices b of $\mathbb{Z}_n$ as for any b we can always find an α in Γ such that $0\alpha b = 0$. And 0 cannot be adjacent to itself as we are considering two distinct vertices. Hence the result. Therefore $deg(0) = n - 1$.

Case 2: if a(arbitrary vertex) is a zero divisor element $\mathbb{Z}_n$.

Now let us consider $\mathbb{Z}_n = 0, 1, 2, ..., n - 1$

and $\Gamma =$ Set of non zero zero divisors of $\mathbb{Z}_n$

Since a is a zero divisor element ab is a zero divisor for any b in $\mathbb{Z}_n$ and thus we can always find an α in Γ such thar $a\alpha b = 0$. Hence a is adjacent to all the other vertices except itself. Hence $deg(a) = n - 1$.

Theorem 3: The degree of all the vertices(Unit elements of $\mathbb{Z}_n$) in the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is $n - \phi(n)$.

Proof: Let m be any vertex of $\mathbb{Z}_n$ and $(m, n) = 1$ i.e m is a unit element in $\mathbb{Z}_n$.

Let $\mathbb{Z}_n = 0, 1, 2, 3, ..., n - 1$ And $\Gamma =$ set of non zero zero divisors of $\mathbb{Z}_n$

Let us take an unit element b of $\mathbb{Z}_n$ then $(mb, n) = 1$.

i.e mb is also a unit element.

Hence for m and b to be adjacent there must exist one α in Γ such that $m\alpha b = 0$ and that α must be n which does not belong to Γ.

Hence no two unit elements can be adjacent.

And if b is not a unit element of $\mathbb{Z}_n$ then $(mb, n) \neq 1$

i.e mb is a zero divisor element of $\mathbb{Z}_n$.

Therefore by previous theorem we can conclude that m(unit element of $\mathbb{Z}_n$) is adjacent only to the zero divisor elements of $\mathbb{Z}_n$.

Thus $deg(m) = n - \phi(n)$.

Corollary : Total number of edges in the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is

$$\frac{\{n-\phi(n)\}\{n+\phi(n)-1\}}{2}$$

Proof: There are $\phi(n)$ number of unit elements in $\mathbb{Z}_n$ of degree $n - \phi(n)$ and $n - \phi(n)$ number of non unit elements in $\mathbb{Z}_n$ of degree $n - 1$. Now let us consider there are q number of edges in the zero divisor graph of $\mathbb{Z}_n(\Gamma)$. Then

$2q = \{n - \phi(n)\}\phi(n) + \{n - \phi(n)\}(n - 1)$

with the help of simple calculation we can conclude

$q = \frac{\{n - \phi(n)\}\{n + \phi(n) - 1\}}{2}$

Corollary : The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is semi regular.

Proof: We can divide the vertex set into two components V_1 of unit elements having degree $n - \phi(n)$ and V_2 having degrees $n - 1$. Therefore we can conclude that the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is semi regular.

Theorem 4: The girth of zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is always 3.

Proof: Let us consider an unit element m and a non unit element b then we will always get a cycle $(0, m, b, 0)$. Hence the girth of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is always 3.

Theorem 5: The diameter of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is always two.

Proof: : Since two unit elements are not adjacent in the zero divisor graph of $\mathbb{Z}_n(\Gamma)$. Hence length between any two units elements will be largest.

Let us consider two unit elements m_1 and m_2. Again 0 is adjacent to every element of $\mathbb{Z}_n$ and therefore we will have a path $m_1, 0, m_2$ of length 2. Hence diameter of the zero divisor Graph of $\mathbb{Z}_n(\Gamma)$ is two.

For the study of the Traversability (Hamiltonian and Eulerian) of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ we will require the following result.

Result: : A Graph $G(V, E)$ is Eulerian if and only If the degree of every vertex of a graph $G(V, E)$ is even.

Theorem 6: The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is Hamiltonian when number of zero divisors in $\mathbb{Z}_n$ is greater than the no of unit elements in $\mathbb{Z}_n$ or must be one less than no of unit elements.

Proof: Suppose no of unit elements in $\mathbb{Z}_n$ is less than the number of zero divisors in $\mathbb{Z}_n$.

Let $m_1, m_2, m_3, \ldots, m_{\phi(n)}$ be the unit elements of $\mathbb{Z}_n$ and $b_1, b_2, b_3, \ldots, b_{n-\phi(n)}$, be the non unit elements of $\mathbb{Z}_n$ including 0.

Now we can construct a cycle in such a way that every vertex of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is contained in the cycle exactly once. The cycle can be constructed as follows,

$(0, m_1, b_1, m_2, b_2, \ldots, m_{\phi(n)}, b_{\phi(n)}, \ldots b_{n-\phi(n)}, 0)$. Hence zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is Hamiltonian when no of zero divisors in $\mathbb{Z}_n$ is greater than the no of unit elements in $\mathbb{Z}_n$.

If number of zero divisor of $\mathbb{Z}_n$ is one less than the number of unit elements then we can construct the cycle as:

$(m_1, b_1, m_2, b_2, m_3, b_3, \ldots\ldots\ldots m_{\phi(n)-1}, b_{\phi(n)}, m_{\phi(n)})$ containing every vertex exactly once Hence contains an Hamiltonian cycle and thus Hamiltonian.

If number of zero divisor of $\mathbb{Z}_n$ is less than the number of unit elements then we cannot construct any such cycle as two units elements are not adjacent among themselves. For example The graph of $\mathbb{Z}_9(\Gamma = \{3,6\})$ is non Hamiltonian(from fig). As no of units is 6 and no of zero divisor is 3.

Theorem 7: The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is never Eulerian.

Proof: Let us suppose n is to be even and let b be any zero divisor of $\mathbb{Z}_n$, then,

$deg(b) = n - 1$ which is odd.

Hence The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is not Eulerian when n is even.

Now let us assume that n is odd. Again no of unit elements in $\mathbb{Z}_n$ is always even for $n \geq 3$. Now let us consider an unit element m in $\mathbb{Z}_n$.

$deg(m) = n - \phi(n)$. Which is odd.

Hence we can conclude that the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is never Eulerian.

For the study of planarity of the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ we have used the following results:

Result: A graph is non planar if and only if it contains a subgraph homeomorphic to complete graph K_5 or $K_{3,3}$.

Result: If a graph $G(V,E)$ is planar then $q \leq 3p - 6$ containing p vertices and q edges.

Result: A graph $G(V,E)$ is maximal planar if the graph satisfies

$q = 3p - 6$.

Theorem 8: The zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is planar when $n = 4, 9$.

Proof: From above result we have graph Is non planar if it has a subgraph of K_5 or $K_{3,3}$ and all the vertices must have degree greater than or equal to six.

Now in the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ two zero divisor elements is always adjacent. Hence if the minimum number of zero divisor in $\mathbb{Z}_n$ is 5 then the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ will have a subgraph of K_5.Hence for the above graph to be non planar $n - \phi(n) \geq 5$. which is true for all $n \geq 20$. Hence the zero divisor graph of $\mathbb{Z}_n(\Gamma)$ is non planar for all $n \geq 20$.

Now in $n = 14, 15, 16, 18$ and 20 has no of zero divisors 6, 8, 8, 6, 8 respectively and therefore has a subgraph of K_5. Hence non planar for these values of n

For $n = 4$, we can easily produce a planar representation.

For $n = 9$, the zero divisor graph of $\mathbb{Z}_9(\Gamma = \{3, 6\})$ satisfy the above mention result hence Planar. In fact, the zero divisor graph of $\mathbb{Z}_9(\Gamma = \{3, 6\})$ is a maximal planar graph.

CONCLUSION

The graph structure G of zero divisor graph of $\mathbb{Z}_n$ associated with Γ- semigroup is an interesting field of study. In this paper we studied the different aspects of graph theoretical property such as degree of the vertices , number of edges ,planarity , girth and its traversibility.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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