Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 3866-3886
https://doi.org/10.28919/jmcs/5766
ISSN: 1927-5307

GEOCHROMATIC NUMBER OF CARTESIAN PRODUCT OF SOME GRAPHS

MEDHA ITAGI HUILGOL[1,*], B. DIVYA[2]

[1]Department of Mathematics, Bengaluru City University, Central College Campus, Bengaluru-560 001, India

[2]Department of Mathematics, Bangalore University, Central College Campus, Bengaluru-560 001, India

Abstract. In a graph G, a set $S \subseteq V(G)$ is called a geodetic set if every vertex of G lies on a shortest $u-v$ path for some $u,v \in S$, the minimum cardinality among all geodetic sets is called the geodetic number and is denoted by $g_n(G)$. A set $C \subseteq V(G)$ is called a chromatic set if C contains all vertices of different colors in G, the minimum cardinality among all chromatic sets is called the chromatic number and is denoted by $\chi(G)$. A geochromatic set $S_c \subseteq V(G)$ is both a geodetic set and a chromatic set. The geochromatic number $\chi_{gc}(G)$ of G is the minimum cardinality among all geochromatic sets of G. In this paper we determine the geochromatic number of cartesian product of standard graphs and derive general results, that prove some of the existing results on products as particular cases. Also some of the existing results are shown to be incorrect.

Keywords: geodetic number; chromatic number; geochromatic number; Cartesian product; chromatic set; geodetic set.

2010 AMS Subject Classification: 05C12, 05C99, 05C76.

1. Introduction

It is a natural phenomenon in Mathematics, especially in Graph Theory to combine two different concepts under a single umbrella and study it in contrast to the individual parameters. We see in literature that, the concepts of geodomination [6], distance domination [6], independent

*Corresponding author

E-mail address: medha@bub.ernet.in

Received March 25, 2021

domination [6], connected domination [6], biconnected domination [11], harmonious coloring [9], eccentric coloring [10], etc, dealing with more than one concept. One more such a parameter is that of geochromaticity. The geochromatic number of a graph intertwines coloring and geodeticity, the two most important, widely applied graph parameters individually, which have been studied by researchers extensively. This concept was introduced in [14] and was further studied by [15].

The geodetic number of a graph was mainly introduced to study the distance convexity, and studied in detail by numerous researchers [5], [4], [7], etc. The history of coloring and chromatic number is not new to mankind either, as it dates back to the late 19^{th} century, with the formal quoting of the famous *Four Color Theorem*. Since then, there has been a flooding of articles to prove it. But till today an elegant proof is awaited. Adding to it, many variations of coloring, chromatic polynomial, have enriched the area beyond one's imagination. Thousands of research papers speak volume of growth in the chromatic number related concepts. Several open problems, conjectures make people work on these areas till date. Major reason is that of its applicability in wide variety of fields such as computer science, communication network, scheduling problems, storage problems, placement problems, etc. find many non-graph theorists to show interest in the field.

In this paper we determine the geochromatic number of cartesian product of some standard graphs. Since the geochromatic number is a combination of geodeticity and chromaticity, it acts as a double layered measure that covers all the vertices in a graph containing all color class representations. In a real world network model, a geochromatic set acts as the minimum number of all kinds of facility (emergency service) centers to be located in such a way that every node in the network can be reached using shortest distance paths (geodesics) from these facility centers.

2. Definitions and Preliminary Results

All the terms undefined here are in the sense of Buckley and Harary [3].

Here we consider a finite graph without loops and multiple edges. For any graph G the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of G are denoted by p and q respectively.

Let u and v be vertices of a connected graph G. A shortest $u-v$ path is also called a u,v -

geodesic.

The distance between two vertices u and v is defined as the length of a u,v- geodesic in G and is denoted by $d_G(u,v)$ or $d(u,v)$ if G is clear from the context.

The eccentricity of vertex v in a graph G denoted by $ecc(v)$ is the maximum distance from v to any other vertex of G. The diameter of G, denoted by $diam(G)$ is the maximum eccentricity of vertices in G, and radius is the minimum such eccentricity denoted by $\mathrm{rad}(G)$.

Definition 2.1. [3] *A vertex v of G is a peripheral vertex if $ecc(v) = diam(G)$.*

Definition 2.2. [3] *The set of all peripheral vertices of G is called the periphery and is denoted by $P(G)$. That is, $P(G) = \{v \in V(G) : e(v) = diam(G)\}$.*

Definition 2.3. [3] *A graph G is said to be self-centered if $diam(G) = rad(G)$.*

Definition 2.4. [1] *If each vertex of a graph G has exactly one eccentric vertex, then G is called a unique eccentric vertex graph.*

Definition 2.5. [5] *The (geodesic) interval $I(u,v)$ between u and v is the set of all vertices on all shortest $u-v$ paths. Given a set $S \subseteq V(G))$, its geodetic closure $I[S]$ is the set of all vertices lying on some shortest path joining two vertices of S. Thus,*
$I[S] = \{v \in V(G) : v \in I(x,y), x,y \in S\} = \bigcup_{x,y} I(x,y)$.
A set $S \subseteq V(G)$ is called a geodetic set in G if $I[S] = V(G)$; that is every vertex in G lies on some geodesic between two vertices from S. The geodetic number $g_n(G)$ of a graph G is the minimum cardinality of a geodetic set in G.

Definition 2.6. [12] *A n-vertex coloring of G is an assignment of n colors 1,2,3,....n to the vertices of G. The coloring is proper if no two adjacent vertices have the same color.*

Definition 2.7. [12] *A set $C \subseteq V(G)$ is called chromatic set if C contains all vertices belonging to each color class. Chromatic number of G is the minimum cardinality among all the chromatic sets of G, that is, $\chi(G)$=\{ $min|C_i|/C_i$ is a chromatic set of G\}.*
If $\chi(G)$ = n, then G is said to be n-chromatic where $n \leq p$.

Definition 2.8. [14] *A set S_c of vertices in G is said to be geochromatic set, if S_c is both a geodetic set and a chromatic set. The minimum cardinality of a geochromatic set of G is its geochromatic number (GCN) and is denoted by $\chi_{gc}(G)$. A geochromatic set of size $\chi_{gc}(G)$ is said to be χ_{gc}-set.*

Definition 2.9. [4] *A vertex v in G is an extreme vertex if the subgraph induced by its neighborhood is complete.*

Definition 2.10. [5] *Let G be a graph and let $S = \{x_1, x_2, ..., x_k\}$ be a geodetic set of G, then S is a linear geodetic set if for any $x \in V(G)$ there exists an index i, $1 < i < k$ such that $x \in I[x_i, x_{i+1}]$. Examples of graphs having linear geodetic sets are odd cycles.*

Definition 2.11. [5] *Let G be a graph. If S is a geodetic set of G such that, for all $u \in V(G) \backslash S$, for all $v, w \in S : u \in I[v, w]$ then S is a complete geodetic set of G.*
Examples of graphs having complete minimum geodetic sets are paths, even cycles, complete graphs.

The following results are used in proving our results:

Theorem 2.1. [2] *Every geodetic set of a graph contains its extreme vertices.*

Theorem 2.2. [5] *If G is a non trivial connected graph of order p and diameter d, then $g_n(G) \leq p - d + 1$.*

Theorem 2.3. [1] *If every chromatic set of a graph G contains k vertices, then G has k vertices of degree at least $k - 1$.*

Theorem 2.4. [10] *Every minimum chromatic set of a graph G contains at most $(\Delta(G) + 1)$ vertices.*

Theorem 2.5. [10] *If $G = K_t$, a complete graph on t vertices, then $V(G)$ is the unique chromatic set of G.*

3. Geochromatic Number of Cartesian Product of Graphs

We establish the geochromatic number of graphs resulting from cartesian product of two graphs. The cartesian product of graphs is one of the fundamental types of graph products. We first give the definitions and preliminary results pertaining to cartesian products on chromaticity and geodeticity, then get the results on geochromatic number.

Definition 3.1. [8] *The cartesian product $G\Box H$ of graphs G and H is the graph with vertex set $V(G)\times V(H)$ in which vertices (g,h) and (g',h') are adjacent whenever $gg'\in E(G)$ and $h=h'$ or $g=g'$ and $hh'\in E(H)$.*

By [13] the most important metric property of the cartesian product operation can be written for any two graphs G and H as
$d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(h,h')$.

Theorem 3.1. [8] *For any two graphs G and H, $\chi(G\Box H)$ = $max\{\chi(G),\chi(H)\}$.*

Remark 1. *In the cartesian product color assignment is given as follows, whenever $\chi(G)\geq\chi(H)$, let $g:V(G)\to\{0,1,...,\chi(G)-1\}$ be a coloring of G and $h:V(H)\to\{0,1,...,\chi(H)-1\}$ be a coloring of H. Hence a color assignment can be given for $G\Box H$ as $f:V(G\Box H)\to\{0,1,...,\chi(G)-1\}$, defined by $f(a,x)=g(a)+h(x)(mod \quad \chi(G))$.*

Theorem 3.2. [13] *Let $X=G\Box H$ be the cartesian product of connected graphs G and H and let (g,h), (g',h') be vertices of X then, $I_X[(g,h),(g',h')] = I_G[(g,g')]\times I_H[(h,h')]$. Moreover, $I_X[(g,h),(g',h')] = I_X[(g',h),(g,h')]$.*

Theorem 3.3. [2] *For any graphs G and H, $g_n(G)=m\geq g_n(H)=n\geq 2$, then $m\leq g_n(G\Box H)\leq mn-n$.*

Theorem 3.4. [2] *Let G and H be graphs on at least two vertices with $g_n(G)=m$ and let $g_n(H)=n$. Suppose that both G and H contain linear minimum geodetic sets, then $g_n(G\Box H)\leq\left\lfloor\frac{mn}{2}\right\rfloor$.*

Theorem 3.5. [5] *Let G be a graph on at least two vertices that admits a linear minimum geodetic set and let H be a graph with $g_n(H)=2$, then $g_n(G\Box H)=g_n(G)$.*

Theorem 3.6. [2] *Let G and H be non trivial graphs, both being non trivial graphs having complete minimum geodetic sets. Let H be a graph with $g_n(H) = 2$ then $g_n(G\Box H) = max\{g_n(G), g_n(H)\}$.*

Theorem 3.7. [2] *For any two trees T_1 and T_2, $g_n(T_1\Box T_2) = max\{g_n(T_1), g_n(T_2)\}$.*

Now we prove results on cartesian product of graphs and establish geochromatic number. First one for the simplest grid graphs.

Theorem 3.8. *For the cartesian product of two paths, that is, the grid graphs, the geochromatic number is given by,*

$$\chi_{gc}(P_m\Box P_n) = \begin{cases} 2, & \text{for } m \neq n, \text{ and one of } m \text{ or } n \text{ is even,} \\ 3, & \text{for } m = n, \text{ and for } m \neq n, \text{ with both } m \text{ and } n \text{ odd or both even.} \end{cases}$$

Proof. We know that the cartesian product of two paths is the grid graph. Let G denote $P_m\Box P_n$. By the structure of G, it is clear that it has mn vertices and $2mn - m - n$ edges, with $\Delta(G) = 4$ and $\delta(G) = 2$. And there are exactly four vertices of degree 2 (the corner vertices) and the remaining of degree 3 and 4.

By Theorem 3.1 [8], we know that $\chi(P_m\Box P_n) = 2$. From Theorem 3.7 [2], we have $g_n(P_m\Box P_n) = 2$. Now to find the geochromatic number and sets, we use the similar notation as in [2] and get them depending on the values of m and n. We label the vertices of P_m by $\{a_1, a_2, ..., a_m\}$ and vertices of P_n by $\{b_1, b_2, ..., b_n\}$.

Case 1. $m \neq n$, and one of them is odd.

In this case we get two possible geodetic bases $\{(a_1, b_1), (a_m, b_n)\}$ or $\{(a_1, b_n), (a_m, b_1)\}$ as P_m and P_n have complete minimum geodetic sets. We know that the periphery $P(G)$ is $\{(a_1, b_1), (a_m, b_n)\}$ or $\{(a_1, b_n), (a_m, b_1)\}$. And hence the pair of vertices forming the geodetic bases are at diameter distance. Since m or n is odd, G is bipartite and hence bi-colorable. Also the pairs of vertices in each of the geodetic bases belongs to different color classes, as diameter(G) is odd, thereby forming a chromatic set also, proving $\chi_{gc}(P_m\Box P_n) = 2$, with the same geodetic and chromatic sets.

Case 2. $m = n$ or $m \neq n$, both m and n either odd or even.

As in Case 1, here also we get the geodetic bases as $\{(a_1, b_1), (a_m, b_n)\}$ or $\{(a_1, b_n),$

$(a_m, b_{1)})\}$. For the values of m and n chosen, we see that the diameter of G is even. Hence the vertices of each geodetic base lie in the same color class, not resulting in a chromatic set. As G is bi-colorable, we see that one color class representation will be missed in both the geodetic bases. To get the same, we pick a vertex belonging to the other color class to make a chromatic set. Hence $\chi_{gc}(P_m \Box P_n) = 3$.

Remark 2. *In* [15]*, the authors have claimed that* $\chi_{gc}(P_2 \Box P_n) = 2$ *and* 4*, for the ladder graph, for all n. But by referring to the above result, we see that it is a particular case of the above theorem. Therefore, the result proved by* [15] *stands incorrect, because depending on the value of n, whether it is odd or even,* $\chi_{gc}(P_2 \Box P_n) = 2$ *or* 3 *respectively. Hence we get the following corollary.*

Corollary 3.1. *For the ladder graph* $P_2 \Box P_n$*,* $\chi_{gc}(P_2 \Box P_n) = \begin{cases} 2, & \text{for } n \text{ odd,} \\ 3, & \text{for } n \text{ even.} \end{cases}$

□

Next result deals with one of the famous class of graphs which arise from cartesian product of K_2's, namely the hypercubes.

Theorem 3.9. *For the hypercube of dimension* n*, the geochromatic number is given by*

$$\chi_{gc}(Q_n) = \begin{cases} 2, & \text{for } n \text{ even,} \\ 3, & \text{for } n \text{ odd.} \end{cases}$$

Proof. It is clear that $\chi(Q_n) = 2$, as Q_n is bipartite. Also $g_n(G) = 2$, since Q_n is a unique eccentric vertex graph in which the geodetic base consists of a vertex and its eccentric vertex. We can get geochromatic sets with two different cases, based on the value of n.

Case 1. If n is odd.

Whenever n is odd, the geodetic base consists of the vertices of different color classes as they are at odd distance, making a chromatic set also. Therefore $\chi_{gc}(Q_n) = 2$.

Case 2. If n is even.

Whenever n is even, the diameter is even, hence a geodetic base consists of vertices of the same color class not making a chromatic set. And no pair of vertices belonging to both the color classes make a geodetic set. Hence, we add a vertex from the second color class to a geodetic base, to get a geochromatic set. Therefore, $\chi_{gc}(Q_n) = 3$. $\square$

Theorem 3.10. *For the cartesian product of cycle C_m with path P_n, the geochromatic number is given by, $\chi_{gc}(C_m \square P_n) = 2$ or 3.*

Proof. We label the vertices of C_m by $\{a_1, a_2, a_3, ..., a_m\}$ and the vertices of P_n by $\{b_1, b_2, b_3, ..., b_n\}$. The cycle C_m is bi-colorable, if m is even, containing complete minimum geodetic set and it is 3 colorable if m is odd, containing linear minimum geodetic set. P_n is bi-colorable having a complete minimum geodetic set. The cartesian product $C_m \square P_n$ has $\Delta(G) = 4$ and $\delta(G) = 3$. We find the geochromatic set of $C_m \square P_n$ based on the values of m and n.

Case 1: For $m = 2k$, $k \geq 2$.

By using Theorem 3.1 [8], we get $\chi(C_{2k} \square P_n) = 2$, as both are bipartite. The complete minimum geodetic sets are $S = \{a_i, a_{i+k}\}$, where a_{i+k} is the eccentric vertex of a_i, for $1 \leq i \leq k$ and $T = \{b_1, b_n\}$, respectively. Then, $S \times T = \{(a_i, b_1), (a_{i+k}, b_n)\}$ or $\{(a_i, b_n), (a_{i+k}, b_1)\}$ is the geodetic set of $C_{2k} \square P_n$. Since C_{2k} is self-centered, unique eccentric vertex graph, all vertices are eccentric vertices and $diam(C_{2k}) = d(a_i, a_{i+k}) = k$.

Referring to Remark 1 the color assignments are given by mappings $g : V(C_{2k}) \to \{0, 1\}$ and $h : V(P_n) \to \{0, 1\}$. Hence $f : V(C_{2k} \square P_n) \to \{0, 1\}$, defined by $f(a, x) = g(a) + h(x) (mod \quad 2)$. The color pattern can be written as follows:

$$g(a_i) = \begin{cases} 0, & \text{for all odd } i, \\ 1, & \text{for all even } i. \end{cases}$$

$$h(a_i) = \begin{cases} 0, & \text{for all odd } i, \\ 1, & \text{for all even } i. \end{cases}$$

The color assignment for the vertices of $S \times T$ can be obtained by using the distances as follows, $d((a_i, b_1), (a_j, b_n)) = d(a_i, a_{i+k}) + d(b_1, b_n) = k + n - 1$. Here we get two sub cases to find the geochromatic set.

Subcase (i): If $d((a_i, b_1), (a_{i+k}, b_n))$ or $d((a_i, b_n), (a_{i+k}, b_1))$ is odd, then the geodetic pair of vertices lie in both color classes, making it a chromatic set too. This happens if the following conditions are satisfied by the values i, j, k and n:

(i) k even, i odd, n even; since and $f(a_i,b_1) = g(a_i)+h(b_1) = 0+0 = 0$ and $f(a_{i+k},b_n) = g(a_{i+k})+h(b_n) = 0+1 = 1$.

(ii) k even, i even, n even; since, and $f(a_i,b_1) = 1$, $f(a_{i+k},b_n) = 0$.

(iii) k odd, i odd, n odd; since, and $f(a_i,b_1) = 0$, $f(a_{i+k},b_n) = 1$.

(iv) k odd, i even, n even; since, and $f(a_i,b_1) = 1$ and $f(a_{i+k},b_n) = 0$.

Hence, in all the above conditions $\chi_{gc}(C_{2k}\Box P_n) = 2$.

Subcase (ii): If $d((a_i,b_1),(a_{i+k},b_n))$ or $d((a_i,b_n),(a_{i+k},b_1))$ is even, then these geodetic pair of vertices lie in the same color class, thereby not forming a chromatic set. This happens if the following conditions are satisfied by the values i, j, k and n:

(i) k even, i odd, n odd; since, and $f(a_i,b_1) = g(a_i)+h(b_1) = 0+0 = 0$ and $f(a_{i+k},b_1) = g(a_{i+k})+h(b_n) = 0+0 = 0$.

(ii) k even, i even, n odd; since, and $f(a_i,b_1) = 1$ and $f(a_{i+k},b_n) = 1$.

(iii) k odd, i odd, n even; since, and $f(a_i,b_1) = 0$ and $f(a_{i+k},b_n) = 0$.

(iv) k odd, i even, n odd; since, and $f(a_i,b_1) = 1$ and $f(a_{i+k},b_n) = 1$.

Hence we add a vertex from the missed color class to make it geochromatic set. Therefore, in all the above cases $\chi_{gc}(C_{2k}\Box P_n) = 3$.

Case 2 : For $m = 2k+1$, $k \geq 1$.

By using Theorem 3.1 [8], we have $\chi(C_m\Box P_n) = 3$. The coloring pattern is defined as follows:

$$g(a_i) = \begin{cases} 0, \text{ for all odd } i, \text{ and } i \neq m, \\ 1, \quad \text{for all even } i, \\ 2, \text{ for } i = m. \end{cases}$$

$$h(b_{i'}) = \begin{cases} 0, \text{ for all odd } i', i' \neq n, \\ 1, \text{for all even } i' \text{and } i' = n. \end{cases}$$

We know that, for $m = 2k+1$, the odd cycle has a geodetic set $S = \{a_i, a_{i+1}, a_{i+k+1}\}$ consisting of three vertices, a vertex a_i and its two eccentric vertices a_{i+k} and a_{i+k+1}, such that $d(a_i,a_{i+k}) = d(a_i,a_{i+k+1}) = k$. But the geodetic set T of P_n remains unchanged as $T = \{b_1,b_n\}$. Then, the geodetic set of $C_{2k+1}\Box P_n$ is given by $\{(a_i,b_1),(a_{i+k},b_n),(a_{i+k+1},b_n)\}$ or $\{(a_i,b_n),(a_{i+k},b_1),(a_{i+k+1},b_1)\}$. Hence $g_n(C_{2k+1}\Box P_n) =$

3 and the $diam(C_{2k+1}\Box P_n) = k+(n-1)$. Now we check whether the geodetic sets formed are chromatic sets or not, based on the values of i, k, n.

Subcase (i): Suppose $d((a_i,b_1),(a_{i+k},b_n))$ and $d((a_i,b_1),(a_{i+k+1},b_n))$ or $d((a_i,b_n),(a_{i+k},b_1))$ and $d((a_i,b_n),(a_{i+k+1},b_1))$ are odd, that is the diameter $k+n-1$ is odd. This means that the following cases arise depending on the values of i and in each case we show that the color assignment is at most 3, so as to prove the geochromatic number to be 3.

(i) For k even, n even, i odd and $i=m$, we have, $f(a_i,b_1) = g(a_i)+h(b_1) = 2; f(a_{(i+k)},b_n) = g(a_{(i+k)})+h(b_n) = 1; f(a_{(i+k+1)},b_n) = g(a_{(i+k+1)})+h(b_n) = 2$.

(ii) k **even,** n **even,** i **odd and** $i \neq m$, we have, $f(a_i,b_1) = 0, f(a_{i+k},b_n) = 1, f(a_{i+k+1},b_n) = 2$.

(iii) k even, n even, i even, we have, $f(a_i,b_1) = 1, f(a_{i+k},b_n) = 2, f(a_{i+k+1},b_n) = 1$.

(iv) k odd, n odd, i odd and $i = m$, we have, $f(a_i,b_1) = 2 = f(a_{i+k},b_n)$, $f(a_{i+k+1},b_n) = 0$.

(v) k **odd,** n **odd,** i **odd** $i \neq m$, we get, $f(a_i,b_1) = 0, f(a_{i+k},b_n) = 1$, $f(a_{i+k+1},b_n) = 2$.

(vi) k odd, n odd, i even, we get, $f(a_i,b_1) = 1, f(a_{i+k},b_n) = 0, f(a_{i+k+1},b_n) = 1$.

Observing all the above cases we see that all sets are not geochromatic sets. Hence, we consider the sets containing all color classes. So we get only two chromatic sets (shown in bold, above) to get $\chi_{gc}(C_{2k+1}\Box P_n) = 3$.

Subcase (ii): If $k+n-1$ is even, as in the above case, we get the following depending on i.

(i) For k even, n odd, i odd and $i=m$, we have, $f(a_i,b_1) = 2 = f(a_{i+k},b_n)$, $f(a_{i+k+1},b_n) = 1$.

(ii) k **even,** n **odd,** i **even and** $i \neq m$, we have $f(a_i,b_1) = 2, f(a_{i+k},b_n) = 1, f(a_{i+k+1},b_n) = 0$.

(iii) k even, n odd, i even, $f(a_i,b_1) = 1, f(a_{i+k},b_n) = 1, f(a_{i+k+1},b_n) = 0$.

(iv) k odd, n even, i odd and $i=m$, we get $f(a_i,b_1) = 2 = f(a_{i+k},b_n)$, $f(a_{i+k+1},b_n) = 1$.

(v) k odd, n even, i odd, $i \neq m$, we get, $f(a_i,b_1) = 2 = f(a_{i+k},b_n)$, $f(a_{i+k+1},b_n) = 1$.

(vi) k even, n even, i even we get, $f(a_i,b_1) = 1 = f(a_{i+k},b_n), f(a_{i+k+1},b_n) = 2$.

Here also all sets are not geochromatic sets, we consider the sets containing all color classes for minimality. Hence in this case we get $\chi_{gc}(C_{2k+1}\Box P_n) = 3$.

$\square$

Theorem 3.11. *For the cartesian product of cycle C_m with cycle C_n the geochromatic number is given by, $\chi_{gc}(C_m\Box C_n) = 2, 3$ or 5.*

Proof. We label the vertices of C_m by $V(C_m) = \{a_1,a_2,a_3,...,a_m\}$ and the vertices of C_n by $V(C_n) = \{b_1,b_2,b_3,...,b_n\}$. It is clear that $C_m\Box C_n$ is a 4 regular graph, self-centered graph. The geochromatic sets can be obtained depending on the values of m and n as follows:

Case 1. Both m and n are even.

Let $m = 2k$, $k \geq 2$ and $n = 2l, l \geq 2$ using Theorem 3.1 [8], we get the $\chi(C_{2k}\Box C_{2l}) = 2$, as both are bipartite. Using Theorem 3.7 [2], we get $g_n(C_{2k}\Box C_{2l}) = 2$, as both C_{2k} and C_{2l} have complete minimum geodetic sets.

Let the geodetic base of C_{2k} be denoted as $S = \{a_i,a_{i+k}\}$, where a_{i+k} is the eccentric vertex of a_i as $diam(C_{2k}) = k$. Similarly, $T = \{b_{i'},b_{i'+l}\}$ where $b_{i'+l}$ is the eccentric vertex of $b_{i'}$ as $diam(C_{2k}) = l$, is the geodetic base of C_{2l}.

Now a geodetic base of $C_{2k}\Box C_{2l}$ is of the form $\{(a_i,b_{i'}),(a_{i+k},b_{i'+l})\}$ or $\{(a_i,b_{i'+l}),(a_{i+k},b_{i'})\}$ with $d((a_i,b_{i'}),(a_{i+k},b_{i'+l})) = d((a_i,a_{i+k}) + d(b_{i'},b_{i'+})) = d((a_i,b_{i'+l}),(a_{i+k},b_{i'+l})) = k+l$. The color assignment is as follows to find the chromatic number as well as chromatic sets:

$$g(a_i) = \begin{cases} 0, \text{ for all odd } i, \\ 1, \text{for all even } i. \end{cases}$$

$$h(b_{i'}) = \begin{cases} 0, \text{ for all odd } i', \\ 1, \text{for all even } i'. \end{cases}$$

Subcase (i): If $d((a_i,b_{i'}),\ (a_{i+k},b_{i'+l}))$ or $d((a_i,b_{i'+l}),\ (a_{i+k},b_{i'}))$, that is, the $diam(C_{2k}\Box C_{2l}) = k+l$ is odd, then the geodetic pair of vertices lie in both color classes, making it a chromatic set too. This happens if the following conditions are satisfied by i, i', k, l.

(i) For k even, l odd, i odd, i' odd, we have, $f(a_i,b_{i'}) = g(a_i) + h(b_{i'})(mod \quad 2) = 0$, $f(a_{i+k},b_{i'+l})$ $= g(a_{i+k}) + h(b_{i'+l})(mod \quad 2) = 1$.

(ii) For k even, l odd, i odd, i' even, we have, $f(a_i,b_{i'}) = 1$, $f(a_{i+k},b_{i'+l}) = 0$.

(iii) For k even, l odd, i even, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = 1$.

(iv) For k even, l odd, i odd, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 0$.

Similar argument holds good if k is odd, l is even. Hence in this case we get $\chi_{gc}(C_m \Box C_n) = 2$.

If $d((a_i, b_{i^{'}}), (a_{i+k}, b_{i^{'}+l}))$ or $d((a_i, b_{i^{'}+l}), (a_{i+k}, b_{i^{'}}))$ is even, then we get two more subcases, one in which both k and l are even and the other in which both k and l are odd.

Subcase (ii): Both k and l even the following conditions are satisfied by i, $i^{'}$, k, l.

(i) For k even, l even, i odd, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = g(a_i) + h(b_i^{'})(mod \quad 2) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = g(a_{i+k}) + h(b_{i^{'}+l})(mod \quad 2) = 0$.

(ii) For k even, l even, i odd, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 1$.

(iii) For k even, l even, i even, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 1$.

(iv) For k even, l even, i even, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = 0$.

Hence the geodetic set is not a chromatic set, as geodetic sets contain the same color class vertices. Hence, to get representation from both color classes we add a vertex from the other color class, and to get $\chi_{gc}(C_m \Box C_n) = 3$.

Subcase (iii): Both k and l odd, the following conditions are satisfied by i, $i^{'}$

(i) For i and $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = g(a_i) + h(b_i^{'})(mod \quad 2) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = g(a_{i+k}) + h(b_{i^{'}+l})(mod \quad 2) = 0$.

(ii) For i odd, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 1$.

(iii) For i even, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = 0$.

(iv) For i even, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 1$.

From these cases it is clear that the geodetic sets do not form chromatic sets, since for any value of i, $i^{'}$, k, l we do not get both color class representation. To form a geochromatic set, we add a vertex from the missing color class, therefore, we get $\chi_{gc}(C_m \Box C_n) = 3$.

Case 2. m odd and n even.

Let $m = 2k+1$, $k \geq 1$ and $n = 2l, l \geq 2$. Again by using Theorem 3.1 [8], we have $\chi_{gc}(C_{2k+1} \Box C_{2l}) = 3$ and the color assignment is as follows:

$$g(a_i) = \begin{cases} 0, \text{ for all odd } i, \text{ and } i \neq m, \\ 1, \text{ for all even } i, \\ 2, \text{ for } i = m. \end{cases}$$

$$h(b_{i'}) = \begin{cases} 0, \text{ for all odd } i', \\ 1, \text{ for all even } i'. \end{cases}$$

In C_{2k+1}, the geodetic set is $S = \{a_i, a_{i+k}, a_{i+k+1}\}$, as $g_n(C_{2k+1}) = 3$ and in C_{2l} we get $T = \{b_i, b_{i'+l}\}$ as the geodetic set. By using Theorem 3.7 [2] we get $g_n(C_{2k+1}\Box C_{2l}) = max\{g_n(C_{2k+1}), g_n(C_{2l})\} = 3$. The geodetic set of $C_{2k+1}\Box C_{2l}$ is given by $\{(a_i, b_{i'}), (a_{i+k}, b_{i'+l}), (a_{i+k+1}, b_{i'+l})\}$ or $\{(a_i, b_{i'+l}), (a_{i+k}, b_{i'}), (a_{i+k+1}, b_{i'})\}$ with $d((a_i, b_{i'}), (a_{i+k}, b_{i'+l})) = d(a_i, a_{i+k}) + d(b_{i'}, b_{i'+l}) = d((a_i, a_{i+k+1}) + d(b_{i'}, b_{i'+l})) = k + l$.

Here also subcases arise depending on the parity of k and l.

Subcase (i): If $k+l$ is odd then either k is even and l is odd or vice versa.

First let us consider k even, l odd, then values of i, i' give rise to the following.

(i) **For i odd and $i \neq m$, i' odd**, we have, $f(a_i, b_{i'}) = g(a_i) + h(b_{i'}) (mod \quad 2) = 0$, $f(a_{i+k}, b_{i'+l}) = g(a_{i+k}) + h(b_{i'+l}) (mod \quad 2) = 1$, $f(a_{i+k+1}, b_{i'+l}) = g(a_{i+k+1}) + h(b_{i'+l}) (mod \quad 2) = 2$.

(ii) For i odd $i = m$, i' odd, we have, $f(a_i, b_{i'}) = 2$, $f(a_{i+k}, b_{i'+l}) = 1$, $f(a_{i+k+1}, b_{i'+l}) = 2 = f(a_{i+k+1}, b_{i'+l})$.

(iii) For i odd $i \neq m$, i' even, we have, $f(a_i, b_{i'}) = f(a_{i+k+1}, b_{i'+l}) = 1$, $f(a_{i+k}, b_{i'+1}) = 0$.

(iv) For i odd $i = m$, i' even, we have, $f(a_i, b_{i'}) = 0 = f(a_{i+k}, b_{i'+l})$, $f(a_{i+k+1}, b_{i'+l}) = 1$.

(v) For i even i' even, we have, $f(a_i, b_{i'}) = 2$, $f(a_{i+k}, b_{i'+l}) = 1$, $f(a_{i+k+1}, b_{i'+l}) = 0$.

(vi) For i even i' odd, we have, $f(a_i, b_{i'}) = 1$, $f(a_{i+k}, b_{i'+l}) = 2$, $f(a_{i+k+1}, b_{i'+l}) = 1$.

Hence in two of the above cases a geodetic set is a chromatic set to give $\chi_{gc}(C_{2k+1}\Box C_{2l}) = 3$. Similar argument holds when k is odd and l is even.

If $k+l$ is even, then we get again two more subcases, k and l both even and k and l both odd.

Subcase (ii): Both k and l are even, we have the following conditions for i, i'

(i) For i odd and $i \neq m$, i' odd, we have, $f(a_i, b_{i'}) = g(a_i) + h(b_{i'}) (mod \quad 2) = 0$, $f(a_{i+k}, b_{i'+l}) = g(a_{i+k}) + h(b_{i'+l}) (mod \quad 2) = 0$, $f(a_{i+k+1}, b_{i'+l}) = g(a_{i+k+1} + h(b_{i'+l}) (mod \quad 2) = 1$.

(ii) For i odd, $i = m$, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 2 = f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 1$.

(iii) For i odd, $i \neq m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}})$=1 $= f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 2$.

(iv) For i odd, $i = m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}})$ =0 $= f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 2$.

(v) For i even, $i^{'}$ even, we have, $f(a_i, b_{i^{'}})$=2 $= f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 1$.

(vi) For i even $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 1$ =$f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 0$.

Here no geodetic set is a chromatic set. To make it a chromatic set we need to add one more vertex of missed color class. But, observing all the conditions for $k, l, i, i^{'}$, such an assignment results in a chromatic set having at least 4 vertices. Hence for the values chosen a geochromatic set of minimum cardinality is not possible.

Subcase (iii): Both k and l odd, we have the following conditions for i, $i^{'}$.

(i) **For i odd and $i \neq m$, $i^{'}$ odd**, we have, $f(a_i, b_{i^{'}})$ =$g(a_i) + h(b_{i^{'}})(mod \quad 2) = 0$, $f(a_{i+k}, b_{i^{'}+l})$ =$g(a_{i+k}) + h(b_{i^{'}+l})(mod \quad 2) = 2$, $f(a_{i+k+1}, b_{i^{'}+l})$
= $g(a_{i+k+1}) + h(b_{i^{'}+l})(mod \quad 2) = 1$.

(ii) For i odd, $i = m$, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 2$ =$f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 0$.

(iii) For i odd, $i \neq m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}})$=1 $= f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 0$.

(iv) For i odd, $i = m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 0$, $f(a_{i+k}, b_{i^{'}+l}) = 1$, $f(a_{i+k+1}, b_{i^{'}+l}) = 2$**.**

(v) For i even, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 2$, $f(a_{i+k}, b_{i^{'}+l})$= 1 =$f(a_{i+k+1}, b_{i^{'}+l})$.

(vi) For i even, $i^{'}$ odd, we have, $f(a_i, b_{i^{'}}) = 1$ =$f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l}) = 2$.

By choosing a geodetic set containing all three color classes (in bold) we get a geochromatic set $\chi_{gc}(C_m \square C_n) = 3$.

Case 3: If m is even and n odd.

The analysis is similar to Case 2, hence we can prove $\chi_{gc}(C_m \square C_n) = 3 = \chi_g c(C_{2k} \square C_{2l+1})$.

Case 4: Both m and n odd.

By using [8], the coloring pattern is given by,

$$g(a_i) = h(b_i) = \begin{cases} 0, \text{ for all odd } i, \text{ and } i \neq m, \text{ and for all } i^{'}, i^{'} \neq n, \\ 1, \text{ for all even } i \text{ and } i^{'}, \\ 2, \text{ for } i = m, i \neq n. \end{cases}$$

By using [2] for any two graphs G and H, with $g_n(G) = k_1 \geq g_n(H) = k_2 \geq 2$, then we get $g_n(G\Box H) \leq k_1k_2 - k_2$. Since we have $G \approx C_m$ and $H \approx C_n$ where both C_m and C_n have linear minimum geodetic sets, we get the geodetic number as $g_n(C_m\Box C_n) \leq 3.3 - 3 = 6$. Hence $g_n(C_m\Box C_n) \leq 6$.

In C_{2k+1}, the geodetic set is $S = \{a_i, a_{i+k}, a_{i+k+1}\}$, as $g_n(C_{2k+1}) = 3$, where a_i is any arbitrary vertex and a_{i+k} and a_{i+k+1} are its two eccentric vertices. Similarly, in C_{2l+1} we have $T = \{b_{i'}, b_{i'+l}, b_{i'+l+1}\}$ as a geodetic set. Then by using Theorem 3.7 [2], the geodetic set of $C_{2k+1}\Box C_{2l+1}$ is given by $S\times T = \{(a_i,b_{i'}), (a_{i+k},b_{i'+l}), (a_{i+k},b_{i'+l+1}), (a_{i+k+1},b_{i'+l+1}), (a_{i+k+1},b_{i'+l+1})\}$ with $d((a_i,b_{i'}), (a_{i+k},b_{i'+l})) = k + l = d((a_i,b_{i'}), (a_{i+k+1},b_{i'+l+1})) = d((a_i,b_{i'}), (a_{i+k},b_{i'+l+1})) = d((a_i,b_{i'}), (a_{i+k+1},b_{i'+l}))$. Here also subcases arise depending on the parity of k and l.

Subcase (i): If $k+l$ is odd, then either k is even and l is odd or vice versa. First let us consider k even, l odd, then values of i, i' give rise to the following.

(i) For i odd and $i \neq m$, i' odd, $i' \neq n$, we have, $f(a_i,b_{i'}) = g(a_i)+h(b_{i'}) = 0$, $f(a_{i+k},b_{i'+l}) = g(a_{i+k})+h(b_{i'+l}) = 2$, $f(a_{i+k},b_{i'+l+1}) = g(a_{i+k})+h(b_{i'+l+1}) = 0$, $f(a_{i+k+1},b_{i'+l}) = g(a_{i+k+1})+h(b_{i'+l}) = 2$, $f(a_{i+k+1},b_{i'+l+1}) = g(a_{i+k+1})+h(b_{i'+l+1}) = 1$.

(ii) For i odd and $i \neq m$, i' odd, $i' = n$, we have, $f(a_i,b_{i'}) = 2$, $f(a_{i+k},b_{i'+l}) = 1$, $f(a_{i+k},b_{i'+l+1}) = 2 = f(a_{i+k+1},b_{i'+l}), f(a_{i+k+1},b_{i'+l+1}) = 0$.

(iii) For i odd and $i = m$, i' odd and $i' \neq n$, we have, $f(a_i,b_{i'}) = 2 = f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k},b_{i'+l}) = 0 = f(a_{i+k},b_{i'+l+1})$, $f(a_{i+k+1},b_{i'+l+1}) = 1$.

(iv) For i odd and $i = m$, i' odd and $i' = n$, we have, $f(a_i,b_{i'}) = 1 = f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l+1}) = 1$, $f(a_{i+k+1},b_{i+l}) = 2$, $f(a_{i+k+1},b_{i'+l+1}) = 0$.

(v) For i odd and $i \neq m$, i' even, we have $f(a_i,b_{i'}) = 0 = f(a_{i+k},b_{i'+l}) = f(a_{i+k},b_{i'+l+1})$, $f(a_{i+k+1},b_{i'+l}) = 1$, $f(a_{i+k+1},b_{i'+l+1}) = 2$.

(vi) For i odd and $i = m$, i' even, we have $f(a_i,b_{i'}) = 0$, $(a_{i+k+1},b_{i'+l}) = 1$, $f(a_{i+k},b_{i'+l}) = 2 = f(a_{i+k},b_{i'+l+1}) = f(a_{i+k+1},b_{i'+l+1})$.

(vii) For i even and i' odd, $i' \neq n$, we have, $f(a_{i+k+1},b_{i'+l+1}) = 0$, $f(a_i,b_{i'}) = 1 = f(a_{i+k}, b_{i'+l+1}) = f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k},b_{i'+l}) = 2$.

(viii) For i even and $i^{'}$ odd, $i^{'}=n$, we have, $f(a_i,b_{i'})=0$, $f(a_{i+k},b_{i'+l+1})$ =1 $=f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k+1},b_{i'+l+1})=f(a_{i+k},b_{i'+l})=2$.

(ix) For i even and $i^{'}$ even, we have, $f(a_i,b_{i'})$=2 $=f(a_{i+k},b_{i'+l+1})$, $f(a_{i+k+1},b_{i'+l})=0$, $f(a_{i+k+1},b_{i'+l+1})$=1 $=f(a_{i+k},b_{i'+l})$.

Subcase (ii): If $k+l$ is even with k even and l even.

(i) For i odd and $i\neq m$, $i^{'}$ odd, $i^{'}\neq n$, we have $f(a_i,b_{i'})$=0$=f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l+1})$ =1 $=f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k+1},b_{i'+l+1})=2$.

(ii) For i odd and $i\neq m$, $i^{'}$ odd and $i^{'}=n$, we have, $f(a_i,b_{i'})$ =2 $=f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l+1})=1$, $f(a_{i+k+1},b_{i'+l})=0$, $f(a_{i+k+1},b_{i'+l+1})=2$.

(iii) For i odd and $i=m$, $i^{'}$ odd and $i^{'}\neq n$, we have, $f(a_{i+k},b_{i'+l+1})=0$, $f(a_{i+k+1},b_{i'+l})=1$, $f(a_i,b_{i'})$=2=$f(a_{i+k},b_{i'+l})=f(a_{i+k+1},b_{i'+l+1})$.

(iv) For i odd and $i=m,i^{'}$ odd and $i^{'}=n$, we have $f(a_i,b_{i'})$ =1 =$f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l})$ = 0 =$f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k+1},b_{i'+l+1})=2$.

(v) For i odd and $i\neq m,i^{'}$ even, we have, $f(a_i,b_{i'})$ =1 =$f(a_{i+k},b_{i'+l})=f(a_{i+k+1},b_{i'+l+1})$, $f(a_{i+k},b_{i'+l+1})=0$, $f(a_{i+k+1},b_{i'+l})=2$.

(vi) For i odd and $i=m$, $i^{'}$ even, we have, $f(a_i,b_{i'})=f(a_{i+k},b_{i'+l})=0$, $f(a_{i+k+1},b_{i'+l})=1$ $=f(a_{i+k+1},b_{i'+l+1})$, $f(a_{i+k},b_{i'+l+1})=2$.

(vii) For i even and $i^{'}$ odd, $\neq n$, we have, $f(a_{i+k+1},b_{i'+l})=0$, $f(a_{i+k},b_{i'+l+1})=2$, $f(a_i,b_{i'})$ $=1=f(a_{i+k},b_{i'+l})=f(a_{i+k+1},b_{i'+l+1})$.

(viii) For i even and $i^{'}$ odd, $i^{'}=n$, we have, $f(a_i,b_{i'})=0=f(a_{i+k},b_{i'+1})$, $f(a_{i+k},b_{i'+l+1})=2$ $=f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k+1},b_{i'+l+1})=1$.

(ix) For i even and $i^{'}$even, we have, $f(a_i,b_{i'})=2=f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l+1})=1$, $f(a_{i+k+1},b_{i'+l})=1$, $f(a_{i+k+1},b_{i'+l+1})=0$.

Subcase (iii): If $k+l$ is even with k odd and l odd.

(i) For i odd and $i\neq m$, $i^{'}$ odd, $i^{'}\neq n$, we have, $f(a_i,b_{i'})=0=f(a_{i+k+1)},b_{i'+l+1})$, $f(a_{i+k},b_{i'+l+1})$ $=1=f(a_{i+k+1},b_{i'+l})$, $f(a_{i+k},b_{i'+l})=2$.

(ii) For i odd and $i\neq m,i^{'}$ odd and $i^{'}=n$, we have, $f(a_i,b_{i'})=2=f(a_{i+k},b_{i'+l})$, $f(a_{i+k},b_{i'+l+1})=$ 0, $f(a_{i+k+1},b_{i'+l})=1$, $f(a_{i+k+1},b_{i'+l+1})=2$.

(iii) For i odd and $i = m, i^{'}$ odd and $i^{'} \neq n$, we have, $f(a_i, b_{i^{'}}) = 0 = f(a_{i+k+1}, b_{i^{'}+l})$, $f(a_{i+k}, b_{i^{'}+l+1}) = 1$, $f(a_{i+k}, b_{i^{'}+l}) = 2 = f(a_{i+k+1}, b_{i^{'}+l+1})$.

(iv) For i odd and $i = m$, $i^{'}$ odd and $i^{'} = n$, we have, $f(a_{i+k}, b_{i^{'}+l}) = 2$, $f(a_i, b_{i^{'}}) = 0 = f(a_{i+k}, b_{i^{'}+l+1}) = f(a_{i+k+1}, b_{i^{'}+l})$, $f(a_{i+k+1}, b_{i^{'}+l+1}) = 1$.

(v) For i odd and $i \neq m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 1 = f(a_{i+k+1}, b_{i^{'}+l+1})$, $f(a_{i+k}, b_{i^{'}+l}) = 0 = f(a_{i+k+1}, b_{i^{'}+l})$, $f(a_{i+k}, b_{i^{'}+l+1}) = 2$.

(vi) For i odd and $i = m$, $i^{'}$ even, we have, $f(a_i, b_{i^{'}}) = 0 = f(a_{i+k+1}, b_{i^{'}+l+1})$, $f(a_{i+k}, b_{i^{'}+l}) = 1$, $f(a_{i+k}, b_{i^{'}+l+1}) = 2 = f(a_{i+k+1}, b_{i^{'}+l})$.

(vii) For i even and $i^{'}$ odd, $i^{'} \neq n$, we have, $f(a_i, b_{i^{'}}) = 1 = f(a_{i+k}, b_{i^{'}+l})$, $f(a_{i+k}, b_{i^{'}+l+1}) = 0$, $f(a_{i+k+1}, b_{i^{'}+l}) = 2$, $f(a_{i+k+1}, b_{i^{'}+l+1}) = 1$.

(viii) For i even and $i^{'}$ odd, $i^{'} = n$, we have, $f(a_i, b_{i^{'}}) = 0 = f(a_{i+k+1}, b_{i^{'}+l+1})$, $f(a_{i+k}, b_{i^{'}+l}) = 1$, $f(a_{i+k}, b_{i^{'}+l+1}) = 2 = f(a_{i+k+1}, b_{i^{'}+l})$.

(ix) For i even and $i^{'}$ even, we get, $f(a_i, b_{i^{'}}) = 2 = f(a_{i+k+1}, b_{i^{'}+l+1})$, $f(a_{i+k}, b_{i^{'}+l}) = 0$, $f(a_{i+k}, b_{i^{'}+l+1}) = 1 = f(a_{i+k+1}, b_{i^{'}+l})$.

Here all the geodetic sets have cardinality 5 and are chromatic sets. But, $\chi(C_m \Box C_n) = 3$. Hence in all three subcases we have $\chi_{gc}(C_m \Box C_n) = 5$. $\square$

Theorem 3.12. *For the cartesian product of complete graph K_m with path P_n the geochromatic number is given by,*
$$\chi_{gc}(K_m \Box P_n) = \begin{cases} m, & \text{for } n \text{ odd,} \\ m+1, & \text{for } n \text{ even.} \end{cases}$$

Proof. We label the vertices of K_m by $\{a_1, a_2, a_3, ...a_m\}$ and the vertices of P_n by $\{b_1, b_2, b_3, ..., b_n\}$. In the cartesian product of $K_m \Box P_n$, we have $\Delta(K_m \Box P_n) = m+1$ and $\delta(K_m \Box P_n) = m$. The complete graph is m colorable and the path is bicolorable. The complete graph K_m and path P_n contain complete minimum geodetic sets. By Theorem 3.1 [8] $\chi(K_m \Box P_n) = \max\{m, 2\} = m$, for $m \geq 2$ and by Theorem 3.7 [2] $g_n(K_m \Box P_n) = \max\{m, 2\} = m$, for $m \geq 2$.

Now to find geochromatic sets we use the structure of $K_m \Box P_n$ containing m copies of P_n and n copies of K_m. The minimum degree vertices form the periphery, and the periphery is $\{(a_i, b_1), (a_j, b_n)\}$ or $\{(a_i, b_n), (a_j, b_1)\}$ for $1 \leq i \leq m, 1 \leq j \leq n$ and $i \neq j$. It is clear that the periphery forms a geodetic set. Now we check whether such a geodetic set is a chromatic set or not. Here two cases arise based on the value of n.

Case 1: If n is odd.

If n is odd then, $d((a_i,b_1),(a_j,b_n)) = n$ or $d((a_i,b_n),(a_j,b_1)) = n$, we see that the diameter of $K_m \square P_n$ is odd. The geodetic base consists of all color classes resulting in a chromatic set becoming a geochromatic set too. Hence $\chi_{gc}(K_m \square P_n) = m$.

Case 2: If n is even.

If n is even then, $d((a_i,b_1),(a_j,b_n)) = n$ or $d((a_i,b_n),(a_j,b_1)) = n$, we see that the diameter is even. Hence the geodetic set is not a chromatic set as it contains vertices from repeated color classes to make it a chromatic set forming a geochromatic set. Hence $\chi_{gc}(K_m \square P_n) = m+1$. □

Theorem 3.13. *For the cartesian product of complete graph K_m with cycle C_n the geochromatic number is given by,* $\chi_{gc}(K_m \square C_n) = \begin{cases} m, & \text{for } n \text{ even and } n/2 \text{ even,} \\ m+1, & \text{for } n \text{ even and } n/2 \text{ odd,} \\ 2m-1, & \text{for } n \text{ odd.} \end{cases}$

Proof. Let the vertices of K_m be $\{a_1,a_2,a_3,...,a_m\}$ and the vertices of cycle C_n be $\{b_1,b_2,b_3,..,b_n\}$. It is clear from the structure that $K_m \square C_n$ is a regular graph with regularity $(m+1)$ and a self-centered graph.

The complete graph K_m is m colorable having a complete minimum geodetic set. C_n has a complete minimum geodetic set if n is even and has a linear minimum geodetic set if n is odd. And also if n is even then C_n is bicolorable where as C_n is 3 colorable if n is odd.

By Theorem 3.1 [8], we have $\chi(K_m \square C_n) = \begin{cases} max\{m,2\} = m, & \text{if } n \text{ is even,} \\ max\{m,3\} = m, & \text{if } n \text{ odd.} \end{cases}$

By Theorem 3.7 [2], we have $g_n(K_m \square C_n) = \begin{cases} max\{m,2\} = m, & \text{if } n \text{ is even,} \\ max\{m,3\} = m, & \text{if } n \text{ odd.} \end{cases}$

Case 1: If n is even.

In thus case the periphery of $K_m \square P_n$ is $\{(a_i,b_j),(a_{i'},b_{j'})\}$ for $i \neq i'$, $1 \leq i,i' \leq m$ and b_j is eccentric to $b_{j'}$, $1 \leq j,j' \leq n$. Here two cases arise based on the value of $\frac{n}{2}$.

Subcase (i): If $\frac{n}{2}$ is even.

Suppose $\frac{n}{2}$ is even then $d((a_i,b_j),(a_{i'},b_{j'}))$ is odd. Hence the vertices of geodetic base consist of all color classes resulting in a chromatic set. Therefore $\chi_{gc}(K_m \square C_n) = m$.

Subcase (ii): If $\frac{n}{2}$ is odd.

Suppose $\frac{n}{2}$ is odd then $d((a_i,b_j),(a_{i'},b_{j'}))$ is even. Hence the vertices of geodetic set do not form a chromatic set as one of the color class vertex will be missing. Hence we add that to make a geochromatic set, therefore $\chi_{gc}(K_m \square C_n) = m+1$.

Case 2: If n is odd.

If n is odd, then $d((a_i,b_j),(a_{i'},b_{j'}))$, and $d((a_i,b_j),d(a_{i'},b_{j'+1}))$ is even if m odd. And $d((a_i,b_j),(a_{i'},b_{j'}))$, and $d((a_i,b_j),d(a_{i'},b_{j'+1}))$ is odd if m even. Hence, the geodetic sets formed contain all the color class vertices which make a chromatic set, thus forming a geochromatic set. Hence $\chi_{gc}(K_m \square C_n) = 2m-1$.

□

Theorem 3.14. *For the cartesian product of complete graphs K_m with m vertices and K_n with n vertices the geochromatic number is given by,*

$$\chi_{gc}(K_m \square K_n) = \begin{cases} m, & \text{for } m=n,\ m \text{ odd}, \\ m+1, & \text{for } m=n,\ m \text{ even}, \\ max\{m,n\}, & \text{for } m \neq n. \end{cases}$$

Proof. The cartesian product of two complete graphs is a regular, self-centered graph with radius 2. We label the vertices of K_m by $\{a_1,a_2,a_3,...,a_m\}$ and the vertices of K_n by $\{b_1,b_2,b_3,..,b_n\}$. Being complete graphs K_m,K_n are m colorable and n colorable with complete minimum geodetic sets respectively.

By Theorem 3.1 [8], $\chi(K_m \square K_n) = max\{m,n\}$, for $m \geq 2$ and by Theorem 3.7 [2] $g_n(K_m \square K_n)$ = max$\{m,2\}$ for $m \geq 2$.

In $K_m \square K_n$ the coloring pattern will be followed as in Remark 1. We have m copies of K_n and n copies of K_m such that each layer of $V(K_m \square K_n)$ is a K_m. Hence we get $0,1,2,....,(m-1)$ colors in each layer and m such layers exist. Therefore it is an arrangement of $0,1,2,3,.....(m-1)$ in each row and in each column such that no value repeats in that row or column. That is, it is a permutation of $0,1,2,3,.....(m-1)$ m times. A cyclic permutation results in a repetitive $m \times m$ grid. Now we need to select m values from this array. Since a complete graph has a complete minimum geodetic set, from [2], a minimum geodetic set is comprised of the vertices lying on the main diagonal of such a grid. Hence each chromatic set is a geodetic set too, to give $\chi_{gc}(K_m \square K_n) = m$.

If $m = n$ and m is even, then it is impossible to get all distinct m vertices from each color class along the main diagonal. In such a condition, one color class representation is missed. Hence adding it to a chromatic set gives $\chi_{gc}(K_m \square K_n) = m+1$.

If $m \neq n$, then a chromatic set is considered first, and checked for its geodeticity. From[8], it is clear that $\chi(K_m \square K_n) = \max\{m,n\}$, whenever $m \neq n$. Without loss of generality, let $m > n$, a chromatic set is as formed in such a way that each color class contains n number of vertices by following the coloring pattern using Remark 1. Hence we need to choose m vertices from n classes, having each set representation to result in a geochromatic set. This is possible by using the Pigeon hole principle and hence $\chi_{gc}(K_m \square K_n) = \max\{m,n\}$. □

CONCLUSION

In this paper we have determined the exact value of geochromatic number of Cartesian product of graphs. The present study gives insights and works as a powerful tool in modeling real world facility location problems such that any node in a network can be reached in a shortest possible route.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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