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NEW COMMON FIXED POINT RESULTS FOR SINGLE-VALUED MAPPINGS IN CONE b-METRIC SPACES OVER BANACH ALGEBRA WITH GENERALIZED c-DISTANCE

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Abstract. In this work, we obtain common fixed point and fixed point theorems in cone b-metric spaces over Banach algebra with generalized c-distance. The presented theorems extend and generalize several well-known comparable results in literature to generalized c-distance in such space. We present an example to support our extension and generalization theorems.

Keywords: cone b-metric spaces; fixed points; coincidence points; common fixed points; generalized Lipschitz constant; generalized c-distance.

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1. INTRODUCTION

The problems of fixed point theorems, common fixed point theorems, coupled fixed point theorems and common coupled fixed point theorems in cone metric spaces and cone b-metric spaces became not attractive since some authors established the equivalence between some fixed point results between cone metric spaces and metric spaces, also between cone b-metric spaces and b-metric spaces. Further, they assert that all proofs are just repeated as the usual

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metric cases (see [10, 11, 12, 13, 14, 15]).

In 2013, Liu and Xu [1] introduced the concept of cone metric spaces over Banach algebras replacing Banach spaces E by Banach algebras $\mathscr{A}$ as the underlying spaces of cone metric spaces. They proved fixed point theorems of generalized Lipschitz mappings under the condition of normality for cones. Liu and Xu have done a beginning and great work by presented an example to explain and show that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces. After that, Xu and Radenović [2] reproved the results of Liu and Xu [1] without assumption of normality. By following that, Huang and Radenović [3, 4] initiated the notion of a cone b-metric space over Banach algebra with constant $s \geq 1$ and proved some fixed point results and common fixed point results of generalized Lipschitz mappings on cone b-metric spaces over Banach algebras. They provided and build up some topological properties which will be needed to upgrade and prove some results in literature to cone b-metric space over Banach algebras see for example [3, 4, 5, 6, 7, 8, 9]. Note that, in the setting of cone metric spaces over Banach algebras and cone b-metric spaces over Banach algebras the contractive coefficient is vector instead of usual real constant and the relevant multiplications is vector instead of scalar multiplications. Thus, there is no equivalence between b-metric spaces and cone b-metric spaces with Banach algebras.

Recently, Huang et al. [16] introduced the concept of a c-distance in cone metric space over Banach algebra with some properties needed. This distance be greatly generalizes the notion of cone metric in cone metric space over Banach algebra. They extended and developed the Banach contraction mapping principle on c-distance with five generalized Lipschitz constants and prove some common fixed point theorems based on them. Furthermore, they gave an example to claim the generalizations are indeed real generalizations based on the non-equivalence for fixed point theorems between metric spaces and cone metric spaces over Banach algebras. Next, Han and Xu [17] proved some fixed point and common fixed point theorems on c-distance in cone metric spaces over Banach algebras. After that, Han et al. [18] proved fixed point theorems for α-admissible mappings on c-distance in cone metric spaces over Banach algebras.

Recently in 2020, Firozjah et al. [20] initiated generalized c-distance on cone b-metric space over Banach algebras and proved some fixed point results. As a result, there is still both interest and need for research in the field of studying fixed point theorems and common fixed point theorem in the framework of generalized c-distance in cone b-metric spaces over Banach algebras .

In this paper, we prove some common fixed point and fixed point theorems on generalized c-distance in cone b-metric space over Banach algebra. The results obtained extend and generalize the main results in [20, 21, 22]

2. Preliminaries

In this section, we recall some basic definitions and results about cone b-metric space over Banach algebras.

Definition 2.1. A real Banach space $\mathscr{A}$ is called a real Banach algebra if for all $x, y, z \in \mathscr{A}$, $\alpha \in \mathbb{R}$, following properties holds:

(1) $x(yz) = (xy)z$,

(2) If $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$,

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

(4) $\|xy\| \leq \|x\|\|y\|$.

Let $\mathscr{A}$ be a real Banach algebra with a unit e, i.e., multiplicative identity e such that for all $x \in \mathscr{A}$, $ex = xe$. An element $x \in \mathscr{A}$ is said to be invertible if there exists an element $y \in \mathscr{A}$ such that $xy = yx = e$, y is called inverse of x and denoted by x^{-1}.

Definition 2.2. A subset P of $\mathscr{A}$ is called a cone if:

(1) P is nonempty set closed and $P \neq \{\theta\}$,

(2) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,

(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

where θ denote to the zero element in $\mathscr{A}$. For any cone $P \subset \mathscr{A}$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \text{int}P$, where int$P$ denotes the interior of P. If int$P \neq \theta$, then P is called a solid. A cone P is called normal if there exists a number K such that for all $x, y \in \mathscr{A}$:

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|. \tag{2.1}$$

Equivalently, the cone P is normal if

$$(\forall n)\; x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying Condition 2.1 is called the normal constant of P.

Example 2.3. ([19]) Let $\mathscr{A} = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for all $x \in \mathscr{A}$. Define multiplication in $\mathscr{A}$ as just pointwise multiplication. Then $\mathscr{A}$ is a real unital Banach algebra with unit $e = 1$. Let $P = \{x \in E : x(t) \geq 0\}$, this cone is nonnormal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by Condition 2.2 that P is a nonnormal cone.

During this paper, $\mathscr{A}$ is a real Banach algebra with a unit e and the cone P is solid, that is, int$P \neq \emptyset$.

Definition 2.4. ([3, 4]) Let X be a nonempty set and $\mathscr{A}$ be a real Banach Algebra with the constant $s \geq 1$. Suppose the mapping $d : X \times X \longrightarrow \mathscr{A}$ satisfy the following conditions:

(1) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(3) $d(x,y) \preceq s(d(x,y) + d(y,z))$ for all $x, y, z \in X$.

Then d is called a cone b-metric on X and (X,d) is called a cone b-metric space over Banach algebra $\mathscr{A}$.

Definition 2.5. ([3, 4]) Let (X,d) be a cone b-metric space over Banach algebra $\mathscr{A}$, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \longrightarrow x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone b-metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

Definition 2.6. ([3, 4]) Let $\mathscr{A}$ be a real Banach algebra with a solid cone P. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a positive integer N such that $u_n \ll c$ for all $n > N$.

The following lemma is helpful to prove our results.

Lemma 2.7. ([3, 4]) Let $\mathscr{A}$ be a real Banach algebra with a unit e and P a solid cone. The following properties holds:

(1) the spectral radius $\rho(u)$ of $u \in \mathscr{A}$ is holds

$$\rho(u) = \lim_{n\to\infty} \| u^n \|^{\frac{1}{n}} = \inf \| u^n \|^{\frac{1}{n}},$$

and if $\rho(u) < 1$ then $e - u$ is invertible in $\mathscr{A}$. Moreover, $(e-u)^{-1} = \sum_{i=1}^{\infty} u^i$ and $\rho((e-u))^{-1} < \frac{1}{1-\rho(u)}$,

(2) if $u, v \in \mathscr{A}$ and u commutes with v, then $\rho(u+v) \leq \rho(u) + \rho(v)$ and $\rho(uv) \leq \rho(u)\rho(v)$,

(3) if $u \preceq ku$ where $u, k \in P$ and $\rho(k) < 1$, then $u = \theta$ and

(4) if $\rho(u) < 1$ then, $\{u^n\}$ is a c-sequence. Further, if $\beta \in P$, then $\{\beta u^n\}$ is a c-sequence.

Let $f : X \longrightarrow X$ and $\mathrm{g} : X \longrightarrow X$ be two mappings and let $x \in X$. Recall that, if $w = \mathrm{g}x = fx$ then we called x is a coincidence point of mappings and w is a point of coincidence. If $x = \mathrm{g}x = fx$, then we called x is a common fixed point of f and g. The mappings f and g are called weakly compatible if $\mathrm{g}fx = f\mathrm{g}x$ whenever $\mathrm{g}x = fx$.

Huang and Radenović [4] proved the following common fixed point and fixed point results in cone b-metric spaces over Banach algebra for weakly compatible mappings.

Theorem 2.8. Let (X,d) be a cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and P be a solid cone on $\mathscr{A}$. Let $k_i \in P(i=1,...,5)$ be generalized Lipschitz constants with $2s\rho(k_1)+(s+1)\rho(k_2+k_3+sk_4+sk_5)<2$. Supose that k_1 commutes with $k_2+k_3+sk_4+sk_5$ and the mappings $f: X \longrightarrow X$ and $g: X \longrightarrow X$ satisfy that

$$d(fx,fy) \preceq k_1 d(gx,gy)+k_2 d(gx,fx)+k_3 d(fy,gy)+k_4 d(gx,fy)+k_5 d(fx,gy),$$

for all $x,y \in X$. If the range of g contains the range of f and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Definition 2.9. [20] Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$ over Banach algebra $\mathscr{A}$. A function $q: X \times X \longrightarrow \mathscr{A}$ is called a generalized c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x,y)$ for all $x,y \in X$,

(q2) $q(x,z) \preceq s(q(x,y)+q(y,z))$ for all $x,y,z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x,y_n) \preceq u$ for some $u=u_x \in P$, then $q(x,y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.10. Let $X=[0,1]$ and $\mathscr{A}=C^1_{\mathbb{R}}[0,1]$ with $\|u\|=\|u\|_\infty+\|u'\|_\infty, u \in \mathscr{A}$. Define multiplication in $\mathscr{A}$ as just pointwise multiplication. Then $\mathscr{A}$ is a real unital Banach algebra with unit $e=1$. Let $P=\{u \in \mathscr{A}: u(t) \geq 0 \text{ on } [0,1]\}$. It is well known that this cone is solid but it is not normal (see Example 2.3). Define a cone b-metric $d: X \times X \to \mathscr{A}$ by $d(x,y)(t)=|x-y|^2 e^t$. Then (X,d) is a complete cone b-metric space with the coefficient $s=2$ over Banach algebra $\mathscr{A}$. Define a mapping $q: X \times X \longrightarrow \mathscr{A}$ by $q(x,y)(t) := y^2 \cdot e^t$ for all $x,y \in X$. Then q is a generalized c-distance on X (see [22]).

The following lemma is useful in our work.

Lemma 2.11. [20] Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$ relative to a solid cone P and q is a generalized c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x,y,z \in X$. Suppose that u_n is a c-sequence in P. Then the following hold:

(1) If $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $y = z$.

(2) If $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $\{y_n\}$ converges to z.

(3) If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

(4) If $q(y,x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

3. Common Fixed Point Results

In this section, we prove some common fixed point results under the concept of generalized c-distance in cone b-metric spaces over Banach algebra for two self-mappings satisfying the contractive condition in the case of generalized Lipschitz constant $\rho(h) \in [0,1/s)$.

Theorem 3.1. Let (X,d) be a cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ and $\mathrm{g} : X \longrightarrow X$ be two self mappings such that the following contractive condition holds for all $x,y \in X$:

$$q(fx,fy) \preceq a_1 q(\mathrm{g}x,\mathrm{g}y) + a_2 q(\mathrm{g}x,fx) + a_3 q(\mathrm{g}y,fy) + a_4 q(\mathrm{g}x,fy),$$

where $a_1,a_2,a_3,a_4 \in P$ are generalized Lipschitz constants such that $(a_3 + sa_4)$ commutes with $(a_1 + a_2 + sa_4)$ and

$$\rho(a_3 + sa_4) + \rho(sa_1 + sa_2 + s^2 a_4) < 1. \tag{3.1}$$

If $f(X) \subseteq \mathrm{g}(X)$ and $\mathrm{g}(X)$ is a complete subspace of X, then f and g have a coincidence point t in X. Further, if $p = \mathrm{g}t = ft$ then $q(p,p) = \theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $\mathrm{g}x_1 = fx_0$. This can be done because $f(X) \subseteq \mathrm{g}(X)$. Continuing this process we obtain a sequence $\{x_n\}$ in X such that $gx_{n+1} = fx_n$. Then we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, fx_{n-1}) + a_3 q(gx_n, fx_n) \\ &\quad + a_4 q(gx_{n-1}, fx_n) \\ &= a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, gx_n) + a_3 q(gx_n, gx_{n+1}) \\ &\quad + a_4 q(gx_{n-1}, gx_{n+1}) \\ &\preceq a_1 q(gx_{n-1}, gx_n) + a_2 q(gx_{n-1}, gx_n) + a_3 q(gx_n, gx_{n+1}) \\ &\quad + s a_4 (q(gx_{n-1}, gx_n) + q(gx_n, gx_{n+1})). \end{aligned}$$

Hence

$$(e - (a_3 + sa_4)) q(gx_n, gx_{n+1}) \preceq (a_1 + a_2 + sa_4) q(gx_{n-1}, gx_n). \tag{3.2}$$

Since Condition (3.1) shows that $\rho(a_3 + sa_4) < 1$, then Lemma 2.7 (1) shows that $(e - (a_3 + sa_4))$ is invertible. Furthermore, $(e - (a_3 + sa_4))^{-1} = \sum_{i=1}^{\infty} (a_3 + sa_4)^i$ and $\rho((e - (a_3 + sa_4)))^{-1} < \frac{1}{1 - \rho(a_3 + sa_4)}$.

Multiple both sides of (3.2) by $(e - (a_3 + sa_4))^{-1}$, we arrive at

$$q(gx_n, gx_{n+1}) \preceq (e - (a_3 + sa_4))^{-1} (a_1 + a_2 + sa_4) q(gx_{n-1}, gx_n). \tag{3.3}$$

Put

$$h = (e - (a_3 + sa_4))^{-1} (a_1 + a_2 + sa_4).$$

Then we have

$$\begin{aligned} q(gx_n, gx_{n+1}) &\preceq h q(gx_{n-1}, gx_n) \\ &\preceq h^2 q(gx_{n-2}, gx_{n-1}) \\ &\;\;\vdots \\ &\preceq h^n q(gx_0, gx_1). \end{aligned} \tag{3.4}$$

Since (a_3+sa_4) commutes with $(a_1+a_2+sa_4)$, one can see that

$$(e-(a_3+sa_4))^{-1}(a_1+a_2+sa_4)=(a_1+a_2+sa_4)(e-(a_3+sa_4))^{-1}, \tag{3.5}$$

which means that $(e-(a_3+sa_4))^{-1}$ commutes with $(a_1+a_2+sa_4)$. Now, apply (3.1), Lemma 2.7(1) and (2), we obtain:

$$\begin{aligned}\rho(h) &= \rho((e-(a_3+sa_4))^{-1}(a_1+a_2+sa_4)) \\ &\leq \rho(e-(a_3+sa_4))^{-1}\rho(a_1+a_2+sa_4) \\ &\leq \frac{1}{1-\rho(a_3+sa_4)}\rho(a_1+a_2+sa_4) \\ &< \frac{1}{s}. \qquad (3.6)\end{aligned}$$

Thus $\rho(sh)<1$, which means that $e-sh$ is invertible.

Let $m,n\in\mathbb{N}$ such that $m>n\geq 1$. Then we have

$$\begin{aligned}q(gx_n,gx_m) &\preceq sq(gx_n,gx_{n+1})+s^2q(gx_{n+1},gx_{n+2})+\ldots+s^{m-n}q(gx_{m-1},gx_m) \\ &\preceq sh^nq(gx_0,gx_1)+s^2h^{n+1}q(gx_0,gx_1)+\ldots+s^{m-n}h^{m-1}q(gx_0,gx_1) \\ &\preceq (sh^n+s^2h^{n+1}+\ldots+s^{m-n}h^{m-1})q(gx_0,gx_1) \\ &= sh^n(1+sh+(sh)^2+\ldots+(sh)^{m-n-1})q(gx_0,gx_1) \\ &\preceq sh^n(1-sh)^{-1}q(gx_0,gx_1).\end{aligned}$$

Since $\rho(h)<\frac{1}{s}<1$ and $\|h^n\|\longrightarrow\theta$ (as $n\longrightarrow\infty$), Lemma 2.7 (4) shows that $\{h^n\}$ is a c-sequence. Thus, Lemma 2.11 (3) shows that $\{x_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there exists $t\in X$ such that $gx_n\longrightarrow gt$ as $n\longrightarrow\infty$. By q3 we have:

$$q(gx_n,gt)\preceq\frac{s^2h^n}{1-sh}q(gx_0,gx_1). \tag{3.7}$$

Note that, from (3.4)

$$q(gx_n,gx_{n+1})=q(fx_{n-1},fx_n)\preceq hq(gx_{n-1},gx_n). \tag{3.8}$$

On the other hand, by applying (3.8) with $x_{n+1} = t$, we have

$$\begin{aligned} q(gx_n, ft) &= q(fx_{n-1}, ft) \\ &\preceq hq(gx_{n-1}, gt) \\ &\preceq h\frac{s^2h^{n-1}}{1-sh}q(gx_0, gx_1) \\ &= \frac{s^2h^n}{1-sh}q(gx_0, gx_1). \end{aligned} \tag{3.9}$$

Observe that, the right sides of (3.7) and (3.9) are c-sequences. Thus, Lemma 2.11 (1) shows that $gt = ft$. Therefore, t is a coincidence point of f and g and p is a point of coincidence of f and g where $p = gt = ft$ for some t in X.

Suppose that $p = gt = ft$. Then we have

$$\begin{aligned} q(p,p) &= q(ft, ft) \\ &\preceq a_1q(gt, gt) + a_2q(gt, ft) + a_3q(gt, ft) + a_4q(gt, ft) \\ &= a_1q(p,p) + a_2q(p,p + a_3q(p,p) + a_4q(p,p) \\ &= (a_1 + a_2 + a_3 + a_4)q(p,p) \\ &\preceq (a_3 + sa_4 + sa_1 + sa_2 + s^2a_4)q(p,p). \end{aligned}$$

Since $\rho(a_3 + sa_4 + sa_1 + sa_2 + s^2a_4) < \rho(a_3 + sa_4) + \rho(sa_1 + sa_2 + s^2a_4) < 1$, Lemma 2.7 (3) shows that $q(p,p) = \theta$.

Finally, suppose the point of coincidence is not unique. Thus, there exists $z \in X$ such that $u = fz = gz$ where is a point of coincidence. Then we have

$$\begin{aligned} q(p,u) &= q(ft, fz) \\ &\preceq a_1q(gt, gz) + a_2q(gt, ft) + a_3q(gz, fz) + a_4q(gt, fz) \\ &= a_1q(p,u) + a_2q(p,p) + a_3q(u,u) + a_4q(p,u) \\ &= a_1q(p,u) + a_4q(p,u) \\ &= (a_1 + a_4)q(p,u) \\ &\preceq (a_3 + sa_4 + sa_1 + sa_2 + s^2a_4)q(p,u) \end{aligned}$$

Since $\rho(a_3+sa_4+sa_1+sa_2+s^2a_4) < \rho(a_3+sa_4)+\rho(sa_1+sa_2+s^2a_4) < 1$, Lemma 2.7 (3) shows that $q(p,u)=\theta$. Also, we have $q(p,p)=\theta$. Thus, Lemma 2.11 (1) shows that $p=u$. Therefore, p is the unique point of coincidence. Now, let $p=\mathrm{g}t=ft$. Since f and g are weakly compatible, we have

$$\mathrm{g}p=\mathrm{gg}t=\mathrm{g}ft=f\mathrm{g}t=fp.$$

Hence, gp is a point of coincidence. The uniqueness of the point of coincidence implies that $\mathrm{g}p=\mathrm{g}t$. Therefore, $p=\mathrm{g}p=fp$. Hence, p is the unique common fixed point of f and g. □

Now, we present one example (in the cases of nonnormal cone) to support Theorem 3.1. The conditions of Theorem 3.1 are achieved, but Theorem 2.8 of Huang and Radenović ([4], Theorem 2.9) is not applicable.

Example 3.2. Consider Example 2.10. Define the mappings $f: X \longrightarrow X$ by $fx=\frac{x^2}{4}$ and $\mathrm{g}: X \longrightarrow X$ by $\mathrm{g}x=\frac{x}{2}$ for all $x\in X$. Clear that $f(X)\subseteq \mathrm{g}(X)$ and $\mathrm{g}(X)$ is a complete subset of X. We have

$$\begin{aligned} d(fx,fy)(t) &= \left|\frac{x^2}{4}-\frac{y^2}{4}\right|^2\cdot e^t \\ &= \left|(\frac{x}{2}-\frac{y}{2})(\frac{x}{2}+\frac{y}{2})\right|^2\cdot e^t \\ &\preceq \left|(\frac{x}{2}-\frac{y}{2})\right|^2\cdot e^t \\ &= k_1 d(\mathrm{g}x,\mathrm{g}y)(t). \end{aligned}$$

Note that, $k_1=1$ so, $\rho(k_1)=1$. Since $2s\rho(k_1)=4 \nless 2$ then, we can not apply Theorem 2.8 of Huang and Radenović ([4], Theorem 2.9) for this example on a cone b-metric space over Banach algebra. To check this example on generalized c-distance, let $a_1=\frac{1}{4}$, $a_2=\frac{1}{40}+\frac{1}{40}t=a_3=a_4$. Clearly that $\rho(a_3+sa_4)+\rho(sa_1+sa_2+s^2a_4)=\frac{19}{20}<1$. Now, we have:

$$
\begin{aligned}
q(fx,fy)(t) &= (fy)^2\cdot e^t \\
&= \frac{y^4}{16}\cdot e^t \\
&= \frac{y^2}{4}\frac{y^2}{4}\cdot e^t \\
&\preceq \frac{1}{4}\frac{y^2}{4}\cdot e^t \\
&= \frac{1}{4}(\frac{y}{2})^2\cdot e^t \\
&\preceq a_1(\frac{y}{2})^2\cdot e^t + a_2\frac{x^4}{16}\cdot e^t + a_3\frac{y^4}{16}\cdot e^t + a_4\frac{y^4}{16}\cdot e^t \\
&= a_1q(\mathrm{g}x,\mathrm{g}y)(t) + a_2q(\mathrm{g}x,fx)(t) + a_3q(\mathrm{g}y,fy)(t) + a_4q(\mathrm{g}x,fy)(t).
\end{aligned}
$$

Also, f and g are weakly compatible at $x=0$. Therefore, all conditions of Theorem 3.1 are satisfied. Hence, f and g have a unique common fixed point $x=0$ and $f(0)=\mathrm{g}(0)=0$ with $q(0,0)=0$.

Corollary 3.3. Let (X,d) be a cone b-metric space over Banach algebra with the coefficient $s\geq 1$ and q is a generalized c-distance on X. Let $f:X\longrightarrow X$ and $\mathrm{g}:X\longrightarrow X$ be two self mappings such that the following contractive condition holds for all $x,y\in X$:

$$q(fx,fy)\preceq kq(\mathrm{g}x,\mathrm{g}y),$$

where $k\in P$ is a generalized Lipschitz constant and $\rho(k)<\frac{1}{s}$. If $f(X)\subseteq \mathrm{g}(X)$ and $\mathrm{g}(X)$ is a complete subspace of X, then f and g have a coincidence point t in X. Further, if $p=\mathrm{g}t=ft$ then $q(p,p)=\theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. In Theorem 3.1, put $k=a_1, a_2=a_3=a_4=\theta$. The proof is complete. □

Corollary 3.4. Let (X,d) be a cone b-metric space over Banach algebra with the coefficient $s\geq 1$ and q is a generalized c-distance on X. Let $f:X\longrightarrow X$ and $\mathrm{g}:X\longrightarrow X$ be two self mappings such that the following contractive condition holds for all $x,y\in X$:

$$q(fx,fy)\preceq kq(\mathrm{g}x,fx)+lq(\mathrm{g}y,fy),$$

where $k,l \in P$ are generalized Lipschitz constants such that l commutes with k and $\rho(l)+\rho(sk)<1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point t in X. Further, if $p = gt = ft$ then $q(p,p)=\theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. In Theorem 3.1, put $k=a_2, l=a_3, a_1=a_4=\theta$. The proof is complete. □

Corollary 3.5. Let (X,d) be a cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two self mappings such that the following contractive condition holds for all $x,y \in X$:

$$q(fx,fy) \preceq kq(gx,gy)+lq(gx,fx)+rq(gy,fy),$$

where $k,l,r \in P$ are generalized Lipschitz constants such that r commutes with $(k+l)$ and $\rho(r)+\rho(sk+sl)<1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a coincidence point t in X. Further, if $p = gt = ft$ then $q(p,p)=\theta$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. In Theorem 3.1, put $k=a_1, l=a_2, r=a_3, a_4=\theta$. The proof is complete. □

Next, we have another theorem with different contractive condition to prove the existence and uniqueness of the common fixed point of two self mappings $f : X \longrightarrow X$ and $g : X \longrightarrow X$ with out assumption the weakly compatible for f and g.

Theorem 3.6. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two self mappings such that the following conditions hold for all $x,y \in X$:

$$q(fx,gy) \preceq a_1q(x,y)+a_2q(x,fx)+a_3q(y,gy)+a_4q(x,gy)+a_5q(y,fx) \tag{3.10}$$

and

$$q(gy,fx) \preceq a_1q(y,x)+a_2q(fx,x)+a_3q(gy,y)+a_4q(gy,x)+a_5q(fx,y), \tag{3.11}$$

where $a_1, a_2, a_3, a_4 \in P$ are generalized Lipschitz constants such that $(a_3 + sa_4)$ commutes with $(a_1 + a_2 + sa_4 + 2sa_5)$ and

$$\rho(a_3 + sa_4) + \rho(sa_1 + sa_2 + s^2a_4 + 2s^2a_5) < 1. \tag{3.12}$$

Then f and g have a common fixed point. In addition, if $u = \mathrm{g}u = fu$ then $q(u,u) = \theta$. Furthermore, the common fixed point is unique.

Proof. Let x_0 be an arbitrary point in X. Set $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = \mathrm{g}x_{2n+1}$. Apply Condition (3.10), then we have:

$$\begin{aligned}
q(x_{2n+1}, x_{2n+2}) &= q(fx_{2n}, \mathrm{g}x_{2n+1}) \\
&\preceq a_1 q(x_{2n}, x_{2n+1}) + a_2 q(x_{2n}, fx_{2n}) + a_3 q(x_{2n+1}, \mathrm{g}x_{2n+1}) \\
&\quad + a_4 q(x_{2n}, \mathrm{g}x_{2n+1}) + a_5 q(x_{2n+1}, fx_{2n}) \\
&= a_1 q(x_{2n}, x_{2n+1}) + a_2 q(x_{2n}, x_{2n+1}) + a_3 q(x_{2n+1}, x_{2n+2}) \\
&\quad + a_4 q(x_{2n}, x_{2n+2}) + a_5 q(x_{2n+1}, x_{2n+1}) \\
&\preceq a_1 q(x_{2n}, x_{2n+1}) + a_2 q(x_{2n}, x_{2n+1}) + a_3 q(x_{2n+1}, x_{2n+2}) \\
&\quad + sa_4 (q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2})) \\
&\quad + sa_5 (q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})) \\
&= (a_1 + a_2 + sa_4 + sa_5) q(x_{2n}, x_{2n+1}) + (a_3 + sa_4) q(x_{2n+1}, x_{2n+2}) \\
&\quad + sa_5 q(x_{2n+1}, x_{2n}).
\end{aligned}$$

Hence

$$(e - \alpha) q(x_{2n+1}, x_{2n+2}) \preceq \beta q(x_{2n}, x_{2n+1}) + sa_5 q(x_{2n+1}, x_{2n}), \tag{3.13}$$

where $\alpha = a_3 + sa_4$ and $\beta = a_1 + a_2 + sa_4 + 2sa_5$. Again by applying Condition (3.11), then we have

$$\begin{aligned}
q(x_{2n+2}, x_{2n+1}) &= q(gx_{2n+1}, fx_{2n}) \\
&\preceq a_1 q(x_{2n+1}, x_{2n}) + a_2 q(fx_{2n}, x_{2n}) + a_3 q(gx_{2n+1}, x_{2n+1}) \\
&\quad + a_4 q(gx_{2n+1}, x_{2n}) + a_5 q(fx_{2n}, x_{2n+1}) \\
&= a_1 q(x_{2n+1}, x_{2n}) + a_2 q(x_{2n+1}, x_{2n}) + a_3 q(x_{2n+2}, x_{2n+1}) \\
&\quad + a_4 q(x_{2n+2}, x_{2n}) + a_5 q(x_{2n+1}, x_{2n+1}) \\
&\preceq a_1 q(x_{2n+1}, x_{2n}) + a_2 q(x_{2n+1}, x_{2n}) + a_3 q(x_{2n+2}, x_{2n+1}) \\
&\quad + sa_4 (q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n})) \\
&\quad + sa_5 (q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})) \\
&= (a_1 + a_2 + sa_4 + sa_5) q(x_{2n+1}, x_{2n}) + (a_3 + sa_4) q(x_{2n+2}, x_{2n+1}) \\
&\quad + sa_5 q(x_{2n}, x_{2n+1}).
\end{aligned}$$

Hence

$$(e - \alpha) q(x_{2n+2}, x_{2n+1}) \preceq \beta q(x_{2n+1}, x_{2n}) + sa_5 q(x_{2n}, x_{2n+1}). \tag{3.14}$$

Add (3.13) to (3.14), we get

$$(e - \alpha)(q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2})) \preceq \beta (q(x_{2n+1}, x_{2n}) + q(x_{2n}, x_{2n+1})). \tag{3.15}$$

Repeating the above processes respectively, we obtaining:

$$(e - \alpha) q(x_{2n+2}, x_{2n+3}) \preceq \beta q(x_{2n+1}, x_{2n+2}) + sa_5 q(x_{2n+2}, x_{2n+1}), \tag{3.16}$$

and

$$(e - \alpha) q(x_{2n+3}, x_{2n+2}) \preceq \beta q(x_{2n+2}, x_{2n+1}) + sa_5 q(x_{2n+1}, x_{2n+2}). \tag{3.17}$$

Add (3.16) to (3.17), we get

$$(e - \alpha)(q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+3})) \preceq \beta (q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2})). \tag{3.18}$$

Now, from (3.15) and (3.18), we can deduce that

$$(e-\alpha)(q(x_{n+1},x_n)+q(x_n,x_{n+1})) \preceq \beta(q(x_n,x_{n-1})+q(x_{n-1},x_n)).$$

Put,

$$u_{n+1}=q(x_{n+1},x_n)+q(x_n,x_{n+1}), \tag{3.19}$$

then we have

$$(e-\alpha)u_{n+1} \preceq \beta u_n. \tag{3.20}$$

Since Condition (3.12) shows that $\rho(\alpha)<1$, then Lemma 2.7 (1) shows that $(e-\alpha)$ is invertible. Furthermore, $(e-\alpha)^{-1}=\sum_{i=1}^{\infty}\alpha^i$ and $\rho((e-\alpha))^{-1}<\frac{1}{1-\rho(\alpha)}$.

On the other hand, multiple both sides of (3.20) by $(e-\alpha)^{-1}$, we arrive at

$$u_{n+1}) \preceq (e-\alpha)^{-1}\beta u_n.$$

Put $h=(e-\alpha)^{-1}\beta$. Then

$$u_{n+1} \preceq hu_n \preceq h^2u_{n-1} \preceq \ldots \preceq h^n u_1, \tag{3.21}$$

where $u_1=q(x_1,x_0)+q(x_0,x_1)$. Note that, from (3.19) and (3.21) one can see that

$$q(x_n,x_{n+1}) \preceq u_{n+1} \preceq h^n u_1, \tag{3.22}$$

and

$$q(x_{n+1},x_n) \preceq u_{n+1} \preceq h^n u_1. \tag{3.23}$$

Since α commutes with β, we can show that

$$(e-\alpha)^{-1}\beta=\beta(e-\alpha)^{-1},$$

which means that $(e-\alpha)^{-1}$ commutes with β. Now using (3.12), Lemma 2.7(1) and (2), we obtain:

$$\begin{aligned}
\rho(h) &= \rho((e-\alpha)^{-1}\beta) \\
&= \rho((e-(a_3+sa_4))^{-1}(a_1+a_2+sa_4+2sa_5)) \\
&\leq \rho(e-(a_3+sa_4))^{-1}\rho(a_1+a_2+sa_4+2sa_5) \\
&\leq \frac{1}{1-\rho(a_3+sa_4)}\rho(a_1+a_2+sa_4+2sa_5) \\
&< \frac{1}{s}. \qquad (3.24)
\end{aligned}$$

Thus $\rho(sh) < 1$, which means that $e - sh$ is invertible.

Let $m > n \geq 1$ with. Then it follows that

$$\begin{aligned}
q(x_n,x_m) &\preceq sq(x_n,x_{n+1}) + s^2q(x_{n+1},x_{n+2}) + \ldots + s^{m-n}q(x_{m-1},x_m) \\
&\preceq sh^nu_1 + s^2h^{n+1}u_1 + \ldots + s^{m-n}h^{m-1}u_1 \\
&\preceq (sh^n + s^2h^{n+1} + \ldots + s^{m-n}h^{m-1})u_1 \\
&= sh^n(1 + sh + (sh)^2 + \ldots + (sh)^{m-n-1})u_1 \\
&\preceq sh^n(1-sh)^{-1}u_1).
\end{aligned}$$

Since $\rho(h) < \frac{1}{s} < 1$ and $\|h^n\| \longrightarrow \theta$ (as $n \longrightarrow \infty$), Lemma 2.7 (4) shows that $\{h^n\}$ is a c-sequence. Thus, Lemma 2.11 (3) shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow \infty$. By q3 we have:

$$q(x_n,x) \preceq \frac{s^2h^n}{1-sh}u_1. \qquad (3.25)$$

Note that, from (3.22) and (3.23) one can see that

$$\begin{aligned} q(fx_{n-1}, gx_n) &= q(x_n, x_{n+1}) \\ &\preceq hu_n \\ &= h(q(x_n, x_{n-1}) + q(x_{n-1}, x_n)) \\ &\preceq h(h^{n-1}u_1 + h^{n-1}u_1) \\ &= 2h^n u_1. \end{aligned} \tag{3.26}$$

Put $x_n = u$ in (3.26), we get

$$\begin{aligned} q(x_n, gu) &= q(fx_{n-1}, gu) \\ &\preceq 2h^n u_1. \end{aligned} \tag{3.27}$$

Observe that, the right sides of (3.25) and (3.27) are c-sequences. Thus, Lemma 2.11 (1) shows that $gu = u$. By the same way we can see that $fu = u$. Therefore, $gu = fu = u$ and u is a common fixed point of f and g.

To show that $q(u,u) = \theta$. Suppose that $gu = fu = u$ and apply Condition (3.10), we have

$$\begin{aligned} q(u,u) &= q(fu, gu) \\ &\preceq a_1 q(u,u) + a_2 q(u, fu) + a_3 q(u, gu) + a_4 q(u, gu) + a_5 q(u, fu) \\ &= a_1 q(u,u) + a_2 q(u,u) + a_3 q(u,u) + a_4 q(u,u) + a_5 q(u,u) \\ &= (a_1 + a_2 + a_3 + a_4 + a_5) q(u,u) \\ &\preceq (a_3 + sa_4 + sa_1 + sa_2 + s^2 a_4 + 2s^2 a_4) q(u,u). \end{aligned}$$

Since $\rho(a_3 + sa_4 + sa_1 + sa_2 + s^2a_4 + 2s^2a_5) < \rho(a_3 + sa_4) + \rho(sa_1 + sa_2 + s^2a_4 + 2s^2a_5) < 1$, Lemma 2.7 (3) shows that $q(u,u) = \theta$.

Finally, suppose there is another common fixed point w of f and g such that $w = fw = \mathrm{g}w$ for some w in X. Using Condition (3.10), we have

$$\begin{aligned}
q(u,w) &= q(fu,\mathrm{g}w)\\
&\preceq a_1q(u,w)+a_2q(u,fu)+a_3q(w,\mathrm{g}w)+a_4q(u,\mathrm{g}w)+a_5q(w,fu)\\
&= a_1q(u,w)+a_2q(u,u)+a_3q(w,w)+a_4q(u,w)+a_5q(w,u)\\
&= a_1q(u,w)+a_4q(u,w)+a_5q(w,u)\\
&= (a_1+a_4)q(u,w)+a_5q(w,u). \qquad (3.28)
\end{aligned}$$

Again by Condition (3.11), we have

$$\begin{aligned}
q(w,u) &= q(\mathrm{g}w,fu)\\
&\preceq a_1q(w,u)+a_2q(fw,w)+a_3q(\mathrm{g}w,w)+a_4q(\mathrm{g}w,u)+a_5q(fu,w)\\
&= a_1q(w,u)+a_2q(w,w)+a_3q(w,w)+a_4q(w,u)+a_5q(u,w)\\
&= a_1q(w,u)+a_4q(w,u)+a_5q(u,w)\\
&= (a_1+a_4)q(w,u)+a_5q(u,w). \qquad (3.29)
\end{aligned}$$

Adding (3.28) and (3.29), we get

$$q(w,u)+q(u,w) = (a_1+a_4+a_5)(q(w,u)+q(u,w)). \qquad (3.30)$$

Then

$$(q(w,u)+q(u,w)) \preceq (sa_1+sa_2+s^2a_4+2s^2a_5)(q(w,u)+q(u,w)). \qquad (3.31)$$

Since $\rho(sa_1+sa_2+s^2a_4+2s^2a_5) < 1$, then $q(w,u)+q(u,w) = \theta$. Hence, $q(w,u) = \theta$ and $q(u,w) = \theta$. Also, $q(u,u) = \theta$. Thus, Lemma 2.11 (1) shows that $w = u$. Therefore, u is the unique common fixed point of f and g. □

4. Fixed Point Theorems

Theorem 4.1. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ be a self mapping such that the following contractive condition holds for all $x,y \in X$:

$$q(fx,fy) \preceq a_1 q(x,y) + a_2 q(x,fx) + a_3 q(y,fy) + a_4 q(x,fy),$$

where $a_1,a_2,a_3,a_4 \in P$ are generalized Lipschitz constants such that (a_3+sa_4) commutes with $(a_1+a_2+sa_4)$ and $\rho(a_3+sa_4)+\rho(sa_1+sa_2+s^2a_4) < 1$. Then f has a fixed point $t \in X$ and for any $t \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $p = fp$ then $q(p,p) = \theta$. The fixed point is unique.

Proof. In Theorem 3.1, put $\mathrm{g}(x) = x$. The proof is complete. □

Remark 4.2. In Theorem 4.1, if $a_3 = \theta$, then we get Theorem 2 of Firozjah et al. [20].

Corollary 4.3. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ be a self mapping such that the following contractive condition holds for all $x,y \in X$:

$$q(fx,fy) \preceq kq(x,y),$$

where $k \in P$ is generalized Lipschitz constant such that $\rho(k) < \frac{1}{s}$. Then f has a fixed point $t \in X$ and for any $t \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $p = fp$ then $q(p,p) = \theta$. The fixed point is unique.

Proof. In Theorem 4.1, put $k = a_1, a_2 = a_3 = a_4 = \theta$. The proof is complete. □

Corollary 4.4. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ be a self mapping such that the following contractive condition holds for all $x,y \in X$:

$$q(fx,fy) \preceq kq(x,fx) + lq(y,fy),$$

where $k,l \in P$ are generalized Lipschitz constants such that l commutes with k and $\rho(l) + \rho(sk) < 1$. Then f has a fixed point $t \in X$ and for any $t \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $p = fp$ then $q(p,p) = \theta$. The fixed point is unique.

Proof. In Theorem 4.1, put $k = a_2, l = a_3, a_1 = a_4 = \theta$. The proof is complete. □

Corollary 4.5. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$ and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ be a self mapping such that the following contractive condition holds for all $x,y \in X$:

$$q(fx, fy) \preceq kq(x,y) + lq(x, fx) + rq(y, fy),$$

where $k,l,r \in P$ are generalized Lipschitz constants such that r commutes with (k,l) and $\rho(r) + \rho(sk+sl) < 1$. Then f has a fixed point $t \in X$ and for any $t \in X$, iterative sequence $\{f^n x\}$ converges to the fixed point. If $p = fp$ then $q(p,p) = \theta$. The fixed point is unique.

Proof. In Theorem 4.1, put $k = a_1, l = a_2, r = a_3, a_4 = \theta$. The proof is complete. □

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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