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ON CHROMATIC *D*-POLYNOMIALS OF MYCIELSKIAN OF PATHS AND CYCLES

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Abstract. Graph colouring is an assignment of colours, labels or weights to elements of a graph subject to certain constraints. Coloring the vertices of a graph in such a way that adjacent vertices are having different colours is called proper vertex colouring. A proper vertex colouring using minimum parameters of colours is studied extensively in recent literature. In this paper, we define new polynomials called chromatic *D*-polynomial and modified chromatic *D*-polynomial in terms of minimal parameter colouring and structural characteristics of graphs such as distances and degrees of vertices.

Keywords: graph colouring; Mycielskian of a graph; chromatic *D*-polynomials.

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1. INTRODUCTION

For all terms and definitions, not defined specifically in this paper, we refer to [1, 2, 5, 12]. Moreover, for notions and norms in graph colouring, see [3, 6, 7]. Unless mentioned otherwise, all graphs considered here are undirected, simple, finite and connected.

1.1. Graph Colouring. Graph colouring is an assignment of colours, labels or weights to the elements of graphs subject to certain conditions. The field of graph colouring has been a

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fascinating research area for mathematicians since its inception. A vertex colouring consisting of the colours with minimum subscripts may be called a *minimum parameter colouring* (see [9]). By graph colouring, we mean vertex colouring of graphs in this paper.

If we colour the vertices of G in such a way that c_1 is assigned to maximum possible number of vertices, then c_2 is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^--*colouring* of G. In a similar manner, if c_ℓ is assigned to maximum possible number of vertices, then $c_{\ell-1}$ is assigned to maximum possible number of remaining uncoloured vertices and proceed in this manner until all vertices are coloured, then such a colouring is called a χ^+-*colouring* of G.

1.2. Mycielskian of a Graph. The notion of Mycielski graph of a given graph G is defined in [8] as given below.

Definition 1.1 (Mycielskian of a Graph)**.** [8] Let G be a graph with the vertex set $V(G) = \{v_1,\dots,v_n\}$. The *Mycielski graph* or the *Mycielskian* of a graph G, denoted by $\mu(G)$, is the graph with vertex set $V(\mu(G)) = \{v_1,v_2,\dots,v_n,u_1,u_2,\dots,u_n,w\}$ such that $v_iv_j \in E(\mu(G)) \iff v_iv_j \in E(G)$, $v_iu_j \in E(\mu(G)) \iff v_iv_j \in E(G)$ and $u_iw \in E(\mu(G))$ for all $i = 1,\dots,n$.

The Mycielskian of a graph G is usually denoted by $\mu(G)$. But for the sake of usage we use the notation $\breve{G}$ instead.

1.3. Chromatic D-Polynomial of Graphs. The notion of the chromatic D-polynomial of graphs, with respect to a given colouring, has been introduced in [10] as follows:

Definition 1.2 (Chromatic D-Polynomial of a Graph)**.** [10] Let G be a connected graph with chromatic number $\chi(G)$, then *the chromatic D-polynomial* of G, denoted by $D_\phi(G,x,y)$, is defined as

$$D_\phi(G,x,y) = \sum_{v_i,v_j\in V(G)} d(v_i,v_j)x^{\zeta(v_i)}y^{\zeta(v_j)}, i<j.$$

The chromatic D-polynomials corresponding to χ^--colouring and χ^+-colouring of a graph G are denoted by $D_{\varphi^-}(\breve{G},x,y)$ and $D_{\varphi^+}(\breve{G},x,y)$ respectively.

The chromatic D-polynomials of certain graph classes have been determined in [10]. Motivated by the studies mentioned above, in this paper, we determine the chromatic D-polynomials of the Mycielskian of two fundamental graph classes.

2. Chromatic D-Polynomials of Mycielski Graphs

In this section, we discuss the chromatic D-polynomial of Mycielskian of paths and cycles, using the φ^- and φ^+colouring.

Theorem 2.1. *For the Mycielskian of a path P_n, we have*

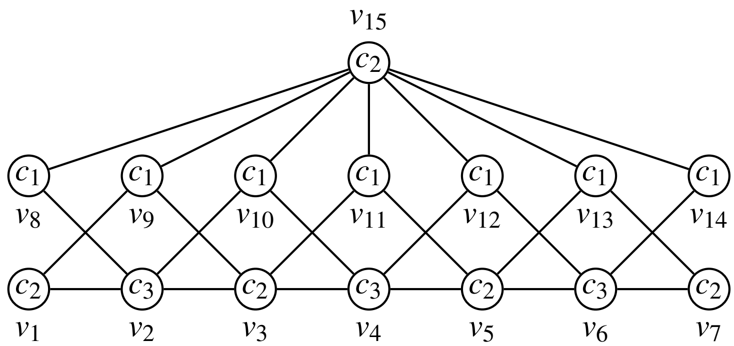
$$D_{\varphi^-}(\breve{P}_n,x,y)=\begin{cases} n(n-1)xy+nxy^2+\frac{3n^2-7n+8}{2}(x^2y+x^3y)+\frac{n^2-2n+4}{2}x^2y^2+ \\ \frac{n^2-2n+2}{2}x^2y^3+\frac{n^2-6n+10}{2}x^3y^2+\frac{n^2-4n+4}{2}x^3y^3, \quad \text{if } n \text{ is even,} \\ n(n-1)xy+nxy^2+(13n+16)x^2y+\frac{n^2+3}{2}x^2y^2+(13n-11)x^3y+ \\ \frac{7n^2-36n+69}{8}x^3y^2+\frac{7n^2-40n+65}{8}x^2y^3+\frac{n^2-6n+9}{2}x^3y^3, \quad \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider the Mycielskian of a path, $\breve{P}_n$ on $2n+1$ vertices and $4n-3$ edges. It is clear that $\chi(\breve{P}_n)=3$. The largest independent set $\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$ is named as U and the root vertex is v_{2n+1}. The vertices of the path $\{v_1,v_2,\ldots,v_n\}$ are named V. Since $\breve{P}_n$ has diameter 4, the distance $d(v_i,v_j)$ can vary from 1 to 4. Now consider the following cases:

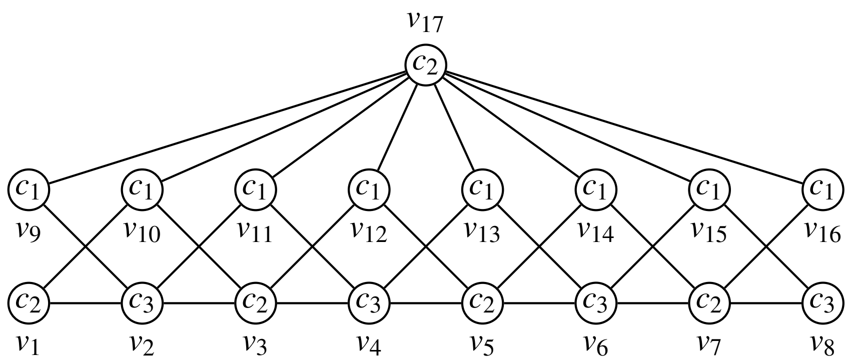
Part (i): When n is even. Then, according to the rules of φ^- colouring, the independent set $U=\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$ is coloured with the colour c_1. Now, the vertices in V are alternatively coloured with c_2 and c_3. Since the root vertex v_{2n+1} is adjacent to all other vertices of U and v_{2n+1} is not adjacent to any vertex in V, v_{2n+1} can have the colour c_2. For the Mycielskian of the path, $\theta(c_1)=n$, $\theta(c_2)=\frac{n+2}{2}$ and $\theta(c_3)=\frac{n}{2}$. The possible colour pairs with appropriate distances and their numbers in $\breve{P}_n$ are listed in the following table.

When n is even, the chromatic D-polynomial can be determined from the values of the Table - 1. Thus, we have $D_{\varphi^-}(\breve{P}_n,x,y)=nxy^2+n-1(x^2y+x^3y)+\frac{n}{2}x^2y^3+\frac{(n-2)}{2}x^3y^2+n(n-1)xy+(3n-4)(x^2y+x^3y)+2(n-1)x^2y^2+(n-2)x^3y^3+\frac{3(n-2)(n-3)}{2}(x^2y+x^3y)+\frac{3(n-2)}{2}x^2y^3+\frac{(n-4)(n-6)}{2}x^3y^2+\frac{3(n-4)}{2}x^3y^2+\frac{(n-2)(n-4)}{2}(x^2y^2+x^3y^3+x^2y^3)$.

Further simplifications gives the result as $D_{\varphi^-}(\breve{P}_n,x,y)=n(n-1)xy+nxy^2+\frac{3n^2-7n+8}{2}(x^2y+x^3y)+\frac{n^2-2n+4}{2}x^2y^2+\frac{n^2-2n+2}{2}$.



(A) The Mycielski graph $\breve{P}_7$ with φ^-colouring



(B) The Mycielski graph $\breve{P}_8$ with φ^-colouring

FIGURE 1

Part (ii): Assume n is odd. According to the rules of φ^- colouring, the independent set $U = \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$ is coloured with the colour c_1. Now, the vertices in V are alternatively coloured with c_2 and c_3. Also, v_{2n+1} can have the colour c_2. Thus, $\theta(c_1) = n$, $\theta(c_2) = \frac{n+3}{2}$ and $\theta(c_3) = \frac{n-1}{2}$. The following table analyses the possible distances in terms of the colour pairs required.

From the Table - 2, and the definitions of chromatic D-polynomials the result follows as: $D_{\varphi^-}(\breve{P}_n, x, y) = nxy^2 + (n-1)(x^2y + x^3y) + \frac{n-1}{2}(x^2y^3 + x^3y^2) + n(n-1)xy + (3n-1)x^2y + 2nx^2y^2 + (3n-7)x^3y + 2(n-2)x^3y^2 + (n-3)x^3y^3 + 9(n+2)x^2y + \frac{3(n-1)(n-3)}{8}x^2y^3 + 3(3n-1)x^3y + \frac{3(n-3)(n-5)}{8}x^3y^2 + \frac{(n-1)(n-3)}{2}x^2y^2 + \frac{(n-3)(n-5)}{2}(x^2y^3 + x^3y^2 + x^3y^3)$.

Further simplifications gives the result as $D_{\varphi^-}(\breve{P}_n, x, y) = n(n-1)xy + nxy^2 + \frac{7n^2-36n+69}{8}x^3y^2 + \frac{n^2+3}{2}x^2y^2 + \frac{7n^2-40n+65}{8}x^2y^3 + (13n-11)x^3y + (13n+16)x^2y + \frac{n^2-6n+9}{2}x^3y^3$. $\square$

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_1, c_2)	n
	(c_2, c_1)	$n-1$
	(c_3, c_1)	$n-1$
	(c_2, c_3)	$\frac{n}{2}$
	(c_3, c_2)	$\frac{n-2}{2}$
2	(c_1, c_1)	$\frac{n(n-1)}{2}$
	(c_2, c_1)	$\frac{3n-4}{2}$
	(c_3, c_1)	$\frac{3n-4}{2}$
	(c_2, c_2)	$n-1$
	(c_3, c_3)	$\frac{n-2}{2}$
3	(c_2, c_1)	$\frac{(n-2)(n-3)}{2}$
	(c_3, c_1)	$\frac{(n-2)(n-3)}{2}$
	(c_2, c_3)	$\frac{n-2}{2}$
	(c_3, c_2)	$\frac{n-4}{2}$
4	(c_2, c_2)	$\frac{(n-2)(n-4)}{8}$
	(c_3, c_3)	$\frac{(n-2)(n-4)}{8}$
	(c_2, c_3)	$\frac{(n-2)(n-4)}{8}$
	(c_3, c_2)	$\frac{(n-4)(n-6)}{8}$

TABLE 1. Table (CDP) for $\breve{P}_n$ when n even, with φ^-colouring

The following theorem discourses about the chromatic D-polynomial of the Mycielskian of paths with φ^+colouring.

Theorem 2.2. *For the Mycielskian of a path P_n, we have*

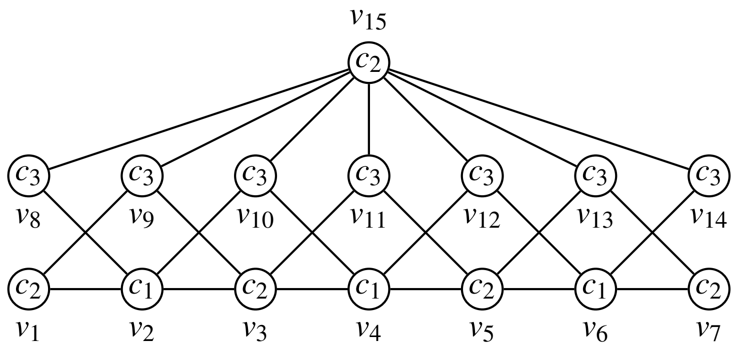
$$D_{\varphi^+}(\breve{P}_n, x, y) = \begin{cases} n(n-1)x^3y^3 + nx^3y^2 + \frac{3n^2-7n+8}{2}(x^2y^3 + xy^3) + \frac{n^2-2n+4}{2}x^2y^2 + \\ \frac{n^2-2n+2}{2}x^2y + \frac{n^2-6n+10}{2}xy^2 + \frac{n^2-4n+4}{2}xy, \quad \textit{if } n \textit{ is even}, \\ n(n-1)x^3y^3 + nx^3y^2 + \frac{7n^2-36n+69}{8}xy^2 + \frac{n^2+3}{2}x^2y^2 + \frac{7n^2-40n+65}{8}x^2y \\ +(13n-11)xy^3 + (13n+16)x^2y^3 + \frac{n^2-6n+9}{2}xy, \quad \textit{if } n \textit{ is odd}. \end{cases}$$

Distance $d(v_i,v_j)$	Colour pairs	Number of pairs
1	(c_1,c_2)	n
	(c_2,c_1)	$n-1$
	(c_3,c_1)	$n-1$
	(c_2,c_3)	$\frac{n-1}{2}$
	(c_3,c_2)	$\frac{n-1}{2}$
2	(c_1,c_1)	$\frac{n(n-1)}{2}$
	(c_2,c_1)	$\frac{3n-1}{2}$
	(c_2,c_2)	n
	(c_3,c_1)	$\frac{3n-7}{2}$
	(c_3,c_2)	$n-2$
	(c_3,c_3)	$\frac{(n-3)}{2}$
3	(c_2,c_1)	$3(n+2)$
	(c_3,c_1)	$3n-1$
	(c_2,c_3)	$\frac{(n-1)(n-3)}{8}$
	(c_3,c_2)	$\frac{(n-3)(n-5)}{8}$
4	(c_2,c_2)	$\frac{(n-1)(n-3)}{8}$
	(c_2,c_3)	$\frac{(n-3)(n-5)}{8}$
	(c_3,c_2)	$\frac{(n-3)(n-5)}{8}$
	(c_3,c_3)	$\frac{(n-3)(n-5)}{8}$

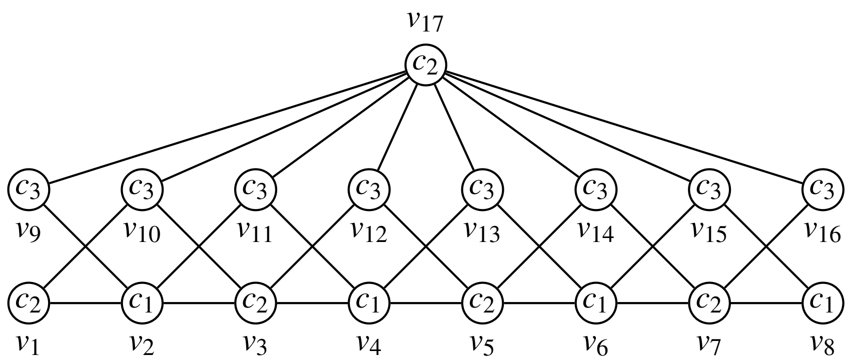
TABLE 2. Table (CDP) for $\breve{P}_n$ when n odd, with φ^-colouring

Proof. Consider the Mycielskian of a path, $\breve{P}_n$ on $2n+1$ vertices and $4n-3$ edges. It is clear that $\chi(\breve{P}_n)=3$. The largest independent set $\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$ is named as U and the root vertex is v_{2n+1}. The vertices of the path $\{v_1,v_2,\ldots,v_n\}$ are named V. Since $\breve{P}_n$ has diameter 4, the distance $d(v_i,v_j)$ can vary from 1 to 4. Now consider the following cases:

Part (i): When n is even. Then according to the rules of φ^+ colouring, the independent set U is coloured with the colour c_3. Now, the vertices in V are alternatively coloured with c_2 and c_1. Since the root vertex v_{2n+1} is adjacent to all vertices of U and v_{2n+1} is not adjacent to any vertex



(A) The Mycielski graph $\breve{P}_7$ with φ^+colouring



(B) The Mycielski graph $\breve{P}_8$ with φ^+colouring

FIGURE 2

in V, v_{2n+1} can have the colour c_2. For the Mycielskian of the path, $\theta(c_3) = n$, $\theta(c_2) = \frac{n+2}{2}$ and $\theta(c_1) = \frac{n}{2}$. The possible colour pairs with appropriate distances and their numbers in $\breve{P}_n$ are listed in the following table.

When n is even, the chromatic D-polynomial can be determined from the values of the Table - 3. Thus, $D_{\varphi^+}(\breve{P}_n,x,y) = nx^3y^2 + n-1(x^2y^3+xy^3) + \frac{n}{2}x^2y + \frac{(n-2)}{2}xy^2 + n(n-1)x^3y^3 + (3n-4)(x^2y+x^3y) + 2(n-1)x^2y^2 + (n-2)x^3y + \frac{3(n-2)(n-3)}{2}(x^2y^3+x^3y^3) + \frac{3(n-2)}{2}x^2y + \frac{3(n-4)}{2}xy^2 + \frac{(n-2)(n-4)}{2}(x^2y^2+xy+x^2y) + \frac{(n-4)(n-6)}{2}xy^2$.

Further simplifications gives the result as: $D_{\varphi^+}(\breve{P}_n,x,y) = n(n-1)x^3y^3 + nx^3y^2 + \frac{3n^2-7n+8}{2}(x^2y^3+xy^3) + \frac{n^2-2n+4}{2}x^2y^2 + \frac{n^2-2n+2}{2}x^2y + \frac{n^2-6n+10}{2}xy^2 + \frac{n^2-4n+4}{2}xy$.

Part (ii): When n is odd. According to the rules of φ^+colouring, the independent set U is coloured with the colour c_3. Now, the vertices in V are alternatively coloured with c_2 and c_1. Also, v_{2n+1} can have the colour c_2. Thus, $\theta(c_3) = n$, $\theta(c_2) = \frac{n+3}{2}$ and $\theta(c_1) = \frac{n-1}{2}$. The following table analyses the possible distances in terms of the colour pairs required.

Distance $d(v_i,v_j)$	Colour pairs	Number of pairs
1	(c_3,c_2)	n
	(c_2,c_3)	$n-1$
	(c_1,c_3)	$n-1$
	(c_2,c_1)	$\frac{n}{2}$
	(c_1,c_2)	$\frac{n-2}{2}$
2	(c_3,c_3)	$\frac{n(n-1)}{2}$
	(c_2,c_3)	$\frac{3n-4}{2}$
	(c_1,c_3)	$\frac{3n-4}{2}$
	(c_2,c_2)	$n-1$
	(c_1,c_1)	$\frac{n-2}{2}$
3	(c_2,c_3)	$\frac{(n-2)(n-3)}{2}$
	(c_1,c_3)	$\frac{(n-2)(n-3)}{2}$
	(c_2,c_1)	$\frac{n-2}{2}$
	(c_1,c_2)	$\frac{n-4}{2}$
4	(c_2,c_2)	$\frac{(n-2)(n-4)}{8}$
	(c_1,c_1)	$\frac{(n-2)(n-4)}{8}$
	(c_2,c_1)	$\frac{(n-2)(n-4)}{8}$
	(c_1,c_2)	$\frac{(n-4)(n-6)}{8}$

TABLE 3. Table (CDP) for $\check{P}_n$ when n even, with φ^+colouring

From the Table - 4, and the definitions of chromatic D-polynomials will give the result as follows: $D_{\varphi^+}(\check{P}_n,x,y) = nx^3y^2 + (n-1)(x^2y^3+xy^3) + \frac{n-1}{2}(x^2y+xy^2) + n(n-1)x^3y^3 + (3n-1)x^2y^3 + 2nx^2y^2 + (3n-7)xy^3 + 2(n-2)xy^2 + (n-3)xy + 9(n+2)x^2y^3 + \frac{3(n-1)(n-3)}{8}x^2y + 3(3n-1)xy^3 + \frac{3(n-3)(n-5)}{8}xy^2 + \frac{(n-1)(n-3)}{2}x^2y^2 + \frac{(n-3)(n-5)}{2}(x^2y+xy^2+xy)$.

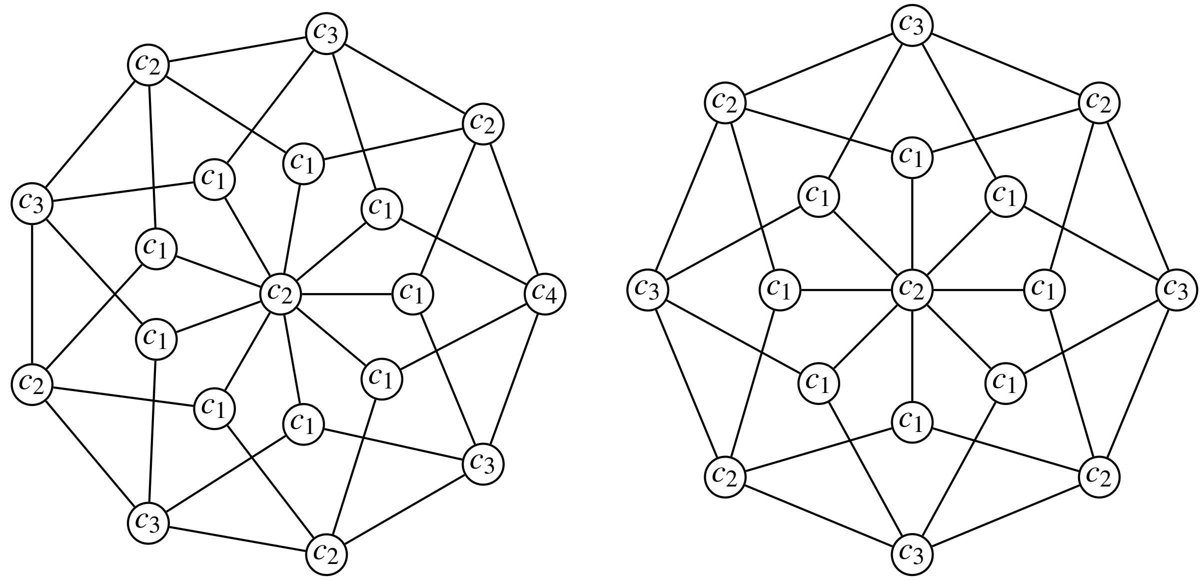
Further simplifications gives the result as $D_{\varphi^+}(\check{P}_n,x,y) = n(n-1)x^3y^3 + nx^3y^2$
$+ \frac{7n^2-36n+69}{8}xy^2 + \frac{n^2+3}{2}x^2y^2 + \frac{7n^2-40n+65}{8}x^2y + (13n-11)xy^3 + (13n+16)x^2y^3 + \frac{n^2-6n+9}{2}xy.$
This completes the proof. □

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_3, c_2)	n
	(c_2, c_3)	$n-1$
	(c_1, c_3)	$n-1$
	(c_2, c_1)	$\frac{n-1}{2}$
	(c_1, c_2)	$\frac{n-1}{2}$
2	(c_3, c_3)	$\frac{n(n-1)}{2}$
	(c_2, c_3)	$\frac{3n-1}{2}$
	(c_2, c_2)	n
	(c_1, c_3)	$\frac{3n-7}{2}$
	(c_1, c_2)	$n-2$
	(c_1, c_1)	$\frac{(n-3)}{2}$
3	(c_2, c_3)	$3(n+2)$
	(c_1, c_3)	$3n-1$
	(c_2, c_1)	$\frac{(n-1)(n-3)}{8}$
	(c_1, c_2)	$\frac{(n-3)(n-5)}{8}$
4	(c_2, c_2)	$\frac{(n-1)(n-3)}{8}$
	(c_2, c_1)	$\frac{(n-3)(n-5)}{8}$
	(c_1, c_2)	$\frac{(n-3)(n-5)}{8}$
	(c_1, c_1)	$\frac{(n-3)(n-5)}{8}$

TABLE 4. Table (CDP) for $\check{P}_n$ when n odd, with φ^+colouring

Theorem 2.3. *For the Mycielskian of a cycle C_n, we have*

$$D_{\varphi^-}(\check{C}_n, x, y) = \begin{cases} nxy^2 + \frac{3n^2-7n}{2}x^2y + \frac{n(n-2)}{2}x^2y^2 + \frac{3n(n-3)}{2}x^3y + \frac{n(n-6)}{2}x^3y^3 + \\ \frac{n^2-18}{2}x^3y^2 + \frac{n^2-2n-14}{2}x^2y^3, \quad \text{if } n \text{ is even,} \\ n(n-1)xy + nxy^2 + \frac{3n^2-10n+7}{2}(x^2y + x^3y) + \frac{n^2-4n+5}{2}x^2y^2 + \frac{n^2-4n-9}{2}x^2y^3 + \\ 2(n-4)x^2y^4 + \frac{n^2-2n-13}{2}x^3y^2 + \frac{n^2-6n+1}{2}x^3y^3 + 2(n-5)x^3y^4 + 3n-7x^4y + \\ 5x^4y^2, \quad \text{if } n \text{ is odd.} \end{cases}$$



(A) The Mycielski graph $\breve{C}_9$ with φ^-colouring (B) The Mycielski graph $\breve{C}_8$ with φ^-colouring

FIGURE 3

Proof. Consider the Mycielskian of a cycle, $\breve{C}_n$ on $2n+1$ vertices and $4n$ edges. When n is even, $\chi(\breve{C}_n)=3$ and when n is odd, $\chi(\breve{C}_n)=4$. The largest independent set $\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$ is named as U and the root vertex is v_{2n+1}. The vertices $\{v_1,v_2,\dots,v_n\}$ on the rim of the cycle C_n are named V. Since $\breve{C}_n$ has diameter 4, the distance $d(v_i,v_j)$ can vary from 1 to 4. Now consider the following cases:

Part (i): When n is even. Then according to the rules of φ^-colouring, the independent set of inner vertex set $U=\{v_{n+1},v_{n+2},\cdots,v_{2n}\}$ is coloured with the colour c_1. Now, the vertices in V are alternatively coloured with c_2 and c_3. Since the root vertex v_{2n+1} is adjacent to all vertices of U and v_{2n+1} is not adjacent to any vertex in V, v_{2n+1} can have the colour c_2. For the Mycielskian of the cycle, $\theta(c_1)=n$, $\theta(c_2)=\frac{n+2}{2}$ and $\theta(c_3)=\frac{n}{2}$. The possible colour pairs with appropriate distances and their numbers in $\breve{C}_n$ are listed in the following table.

When n is even, the chromatic D-polynomial can be determined from the values of the Table - 5. Thus, $D_{\varphi^-}(\breve{C}_n,x,y)=nxy^2+\frac{3n^2-7n}{2}x^2y+\frac{n(n-2)}{2}x^2y^2+\frac{3n(n-3)}{2}x^3y+\frac{n(n-6)}{2}x^3y^3+\frac{n^2-18}{2}x^3y^2+\frac{n^2-2n-14}{2}x^2y^3$.

Part (ii): When n is odd. We apply the φ^-colouring to the graph $\breve{C}_n$ as follows: In this case the largest independent set U receives the colour c_1 and the root vertex is coloured with c_2.

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_1, c_2)	n
	(c_2, c_1)	n
	(c_3, c_1)	n
	(c_2, c_3)	$\frac{n+2}{2}$
	(c_3, c_2)	$\frac{n-2}{2}$
2	(c_1, c_1)	$\frac{n(n-1)}{2}$
	(c_2, c_1)	n
	(c_2, c_2)	n
	(c_3, c_1)	n
	(c_3, c_2)	$\frac{n}{2}$
	(c_2, c_1)	$\frac{n}{2}$
3	(c_2, c_1)	$\frac{n(n-5)}{2}$
	(c_3, c_1)	$\frac{n(n-5)}{2}$
	(c_2, c_3)	$\frac{n}{2}$
	(c_3, c_2)	$\frac{n}{2}$
4	(c_2, c_2)	$\frac{n(n-6)}{8}$
	(c_3, c_3)	$\frac{n(n-6)}{8}$
	(c_2, c_3)	$\frac{(n+2)(n-8)}{8}$
	(c_3, c_2)	$\frac{(n+2)(n-8)}{8}$

TABLE 5. Table (CDP) for $\breve{C}_n$ when n even, with φ^-colouring

Now, C_n is 3 colourable and we can colour the vertices in V using three colours, say c_2, c_3, c_4 such that $\lfloor \frac{n}{2} \rfloor$ vertices have colours c_2 and c_3, while one vertex has colour c_4. Thus, $\theta(c_1) = n$, $\theta(c_2) = \frac{n+1}{2}$, $\theta(c_3) = \frac{n-1}{2}$ and $\theta(c_4) = 1$. The following table analyses the possible distances in terms of the colour pairs required.

Distance $d(v_i,v_j)$	Colour pairs	Number of pairs
1	(c_1,c_2)	n
	(c_2,c_1)	$n-1$
	(c_3,c_1)	$n-1$
	(c_2,c_3)	$\frac{n-1}{2}$
	(c_3,c_2)	$\frac{n-3}{2}$
	(c_4,c_1)	2
	(c_3,c_4)	1
	(c_2,c_4)	1
2	(c_1,c_1)	$\frac{n(n-1)}{2}$
	(c_2,c_1)	$\frac{3(n-1)}{2}$
	(c_3,c_1)	$\frac{3(n-1)}{2}$
	(c_3,c_2)	$\frac{(n-1)}{2}$
	(c_3,c_3)	$\frac{(n-3)}{2}$
	(c_2,c_2)	$n-2$
	(c_4,c_1)	3
	(c_2,c_4)	1
	(c_4,c_2)	1
	(c_2,c_3)	1
3	(c_2,c_1)	$\frac{(n-1)(n-5)}{2}$
	(c_3,c_1)	$\frac{(n-1)(n-5)}{2}$
	(c_2,c_3)	$\frac{(n-5)}{2}$
	(c_3,c_2)	$\frac{(n-5)}{2}$
	(c_4,c_1)	$n-5$
	(c_2,c_2)	1
	(c_2,c_4)	1
	(c_3,c_4)	1
	(c_4,c_2)	1

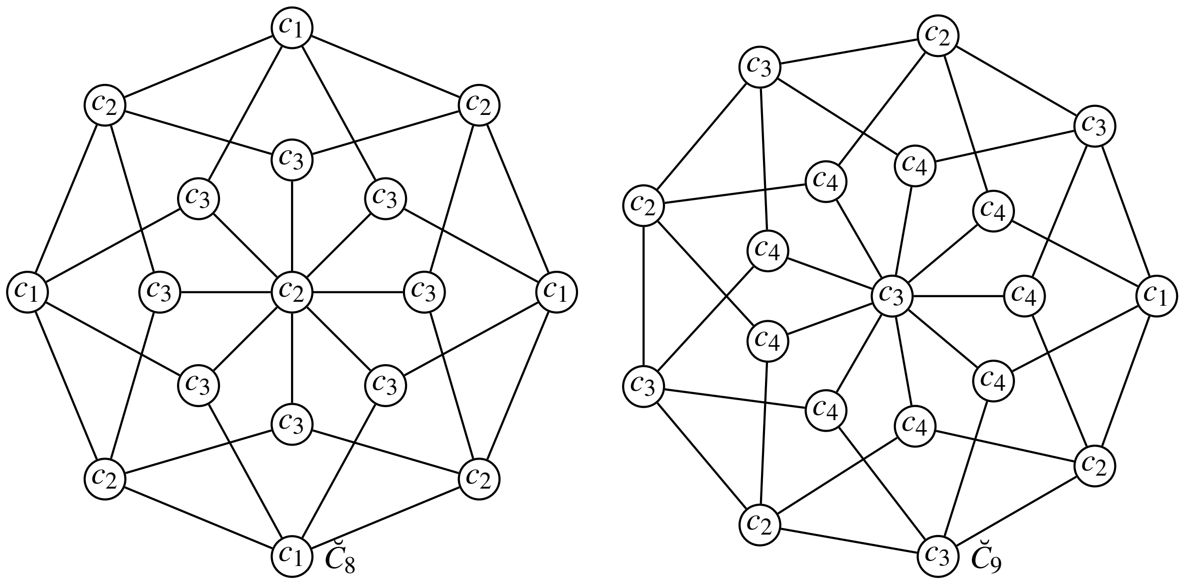
4	(c_2,c_2)	$\frac{(n-1)(n-7)}{8}$
	(c_2,c_3)	$\frac{(n-1)(n-7)}{8}$
	(c_3,c_2)	$\frac{(n-1)(n-7)}{8}$
	(c_3,c_3)	$\frac{(n-1)(n-7)}{8}$
	(c_2,c_4)	$\frac{(n-7)}{2}$
	(c_3,c_4)	$\frac{(n-7)}{2}$

TABLE 6. Table (CDP) for $\breve{C}_n$ when n odd, with φ^-colouring

The Table - 6 and the definitions of chromatic D-polynomials will give the result as follows: $D_{\varphi^-}(\breve{C}_n,x,y) = n(n-1)xy + nxy^2 + \frac{3n^2-10n+7}{2}(x^2y+x^3y) + \frac{n^2-4n+5}{2}x^2y^2 + \frac{n^2-4n-9}{2}x^2y^3 + 2(n-4)x^2y^4 + \frac{n^2-2n-13}{2}x^3y^2 + \frac{n^2-6n+1}{2}x^3y^3 + 2(n-5)x^3y^4 + (3n-7)x^4y + 5x^4y^2$. This completes the proof. □

Theorem 2.4. *For the Mycielskian of a cycle C_n, we have*

$$D_{\varphi^+}(\breve{C}_n,x,y) = \begin{cases} nx^3y^2 + \frac{3n^2-7n}{2}x^2y^3 + \frac{n(n-2)}{2}x^2y^2 + \frac{3n(n-3)}{2}xy^3 + \frac{n(n-6)}{2}xy + \\ \frac{n^2-18}{2}xy^2 + \frac{n^2-2n-14}{2}x^2y, \quad \textit{if } n \textit{ is even,} \\ n(n-1)x^4y^4 + nx^4y^3 + \frac{3n^2-10n+7}{2}(x^3y^4+x^2y^4) + \frac{n^2-4n+5}{2} + x^3y^3 \\ + \frac{n^2-4n-9}{2}x^3y^2 + 2(n-4)x^3y + \frac{n^2-2n-13}{2}x^2y^3 + \frac{n^2-6n+1}{2}x^2y^2 + \\ 2(n-5)x^2y + (3n-7)xy^4 + 5xy^3, \quad \textit{if } n \textit{ is odd.} \end{cases}$$



(A) The Mycielski graph $\breve{C}_8$ with φ^+colouring (B) The Mycielski graph $\breve{C}_9$ with φ^+colouring

FIGURE 4

Proof. Consider the Mycielskian of a cycle, $\breve{C}_n$ on $2n+1$ vertices and $4n$ edges. When n is even, $\chi(\breve{C}_n) = 3$ and when n is odd, $\chi(\breve{C}_n) = 4$. Since $\breve{C}_n$ has diameter 4, the distance $d(v_i, v_j)$ can vary from 1 to 4. Now consider the following cases:

Part (i): When n is even. Then according to the rules of φ^+ colouring, the independent set of inner vertex set $U = \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$ is coloured with the colour c_3. Now, the vertices in V are alternatively coloured with c_2 and c_1. The vertex v_{2n+1} can have the colour c_2. For the Mycielskian of the cycle, $\theta(c_3) = n$, $\theta(c_2) = \frac{n+2}{2}$ and $\theta(c_1) = \frac{n}{2}$. The possible colour pairs with appropriate distances and their numbers in $\breve{C}_n$ are listed in the following table.

Distance $d(v_i,v_j)$	Colour pairs	Number of pairs
1	(c_3,c_2)	n
	(c_2,c_3)	n
	(c_1,c_3)	n
	(c_2,c_1)	$\frac{n+2}{2}$
	(c_1,c_2)	$\frac{n-2}{2}$
2	(c_3,c_3)	$\frac{n(n-1)}{2}$
	(c_2,c_3)	n
	(c_2,c_2)	n
	(c_1,c_3)	n
	(c_1,c_2)	$\frac{n}{2}$
	(c_2,c_3)	$\frac{n}{2}$
3	(c_2,c_3)	$\frac{n(n-5)}{2}$
	(c_1,c_3)	$\frac{n(n-5)}{2}$
	(c_2,c_1)	$\frac{n}{2}$
	(c_1,c_2)	$\frac{n}{2}$
4	(c_2,c_2)	$\frac{n(n-6)}{8}$
	(c_1,c_1)	$\frac{n(n-6)}{8}$
	(c_2,c_1)	$\frac{(n+2)(n-8)}{8}$
	(c_1,c_2)	$\frac{(n+2)(n-8)}{8}$

TABLE 7. Table (CDP) for $\check{C}_n$ when n even, with φ^+colouring

When n is even, the chromatic D-polynomial can be determined from the values of the Table - 7. Thus, we have $D_{\varphi^+}(\check{C}_n,x,y) = nx^3y^2 + \frac{3n^2-7n}{2}x^2y^3 + \frac{n(n-2)}{2}x^2y^2 + \frac{3n(n-3)}{2}xy^3 + \frac{n(n-6)}{2}xy + \frac{n^2-18}{2}xy^2 + \frac{n^2-2n-14}{2}x^2y$.

Part (ii): When n is odd. We apply the φ^+colouring to the graph $\check{C}_n$ as follows: In this case the largest independent set U receives the colour c_4 and the root vertex is coloured with c_3. Now, C_n is 3 colourable and we can colour the vertices in V using three colours, say c_3,c_2,c_1

such that $\lfloor \frac{n}{2} \rfloor$ vertices have colours c_3 and c_2, while one vertex has colour c_1. Thus, $\theta(c_4) = n$, $\theta(c_3) = \frac{n+1}{2}$, $\theta(c_2) = \frac{n-1}{2}$ and $\theta(c_1) = 1$. The Table - 8 analyses the possible distances in terms of the colour pairs required and the definitions of chromatic D-polynomials will give the result completing the proof as follows:

Distance $d(v_i, v_j)$	Colour pairs	Number of pairs
1	(c_4, c_3)	n
	(c_3, c_4)	$n-1$
	(c_2, c_4)	$n-1$
	(c_3, c_2)	$\frac{n-1}{2}$
	(c_2, c_3)	$\frac{n-3}{2}$
	(c_1, c_4)	2
	(c_2, c_1)	1
	(c_3, c_1)	1
2	(c_4, c_4)	$\frac{n(n-1)}{2}$
	(c_3, c_4)	$\frac{3(n-1)}{2}$
	(c_2, c_4)	$\frac{3(n-1)}{2}$
	(c_2, c_3)	$\frac{(n-1)}{2}$
	(c_2, c_2)	$\frac{(n-3)}{2}$
	(c_3, c_3)	$n-2$
	(c_1, c_4)	3
	(c_3, c_1)	1
	(c_1, c_3)	1
	(c_3, c_2)	1
3	(c_3, c_4)	$\frac{(n-1)(n-5)}{2}$
	(c_2, c_4)	$\frac{(n-1)(n-5)}{2}$
	(c_3, c_2)	$\frac{(n-5)}{2}$
	(c_2, c_3)	$\frac{(n-5)}{2}$
	(c_1, c_4)	$n-5$
	(c_3, c_3)	1

	(c_3,c_1)	1
	(c_2,c_1)	1
	(c_1,c_3)	1
4	(c_3,c_3)	$\frac{(n-1)(n-7)}{8}$
	(c_3,c_2)	$\frac{(n-1)(n-7)}{8}$
	(c_2,c_3)	$\frac{(n-1)(n-7)}{8}$
	(c_2,c_2)	$\frac{(n-1)(n-7)}{8}$
	(c_3,c_1)	$\frac{(n-7)}{2}$
	(c_2,c_1)	$\frac{(n-7)}{2}$

TABLE 8. Table (CDP) for $\breve{C}_n$ when n odd, with φ^+colouring

$D_{\varphi^+}(\breve{C}_n,x,y)=n(n-1)x^4y^4+nx^4y^3+\frac{3n^2-10n+7}{2}(x^3y^4+x^2y^4)+\frac{n^2-4n+5}{2}x^3y^3+\frac{n^2-4n-9}{2}x^3y^2+2(n-4)x^3y+\frac{n^2-2n-13}{2}x^2y^3+\frac{n^2-6n+1}{2}x^2y^2+2(n-5)x^2y+(3n-7)xy^4+5xy^3.$ □

3. CONCLUSION

In this article, we have discussed two particular types of colouring related polynomials, called chromatic D-polynomials, of the Mycielskian of paths and cycles. The study seems to be promising for further studies as the polynomial can be computed for many graph classes and classes of derived graphs. The chromatic D-polynomial can be determined for graph operations, graph products and graph powers also. The study on chromatic D-polynomials with respect to different types of graph colourings also seem to be much promising. The concept can be extended to edge colourings and map colourings also.

The chromatic D-polynomials have numerous applications in various fields like Mathematical Chemistry, Distribution Theory, Optimization Techniques etc. Similar studies are possible in various other fields. All these facts highlight the wide scope for further research in this area.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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