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APPROXIMATION OF RADICAL TYPE RATIONAL MAPPINGS

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Abstract. The primary aim of this paper is to solve new radical type rational functional equations. The existence of approximate radical type rational mappings satisfying these equations pertinent to Ulam stability theory are then proved using direct and fixed point methods in quasi β-Banach spaces. An apt example is also demonstrated to prove the non-stability for singular case.

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1. Introduction & Preliminaries

The foundation to the solution of stability problem connected with mathematical equalities is laid down via the following celebrated query in [26]: "Does there exist an approximate solution near to the exact solution of an equation". Mathematically, this query can be stated as follows: "Suppose Y_1 and Y_2 are a group and a metric group with a metric d defined on Y_2, respectively. Let β be a positive constant. Then is it possible to exist a positive constant α such

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that if a function $p : Y_1 \longrightarrow Y_2$ satisfies $d(p(ab), p(a)p(b)) < \alpha$ for all $a, b \in Y_1$, then a homomorphism $P : Y_1 \longrightarrow Y_2$ exists and approximates $d(p(a), P(a)) < \beta$ for all $a \in Y_1$?". If there is an affirmative answer to this query, then the equation is said to hold stability. The foremost affirmative response to this query was provided in [8] by considering Banach spaces instead of metric group. This first partial answer is termed as Ulam-Hyers stability and it involves a positive constant as an upper bound. Later, the responses were extended in different versions by many mathematicians by considering the upper bounds as addition of exponents of norms [17], multiplication of distinct exponents of norms [12], a general control function [6]. These interesting results motivated many researchers to investigate the stability results of a variety of functional equations with solutions as second power, third power, fourth power, fifth power, sixth power, mixed type, rational type and radical type functions. The detailed information is available in [2, 7, 13, 14, 15, 16]. There are many novel, noteworthy, interesting and ground breaking stability results concerning several forms of radical functional equations in [1, 4, 9, 11]. Recently, this research field has attracted towards different forms of rational functional equations and their stability results and applications, (see [3, 10, 18, 19, 20, 21, 22, 23, 24, 25]).

We will recall here some fundamental notions and few basic results concerning quasi β-normed spaces. Suppose $0 < \beta \leq 1$ is a fixed real constant. Let $\mathbb{H}$ denote either $\mathbb{R}$, the set of real numbers or $\mathbb{C}$, the set of complex numbers. Let ϕ be a linear space defined over $\mathbb{H}$. Let $\|\cdot\|$ be a real-valued mapping defined on ϕ satisfying the following properties.

(i) $\|a\| \geq 0$ for every $a \in \phi$ and $\|a\| = 0 \Leftrightarrow a = 0$;

(ii) $\|\mu a\| = |\mu|^{\beta} \cdot \|a\|$ for every $\mu \in \mathbb{H}$ and every $a \in \phi$;

(iii) There exists a constant $\lambda \geq 1$ with the condition that $\|a + b\| \leq \lambda (\|a\| + \|b\|)$, for all $a, b \in \phi$.

Then the mapping $\|\cdot\|$ is said to be a quasi β-norm on ϕ. If $\|\cdot\|$ is a quasi β-norm on ϕ, then the pair $(\phi, \|\cdot\|)$ is called a quasi β-normed space. The minimum value of λ is called the modulus of concavity of $\|\cdot\|$.

Suppose $0 < p \leq 1$. If $\|a + b\|^p \leq \|a\|^p + \|b\|^p$, then a quasi β-norm $\|\cdot\|$ is called a (β, p)-norm for all $a, b \in \phi$. For this instance, a quasi β-Banach space is called a (β, p)-Banach space.

In this article, we focus on the following new radical type rational functional equations

$$h(a+b+2\sqrt{ab}) = \frac{h(a)h(b)}{h(a)+h(b)} \tag{1.1}$$

and

$$h(4a+b+4\sqrt{ab})+h(4a+b-4\sqrt{ab}) = \frac{4h(a)h(b)^2}{4h(b)^2-h(a)^2}. \tag{1.2}$$

We solve the above equations (1.1) and (1.2) for their solutions and prove the existence of approximate radical rational mappings near to their exact solutions using direct and fixed point methods. To prove our main results, we have considered quasi β-Banach spaces as the range of the radical rational mapping satisfying equations (1.1) and (1.2). We have portrayed suitable illustration to disprove the stability for singular case.

2. Mappings Satisfying Equations (1.1) and (1.2)

In this section, we find the mappings satisfying equations (1.1) and (1.2).

Theorem 2.1. *If $h:\mathbb{R}^+ \longrightarrow \mathbb{R}$ is a mapping satisfying equation* (1.1)*, then h is a radical rational mapping of the form $h(a) = \frac{1}{\sqrt{a}}$ for all $a \in \mathbb{R}^+$.*

Proof. Substituting $b = a$ in equation (1.1), we get

$$h(4a) = \frac{1}{2}h(a) = 2^{-1}h(a) \tag{2.1}$$

for all $a \in \mathbb{R}^+$. Now, replacing a by $4a$ in equation (2.1), we obtain

$$h(4^2a) = 2^{-2}h(a) \tag{2.2}$$

for all $a \in \mathbb{R}^+$. Again, replacing a by $4a$ in (2.2), we find that

$$h(4^3a) = 2^{-3}h(a) \tag{2.3}$$

for all $a \in \mathbb{R}^+$. Proceeding further and using mathematical induction on a positive integer n, we arrive at

$$h(4^na) = 2^{-n}h(a),$$

for all $a \in \mathbb{R}^+$. Hence, we conclude that $h(a) = \frac{1}{\sqrt{a}}$ is a solution of equation (1.1). $\square$

Remark 2.2. Applying similar arguments as in Theorem 2.1, it can be verified that $h(a) = \frac{1}{\sqrt{a}}$ satisfies equation (1.2).

Definition 2.3. A mapping $h:\mathbb{R}^+\longrightarrow\mathbb{R}$ is called a radical rational mapping if it satisfies equations (1.1) and (1.2).

3. Hyers' Approach to the Stabilities of Equations (1.1) and (1.2)

In this section, by employing Hyers' approach (direct method) to establish the stability of equations (1.1) and (1.2) in quasi β-Banach spaces with a general control function as an upper bound. Throughout this study, let us consider E to be a complete quasi β-normed space and $\chi:\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ be a function. A mapping $h:\mathbb{R}^+\longrightarrow E$ is called a χ-existence of inexact radical rational mapping, if

$$\|\Delta_1 h(a,b)\|\leq\chi(a,b) \tag{3.1}$$

and

$$\|\Delta_2 h(a,b)\|\leq\chi(a,b) \tag{3.2}$$

where

$$\Delta_1 h(a,b)=h(a+b+2\sqrt{ab})-\frac{h(a)h(b)}{h(a)+h(b)}$$

and

$$\Delta_2 h(a,b)=h(4a+b+4\sqrt{ab})+h(4a+b-4\sqrt{ab})-\frac{4h(a)h(b)^2}{4h(b)^2-h(a)^2}$$

for all $a,b\in\mathbb{R}^+$.

Theorem 3.1. *Let $h:\mathbb{R}^+\longrightarrow E$ be a mapping satisfying* (3.1). *If $\chi:\mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ satisfies the following conditions for all $a,b\in\mathbb{R}^+$,*

$$\mu(a)=\sum_{i=0}^{\infty}(\omega 2^{\beta})^i\chi(4^i a,4^i a)<\infty \tag{3.3}$$

and

$$\lim_{n\to\infty}2^{n\beta}\chi(4^n a,4^n b)=0, \tag{3.4}$$

then there exists a unique solution $H:\mathbb{R}^+\longrightarrow E$ of equation (1.1) *and*

$$\|h(a)-H(a)\|\leq\omega 2^{\beta}\mu(a) \tag{3.5}$$

for all $a\in\mathbb{R}^+$.

Proof. Considering $b = a$ in (3.1), we get

$$\left\|h(4a) - \frac{1}{2}h(a)\right\| \leq \chi(a,a) \tag{3.6}$$

for all $a \in \mathbb{R}^+$. Inequality (3.6) can be written as

$$\|2h(4a) - h(a)\| \leq 2^{\beta}\chi(a,a) \tag{3.7}$$

for all $a \in \mathbb{R}^+$. Then by the induction procedure on a positive integer n, we get

$$\|2^n h(4^n a) - h(a)\| \leq \omega 2^{\beta} \sum_{i=0}^{n-1} (\omega 2^{\beta})^i \chi(4^i a, 4^i a) \tag{3.8}$$

for all $a \in \mathbb{R}^+$. Owing to (3.5) and taking the limit $n \to \infty$ in the above inequality, we can find that the sequence $\{2^n h(4^n a)\}$ turns to be Cauchy. Since E is complete, it converges to a mapping $H : \mathbb{R}^+ \longrightarrow E$ defined by

$$H(a) = \lim_{n\to\infty} 2^n h(4^n a)$$

for all $a \in \mathbb{R}^+$. Next, by the virtue of (3.1) and (3.4), we have

$$\begin{aligned}\|\Delta_1 H(a,b)\| &= \lim_{n\to\infty} 2^{\beta n} \|\Delta_1(4^n a, 4^n b)\| \\ &\leq \lim_{n\to\infty} 2^{\beta n} \chi(4^n a, 4^n b) = 0\end{aligned}$$

for all $a, b \in \mathbb{R}^+$. Therefore H satisfies equation (1.1) on $\mathbb{R}^+$. Next, we claim that H is unique. To justify this claim, let us consider there is an additional radical rational mapping $H' : \mathbb{R}^+ \longrightarrow E$ satisfying (1.1) and (3.5). Then, we have

$$H(4^n a) = \frac{1}{2^n} H(a)$$

and

$$H'(4^n a) = \frac{1}{2^n} H'(a)$$

for all $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Thus

$$\begin{aligned}\left\|H(a) - H'(a)\right\| &= \left\|2^n H(4^n a) - 2^n H'(4^n a)\right\| \\ &\leq \omega 2^{n\beta} \left(\|H(4^n a) - h(4^n a)\| + \left\|h(4^n a) - H'(4^n a)\right\|\right) \\ &\leq 2\omega^2 . 2^{\beta(n+1)} \sum_{i=0}^{\infty} (\omega 2^{\beta})^i \chi(4^{i+n} a, 4^{i+n} a)\end{aligned}$$

for all $a \in \mathbb{R}^+$. Therefore, as $n \to \infty$ in the above inequality, we have $H' = H$ for all $a \in \mathbb{R}^+$. Therefore, H is unique. □

Theorem 3.2. *Let* $h : \mathbb{R}^+ \longrightarrow E$ *be a mapping satisfying* (3.1)*. If* $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *satisfies the following inequality*

$$\mu(a) = \sum_{i=0}^{\infty} \left(\frac{\omega}{2^\beta}\right)^i \chi\left(\frac{a}{4^i}, \frac{a}{4^i}\right) < \infty \tag{3.9}$$

and

$$\lim_{n\to\infty} \frac{1}{2^{n\beta}} \chi\left(\frac{a}{4^n}, \frac{b}{4^n}\right) = 0 \tag{3.10}$$

for all $a, b \in \mathbb{R}^+$*, then there exists a unique solution* $H : \mathbb{R}^+ \longrightarrow E$ *of equation* (1.1) *and*

$$\|h(a) - H(a)\| \leq \omega 2^{-\beta} \mu(a) \tag{3.11}$$

for all $a \in \mathbb{R}^+$.

Proof. The proof is directly obtained by replacing a by $\dfrac{a}{4}$ in the inequality (3.7) of Theorem 3.1 and proceeding with the similar arguments. □

In the following corollaries and theorem, let us assume that $h : \mathbb{R}^+ \longrightarrow E$ is a mapping.

Corollary 3.3. *Let* c *and* d *be any real numbers* $c, d \neq -\dfrac{\beta}{2}$ *and* $\lambda \geq 0$*. Assume that* h *satisfies*

$$\|\Delta_1 h(a,b)\| \leq \lambda\left(|a|^c + |b|^d\right) \tag{3.12}$$

for all $a, b \in \mathbb{R}^+$*, then there exists a unique solution* $H : \mathbb{R}^+ \longrightarrow E$ *of equation* (1.1) *and*

$$\|h(a) - H(a)\| \leq \begin{cases} \lambda\left(\dfrac{|a|^c}{2^{-\beta} - 4^c} + \dfrac{|a|^d}{2^{-\beta} - 4^d}\right) & \text{for } c, d < -\dfrac{\beta}{2} \\ \lambda\left(\dfrac{|a|^c}{4^c - 2^{-\beta}} + \dfrac{|a|^d}{4^d - 2^{-\beta}}\right) & \text{for } c, d > -\dfrac{\beta}{2} \end{cases} \tag{3.13}$$

for all $a \in \mathbb{R}^+$.

Proof. The required result is obtained using Theorems 3.1 and 3.2 by taking $\chi(a,b) = \lambda(|a|^c + |b|^d)$. □

Corollary 3.4. *Let* $c, d \in \mathbb{R}^+ \cup \{0\}$ *with* $c + d \neq -\dfrac{\beta}{2}$ *and* h *satisfies*

$$\|\Delta_1 h(a,b)\| \leq \lambda |a|^c |b|^d$$

for all $a, b \in \mathbb{R}^+$*, then there exists a unique solution* $H : \mathbb{R}^+ \longrightarrow E$ *of equation* (1.1) *and*

$$\|h(a) - H(a)\| \leq \begin{cases} \dfrac{\lambda}{2^{-\beta} - 4^{c+d}} |a|^{c+d} & \text{for } c + d < -\dfrac{\beta}{2} \\ \dfrac{\lambda}{4^{c+d} - 2^{-\beta}} |a|^{c+d} & \text{for } c + d > -\dfrac{\beta}{2} \end{cases} \tag{3.14}$$

for all $a \in \mathbb{R}^+$.

Proof. By replacing $\chi(a,b)$ with $\lambda|a|^c|b|^d$ in Theorems 3.1 and 3.2, the proof follows. □

Theorem 3.5. *Let h satisfies* (3.2)*. If* $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *satisfies the following*

$$\mu(a) = \sum_{i=0}^{\infty} (\omega 3^{\beta})^i \chi(9^i a, 9^i a) < \infty \tag{3.15}$$

and

$$\lim_{n\to\infty} 3^{n\beta} \chi(9^n a, 9^n b) = 0 \tag{3.16}$$

for all $a,b \in \mathbb{R}^+$*, then there exists a unique solution* $H : \mathbb{R}^+ \longrightarrow E$ *of equation* (1.2) *and*

$$\|h(a) - H(a)\| \leq \omega 3^{\beta} \mu(a) \tag{3.17}$$

for all $a \in \mathbb{R}^+$.

Proof. Substituting $b = a$ in (3.2), we get

$$\left\| h(9a) - \frac{1}{3} h(a) \right\| \leq \chi(a,a) \tag{3.18}$$

for all $a \in \mathbb{R}^+$. Now, multiplying 3 on both sides of (3.18), we find that

$$\|3h(9a) - h(a)\| \leq 3^{\beta} \chi(a,a) \tag{3.19}$$

for all $a \in \mathbb{R}^+$. Then by the induction process on a positive integer n, we get

$$\|3^n h(9^n a) - h(a)\| \leq \omega 3^{\beta} \sum_{i=0}^{n-1} (\omega 3^{\beta})^i \chi(9^i a, 9^i a) \tag{3.20}$$

for all $a \in \mathbb{R}^+$. In view of (3.17) and allowing $n \to \infty$ in the above inequality, the sequence $\{3^n h(9^n a)\}$ becomes Cauchy in E. By the completeness of E, it converges to a mapping $H : \mathbb{R}^+ \longrightarrow E$ defined by

$$H(a) = \lim_{n\to\infty} 3^n h(9^n a)$$

for all $a \in \mathbb{R}^+$. The remaining part of the proof is proceeded with parallel arguments as in Theorem 3.1. □

Theorem 3.6. *Let h satisfies* (3.2)*. If* $\chi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ *satisfies the following inequality*

$$\mu(a) = \sum_{i=0}^{\infty} \left(\frac{\omega}{3^{\beta}}\right)^i \chi\left(\frac{a}{9^i}, \frac{a}{9^i}\right) < \infty \tag{3.21}$$

and

$$\lim_{n\to\infty}\frac{1}{3^{n\beta}}\chi\left(\frac{a}{9^n},\frac{b}{9^n}\right)=0 \tag{3.22}$$

for all $a,b\in\mathbb{R}^+$, then there exists a unique solution $H:\mathbb{R}^+\longrightarrow E$ of equation (1.2) *and*

$$\|h(a)-H(a)\|\leq \omega 3^{-\beta}\mu(a) \tag{3.23}$$

for all $a\in\mathbb{R}^+$.

Proof. By the application of Theorem 3.5, the proof is obtained through replacing a by $\frac{a}{9}$ in (3.19). □

Corollary 3.7. *Let there exists real numbers p,q with the condition that $p,q\neq -\frac{\beta}{3}$ and $\lambda\geq 0$. Assume that h satisfies*

$$\|\Delta_2 h(a,b)\|\leq \lambda\left(|a|^p+|b|^q\right) \tag{3.24}$$

for all $a,b\in\mathbb{R}^+$, then there exists a unique solution $H:\mathbb{R}^+\longrightarrow E$ of equation (1.2) *and*

$$\|h(a)-H(a)\|\leq \lambda\left(\frac{|a|^p}{|3^{-\beta}-9^p|}+\frac{|a|^q}{|3^{-\beta}-9^q|}\right) \tag{3.25}$$

for all $a\in\mathbb{R}^+$.

Corollary 3.8. *Let $c,d\in\mathbb{R}^+\cup\{0\}$ with $p+q\neq -\frac{\beta}{3}$ and h satisfies the following inequality*

$$\|\Delta_2 h(a,b)\|\leq \lambda|a|^p|b|^q$$

for all $a,b\in\mathbb{R}^+$. Then there exists a unique solution $H:\mathbb{R}^+\longrightarrow E$ of equation (1.2) *and*

$$\|h(a)-H(a)\|\leq \frac{\lambda}{|3^{-\beta}-9^{p+q}|}|a|^{p+q} \tag{3.26}$$

for all $a\in\mathbb{R}^+$.

4. Approximations of Equations (1.1) and (1.2) in (β,p)-Banach Spaces

In this section, we apply Hyers' method to establish the modified stabilities of equations (1.1) and (1.2) in (β,p)-Banach space. To prove our main results, let us recall fundamental concepts of sub-additive and sub-quadratic functions. Let $\psi:S\longrightarrow T$ be a function with S and $(T,\leq)$ as domain and co-domain, respectively. Let both S and T are closed under the binary operation $+$ and ψ satisfies the following properties:

$$\psi(s+t)\leq\psi(s)+\psi(t)$$

for all $s,t \in S$. Then the function ψ is called as sub-additive. Let $\delta = \pm 1$ be a fixed constant and there exists a constant K with $0 < K < 1$ such that the function $\psi : S \longrightarrow T$ satisfies

$$\delta \psi(s+t) \leq \delta K^{\delta} (\psi(s) + \psi(t))$$

for all $s,t \in S$. Then, the function ψ is called as contractively sub-additive or expansively super-additive if $\delta = 1$ or $\delta = -1$, respectively. Also, it implies that ψ satisfies the succeeding properties:

$$\psi(2^{\delta} s) \leq 2^{\delta} K \psi(s),$$

$$\psi(2^{\delta j} s) \leq (2^{\delta} K)^{j} \psi(s)$$

for all $s \in S$ and $j \geq 1$.

Also, if a function $\phi : S \longrightarrow T$ with $\phi(0) = 0$ satisfies the following property:

$$\phi(s+t) + \phi(s-t) \leq 2\phi(s) + 2\phi(t)$$

for all $s,t \in S$, then it is called as sub-quadratic. Again, if there exists a constant K with $0 < K < 1$ such that the function $\phi : S \longrightarrow T$ with $\phi(0) = 0$ satisfies

$$\delta \phi(s+t) + \delta \phi(s-t) \leq 2\delta K^{\delta} (\phi(s) + \phi(t))$$

for all $s,t \in S$, then we say that ϕ is contractively sub-quadratic or expansively super-quadratic if $\delta = 1$ or $\delta = -1$, respectively. Also, ϕ satisfies the following properties:

$$\phi(2^{\delta} s) \leq 4^{\delta} K \phi(s),$$

$$\phi(2^{\delta j} s) \leq (4^{\delta} K)^{j} \phi(s)$$

for all $s \in S$ and $j \geq 1$.

In the following, we determine the modified stabilities equations (1.1) and (1.2) in (β, p)-Banach space using sub-additive and sub-quadratic functions. In this section, we assume that E to be a (β, p)-Banach space and $h : \mathbb{R}^{+} \longrightarrow E$ be a mapping.

Theorem 4.1. *Assume χ is a contractive sub-additive function with a constant L satisfying $4^{\beta+2} L < 1$. Then there exists a unique solution $H : \mathbb{R}^{+} \longrightarrow E$ of equation* (1.1) *and*

$$\|h(a) - H(a)\| \leq \frac{4^{-\beta}}{\sqrt[p]{4^{-\beta p} - (16L)^{p}}} \zeta(a) \tag{4.1}$$

where $\zeta(a)=\omega 2^{\beta p}\sum_{i=0}^{n-1}(\omega 2^{\beta})^i\chi(4^ia,4^ia)$ *for all* $a\in\mathbb{R}^+$.

Proof. Applying the similar arguments as in Theorem 3.1 for any $n\in\mathbb{N}$, we can show that

$$\left\|h(a)-4^{\frac{n}{2}}h(4^na)\right\|\leq \omega 2^{\beta p}\sum_{i=0}^{n-1}(\omega 2^{\beta})^i\chi(4^ia,4^ia)$$

for all $a\in\mathbb{R}^+$. For $n=2$, we have

$$\|h(a)-4h(16a)\|\leq \omega 2^{\beta p}\chi(a,a)+\omega^2(2^{\beta})^{2p}\chi(4a,4a) \tag{4.2}$$

for all $a\in\mathbb{R}^+$. Then (4.2) takes the following form

$$\|h(a)-4h(16a)\|\leq \zeta(a), \tag{4.3}$$

for all $a\in\mathbb{R}^+$. By the method of induction on a positive integer n, we proceed to have

$$\left\|4^nh(4^{2n}a)-4^{n+1}h(4^{2(n+1)}a)\right\|\leq 4^{n\beta p}\zeta(4^{2n}a),$$

for all $a\in\mathbb{R}^+$ and $n\in\mathbb{N}$. For any $n,l\in\mathbb{N}$, $0\leq l\leq n$, we have

$$\begin{aligned}\left\|4^lh(4^{2l}a)-4^nh(4^{2na})\right\|^p &\leq \sum_{i=1}^{n-1}\left\|4^ih(4^{2i}a)-4^{i+1}h(4^{2(i+1)}a)\right\|^p\\ &\leq \sum_{i=l}^{n-1}4^{i\beta p}\zeta(4^{2i}a)^p\\ &\leq \zeta(a)^p\sum_{i=l}^{n-1}(4^{i\beta p}(4^{2+\beta})L)^{ip} \qquad (4.4)\end{aligned}$$

for all $a\in\mathbb{R}^+$ and $n>l>0$. Thus, the sequence $\left\{4^nh(4^{2n}a)\right\}$ is Cauchy in E and hence it converges to a mapping $H:\mathbb{R}^+\longrightarrow E$ defined as

$$H(a)=\lim_{n\to\infty}4^nh(4^{2n}a)$$

for all $a\in\mathbb{R}^+$. Next, we obtain that

$$\|\Delta_1h(a,b)\|^p\leq\chi(a,b)^p\lim_{n\to\infty}(4^{2+\beta}L)^{np}=0,$$

for all $a,b\in\mathbb{R}^+$ and for this reason the mapping H is a radical rational mapping. Assuming limit as $n\to\infty$ in (4.4) with $l=0$ we assert that

$$\|h(a)-H(a)\|^p\leq\zeta(a)^p\frac{1}{1-(4^{(2+\beta)}L)^p}.$$

Hence, we obtain

$$\|h(a)-H(a)\| \leq \frac{4^{-\beta}}{\sqrt[p]{4^{-\beta p}-(16L)^p}}\zeta(a).$$

Uniqueness: To prove uniqueness, assume now that there is another radical rational mapping H' satisfying (1.1) and (4.1). Hence, we have

$$\begin{aligned}\left\|H^{'}(a)-4^m h(4^{2n}a)\right\|^p &= 4^{m\beta p}\left\|H^{'}(a)-h(4^{2n}a)\right\|^p \\ &\leq \frac{\zeta(a)^p}{4^{-\beta p}-(4^2L)^p}(4^{(2+\beta)}L)^{pm}\end{aligned}$$

for all $a \in \mathbb{R}^+$. Therefore, as $m \to \infty$ in the above inequality, we have $H = H'$ for all $a \in \mathbb{R}^+$ which proves the uniqueness of H. □

Theorem 4.2. *Let a function χ be an expansively super-additive with a constant L satisfying* $4^{2+\beta} < 1$. *Then, there exists a unique solution $H : \mathbb{R}^+ \longrightarrow E$ of equation* (1.1) *and*

$$\|h(a)-H(a)\| \leq \frac{4^{-\beta}}{\sqrt[p]{(16l^{-1})^p-4^{-\beta p}}} \tag{4.5}$$

for all $a \in \mathbb{R}^+$.

Proof. Employing similar steps as in Theorem 4.1, we can have

$$\left\|4^{-n}h(4^{-2n}a)-4^{-(n+1)}h(4^{-2(n+1)}a)\right\| \leq 4^{-n\beta p}\zeta(4^{-2n}a) \tag{4.6}$$

for all $a \in \mathbb{R}^+$. For integers l,n with $n > l > 0$, we can assert that

$$\left\|4^{-l}h(4^{-2l}a)-4^{-n}h(4^{-2n}a)\right\|^p \leq \zeta(a)^p \sum_{i=l+1}^{n}(4^{-(2+\beta)})L)^{pi} \tag{4.7}$$

for all $a \in \mathbb{R}^+$. The remaining part of the proof is analogous to Theorem 4.1. □

5. Fixed Point Method to Stabilities of Equations (1.1) and (1.2)

In this section, we implement fixed point approach to prove the stabilities of equations (1.1) and (1.2) in quasi β-Banach spaces. We presume that the space E to be a quasi β-Banach space with norm $\|\cdot\|_E$. Let K be the modulus of concavity of $\|\cdot\|_E$. In the following results, we assume that $h : \mathbb{R}^+ \longrightarrow E$ to be a mapping.

Lemma 5.1. *Let $j = \pm 1$ be fixed, g be a positive integer with the condition that $V \geq 2$ and $\chi : \mathbb{R}^+ \longrightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with $\chi(V^j a) \leq V^{jg\beta} L \chi(a)$ for all $a \in \mathbb{R}^+$. Let h satisfies*

$$\|h(Va) - V^g h(a)\| \leq \chi(a) \tag{5.1}$$

for all $a \in \mathbb{R}^+$, then there exists a uniquely determined solution $H : \mathbb{R}^+ \longrightarrow E$ of equation (1.1) *such that*

$$H(Va) = V^g H(a)$$

and

$$\|h(a) - H(a)\|_E \leq \frac{1}{V^{g\beta}|1 - L^j|} \chi(a) \tag{5.2}$$

for all $a \in \mathbb{R}^+$.

Theorem 5.2. *Let $j = \pm 1$ be fixed. Let $\gamma : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with $\gamma(4^j a, 4^j b) \leq 2^{-j\beta} L \gamma(a, b)$ for all $a, b \in \mathbb{R}^+$. Let h satisfies*

$$\|\Delta_1 h(a, b)\|_E \leq \gamma(a, b) \tag{5.3}$$

for all $a, b \in \mathbb{R}^+$. Then there exists a unique solution $H : \mathbb{R}^+ \longrightarrow E$ of equation (1.1) *such that*

$$\|h(a) - H(a)\|_E \leq \frac{1}{2^{-\beta}|1 - L^j|}$$

for all $a \in \mathbb{R}^+$.

Proof. Switching b to a in (5.3), we get

$$\left\| h(4a) - \frac{1}{2} h(a) \right\|_E \leq \gamma(a, a)$$

for all $a \in \mathbb{R}^+$. Then, multiplying 2 on both sides of the above inequality, we have

$$\|2h(4a) - h(a)\|_E \leq 2^\beta \gamma(a, a) \tag{5.4}$$

for all $a \in \mathbb{R}^+$. Owing to Lemma (5.1), we find that there is a unique mapping $H : \mathbb{R}^+ \longrightarrow E$ such that $H(4a) = 2^{-1} H(a)$ and

$$\|h(a) - H(a)\|_E \leq \frac{1}{2^{-\beta}|1 - L^j|}$$

for all $a \in \mathbb{R}^+$. It remains show that H is a radical rational mapping. By (5.3), we have

$$\begin{aligned}\|\Delta_1 H(a,b)\|_E &= \left\|2^{jn}\Delta_1 h(4^{jn}a, 4^{jn}b)\right\|_E \\ &\le 2^{jn\beta}\gamma(4^{jn}a, 4^{jn}b) \\ &\le 2^{jn\beta}\left(2^{-j\beta}L\right)^n \gamma(a,b) \\ &= L^n\gamma(a,b)\end{aligned}$$

for all $a,b \in \mathbb{R}^+$ and $n \in \mathbb{N}$. So $\|\Delta_1 H(a,b)\|_E = 0$ for all $a,b \in \mathbb{R}^+$. Thus the mapping $H : \mathbb{R}^+ \longrightarrow E$ is radical rational. □

6. NON-STABILITY OF EQUATION (1.1) FOR SINGULAR CASE

In this section, we illustrate an example to justify that the stability of (1.1) fails for a singular case. Instigated by the excellent example provided in [5], we present the upcoming counter-example to prove the non-stability for the singular case $c = d = -\frac{\beta}{2}$ in Corollary 3.3.

Define a mapping $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$h(a) = \sum_{k=0}^{\infty} \frac{\phi\left(4^{-k}a\right)}{2^k} \tag{6.1}$$

for all $a \in \mathbb{R}^+$, and $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a function given by

$$\phi(a) = \begin{cases} \dfrac{\mu}{a^{\frac{1}{2}}}, & \text{if } a \in (1,\infty) \\ \mu, & \text{otherwise.} \end{cases} \tag{6.2}$$

Then the function h sets out as an apt counter-example for $c,d = -\dfrac{\beta}{2}$ which is explained in the following result.

Theorem 6.1. *The mapping h defined above satisfies*

$$\left| h\left(a+b+2\sqrt{ab}\right) - \frac{h(a)h(b)}{h(a)+h(b)} \right| \le 6\mu\left(\left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}} \right) \tag{6.3}$$

for all $a,b \in \mathbb{R}^+$. Therefore there do not exist a radical rational mapping $B : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and a positive real constant β such that

$$|h(a) - B(a)| \le \frac{\beta}{|a|^{\frac{1}{2}}} \tag{6.4}$$

for all $a \in \mathbb{R}^+$.

Proof. $|h(a)| \leq \sum_{k=0}^{\infty} \frac{\left|\phi\left(4^{-k}a\right)\right|}{\left|2^k\right|} \leq \sum_{k=0}^{\infty} \frac{\mu}{2^k} = \mu\left(1-\frac{1}{2}\right)^{-1} = 2\mu$. Hence h is bounded by 2μ. If $\left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}} \geq 1$, then the value of the expression on the left-hand side of (6.3) is less than $\frac{3}{2}\mu$. Now, let us consider that $0 < \left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}} < 1$. Then, we can find a positive integer r such that

$$\frac{1}{2^{r+1}} \leq \left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}} < \frac{1}{2^r}. \tag{6.5}$$

Hence $\left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}} < \frac{1}{2^r}$ implies

$$4^{\frac{r}{2}}\left|\frac{1}{a}\right|^{\frac{1}{2}} + 4^{\frac{r}{2}}\left|\frac{1}{b}\right|^{\frac{1}{2}} < 1$$

or

$$4^{\frac{r}{2}}\frac{1}{a^{\frac{1}{2}}} + 4^{\frac{r}{2}}\frac{1}{b^{\frac{1}{2}}} < 1$$

or

$$4^{\frac{r-1}{2}}\frac{1}{a^{\frac{1}{2}}} < 1, 4^{\frac{r-1}{2}}\frac{1}{b^{\frac{1}{2}}} < 1$$

and consequently $\frac{1}{4^{r-1}}a, \frac{1}{4^{r-1}}b, \frac{1}{4^{r-1}}(a+b+2\sqrt{ab}) \in (1,\infty)$. Therefore, for each value of $k = 0,1,2,\ldots,r-1$, we obtain $\frac{1}{4^r}a, \frac{1}{4^r}b, \frac{1}{4^r}(a+b+2\sqrt{ab}) \in (1,\infty)$ and $\phi\left(\frac{1}{4^k}(a+b+2\sqrt{ab})\right) - \frac{\phi\left(\frac{1}{4^k}a\right)\phi\left(\frac{1}{4^k}b\right)}{\phi\left(\frac{1}{4^k}a\right)+\phi\left(\frac{1}{4^k}b\right)} = 0$ for $k = 0,1,2,\ldots,r-1$. On account of (6.5) and owing to the definition of h, we obtain

$$\begin{aligned}
&\left|h(a+b+2\sqrt{ab}) - \frac{h(a)h(b)}{h(a)+h(b)}\right| \\
&\qquad \leq \sum_{k=0}^{\infty}\frac{1}{2^k}\left|\phi\left(\frac{1}{4^k}\left(a+b+2\sqrt{ab}\right)\right) - \frac{\phi\left(\frac{1}{4^k}a\right)\phi\left(\frac{1}{4^k}b\right)}{\phi\left(\frac{1}{4^k}a\right)+\phi\left(\frac{1}{4^k}b\right)}\right| \\
&\qquad \leq \sum_{k=r}^{\infty}\frac{1}{2^k}\left|\phi\left(\frac{1}{4^k}\left(a+b+2\sqrt{ab}\right)\right) - \frac{\phi\left(\frac{1}{4^k}a\right)\phi\left(\frac{1}{4^k}b\right)}{\phi\left(\frac{1}{4^k}a\right)+\phi\left(\frac{1}{4^k}b\right)}\right| \\
&\qquad \leq \sum_{k=r}^{\infty}\frac{1}{2^k}\frac{3\mu}{2} = 6\mu\left(\left|\frac{1}{a}\right|^{\frac{1}{2}} + \left|\frac{1}{b}\right|^{\frac{1}{2}}\right).
\end{aligned}$$

Therefore, the inequality (6.3) holds true. Now, we assert that the equation (1.1) is non-stable for $c,d=-\frac{\beta}{2}$ in Corollary 3.3. Let us consider a radical rational mapping $B:\mathbb{R}^+\longrightarrow\mathbb{R}$ satisfying (6.4). Hence, we have

$$|B(a)|\leq(\beta+1)\left|\frac{1}{a^{\frac{1}{2}}}\right|. \tag{6.6}$$

But, we can have a positive integer m with the condition that $m\mu>\beta+1$. If $a\in(1,\infty)$, then $4^{-k}a\in(1,\infty)$ for all $k=0,1,2,\ldots,m-1$ and therefore

$$|B(a)|=\sum_{k=0}^{\infty}\frac{\phi\left(4^{-k}a\right)}{2^k}\geq\sum_{k=0}^{m-1}\frac{1}{2^k}\frac{\mu}{(4^{-k}a)^{\frac{1}{2}}}=m\mu\frac{1}{a^{\frac{1}{2}}}\geq(\beta+1)\frac{1}{a^{\frac{1}{2}}}$$

which contradicts (6.6). Therefore, equation (1.1) is non-stable for $c,d=-\frac{\beta}{2}$ in Corollary 3.3. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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