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PROPERTIES OF STRONGLY PRIME IDEALS AND $\mathscr{C}$-IDEALS IN POSETS

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Abstract. In this paper, the concepts of $\mathscr{C}$-ideal are defined and explored the various properties $\mathscr{C}$-ideals in posets. The equivalent conditions for an ideal to be a $\mathscr{C}$-ideal is obtained. Further the relations between strongly prime ideals and $\mathscr{C}$-ideals are discussed.

Keywords: poset; ideals; strongly prime ideal; strongly m-system; $\mathscr{C}$-ideals.

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1. Introduction

The concept of z-ideals, which are both algebraic and topological objects played a fundamental role in studying the ideal theory of $C(X)$, the ring of continuous real-valued functions on a completely regular Hausdorff space X.

In 1973, Mason[6] studied z-ideals of commutative rings and he proved that maximal ideals, minimal prime ideals and some other important ideals in commutative rings are z-ideals.

An ideal I of a commutative ring R is called a z-ideal if for each $a \in I$, the intersection of all maximal ideals containing a is contained in I.

The concept of z^0-ideals is nothing but the generalization of z-ideals. In 2006, K.Samei[7] studied z^0 -ideals and some special commutative ring.

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Let I and J be two ideals of a commutative ring R. I is said to be a z^J -ideal if $M_a \cap J \subseteq I$, for every $a \in I$, where M_a is the intersection of all maximal ideals containing a.

Whenever $J \nsubseteq I$ and I is a z^J-ideal, we say that I is a relative z-ideal. This special kind of z-ideals introduced and investigated by F. Azarpanah and A. Taherifar in [2].

In 2013, A.R. Aliabad, et.al., have shown that I is a relative z-ideal and the converse is also true for each finitely generated ideal in $C(X)$.

Hence it is natural to study the analogues concept of z-ideals and Z^0-deal in lattices and posets. In this paper, we introduced and studied $\mathscr{C}$-ideals in posets. We discussed the relation between z^J ideal and $\mathscr{C}$-ideals in posets and obtained some characterizations.

2. Preliminaries

Throughout this paper $(X, \leq)$ denotes a poset with least element 0. For basic terminology and notation for posets, we refer [5] and [4]. For $E \subseteq X$, let $E^l = \{x \in X : x \leq e \text{ for all } e \in E\}$ denotes the lower cone of E in X and dually, let $E^u = \{x \in X : e \leq x \text{ for all } e \in E\}$ be the upper cone of E in X.

Let $E, F \subseteq X$, we shall write $(E,F)^l$ instead of $(E \cup F)^l$ and dually for the upper cones. If $E = \{e_1, e_2, ..., e_n\}$ is finite, then we use the notation $(e_1, e_2, ..., e_n)^l$ instead of $(\{e_1, e_2, ..., e_n\})^l$ (and dually).

It is clear that for any subset E of X, we have $E \subseteq E^{ul}$ and $E \subseteq E^{lu}$. If $E \subseteq F$, then $F^l \subseteq E^l$ and $F^u \subseteq E^u$. Moreover, $E^{lul} = E^l$ and $E^{ulu} = E^u$.

Following [8], a non-empty subset K of X is called semi-ideal if $b \in K$ and $a \leq b$, then $a \in K$. A subset K of X is called ideal if $a, b \in K$ implies $(a,b)^{ul} \subseteq K$[5].

A proper semi-ideal (ideal) K of X is called prime if $(a,b)^l \subseteq K$ implies that either $a \in K$ or $b \in K$ [4].

An ideal K of X is called semi-prime if $(a,b)^l \subseteq K$ and $(a,c)^l \subseteq K$ together imply $(a,(b,c)^u)^l \subseteq K$[5]. Given $e \in X$, $(e] = L(e) = \{x \in X : x \leq e\}$ is the principal ideal of X generated by e.

Following [3], an ideal K of X is called strongly prime if $(A^*, B^*)^l \subseteq K$ implies that either $A \subseteq K$ or $B \subseteq K$ for any different proper ideals A, B of K, where $A^* = A \backslash \{0\}$.

Following [3], a non-empty sub-set E of X is called m-system if for any $e_1, e_2 \in E$, there exists $r \in (e_1, e_2)^l$ such that $r \in E$.

As a generalization of m-system, we define the notion of strongly m-system as follows, a non-empty subset E of X is called strongly m-system if $A \cap E \neq \phi$ and $B \cap E \neq \phi$ implies $(A^*, B^*)^l \cap E \neq \phi$ for any proper ideals A, B of X.

It is clear that an ideal K of X is strongly prime if and only if $X \backslash K$ is a strongly m- system of X. Also every strongly m-system is m-system. But the converse need not be true in general.

For an ideal K of X, a strongly prime ideal Q of X is said to be a minimal strongly prime ideal of K if $K \subseteq Q$ and there exists no strongly prime ideal R of X such that $K \subset R \subset Q$.

The set of all strongly prime ideal of X is denoted by $Sspec(X)$ and the set of minimal strongly prime ideals of X is denoted by $Smin(X)$. For any ideal K of X, $SP(K)$ denotes the intersection of all strongly prime ideals of X containing K and $SP(X)$ denotes the intersection all strongly prime ideal of X.

If $K = \{0\}$, then we denote $SP(K) = SP(X)$. From [4], the intersection of all prime semi-ideal of X containing K is K for any semi-ideal K of X. But the intersection of all strongly prime ideal of X containing K need not to be $K$for any ideal K of X[3].

For any subset K of X, we define $\psi(K) = \{Q \in Sspec(X) : K \subseteq Q\}$, $\phi(K) = Sspec(X) \backslash \psi(K)$, $\psi^{'}(K) = \psi(K) \cap Smin(X)$, $\phi^{'}(K) = \phi(K) \cap Smin(X)$ and $[K]$ is the smallest ideal of X containing K. Also $SP(a) = \bigcap\limits_{a \in \psi} \psi$.

For each $a \in X$ and an ideal K of X, we define $X_a(K) = \cap\{Q \in Sspec(X) : Q \in \psi^{'}(K) \cap \psi^{'}(a)\}$.

Following [3], let J be an ideal of X. An ideal I of X containing J is called z^J-ideal if for each $a \in I$, we have $X_a(J) \subseteq I$. Also if I is a z^J - ideal of X, then $X_a(J) \neq X$ for any $a \in I$. Clearly every strongly prime ideal of X is z^J-ideal. But the converse need not be true always.

3. Main Results

Definition 3.1. *Let X be a poset and I be an ideal of X. Then I is called $\mathscr{C}$-ideal of X if $\psi(a) \subseteq \psi(b)$ and $a \in I$ implies $b \in I$.*

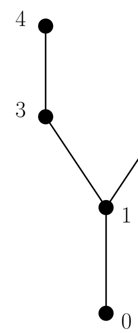
Theorem 3.1. *Every strongly prime ideal is a $\mathscr{C}$-ideal of X.*

Proof: Let S be a strongly prime ideal of X and $\psi(a) \subseteq \psi(b)$, $a \in S$. Since $a \in S$, we have $S \in \psi(a)$ which implies $S \in \psi(b)$. Then $b \in S$. Hence S is $\mathscr{C}$-ideal. □

Corollary 3.1. *Let I be a maximal strongly semi-prime ideal of X. Then I is $\mathscr{C}$-ideal.*

The following example gives the converse of the theorem 3.1 is need not be true in general.

Example 3.1. *Consider $X = \{0,1,2,3,4\}$ and define a relation $\leq$ on X as follows.*

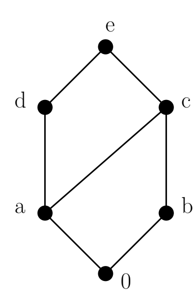


Then $(X,\leq)$ is a poset and $I_1 = \{0,1\}$ is a $\mathscr{C}$-ideal of X. But not a Strongly prime ideal as we take $I_2 = \{0,1,2\}$ and $I_3 = \{0,1,3\}$, we have $L(I_2,I_3) \subseteq I_1$ with $I_2 \nsubseteq I_1$ and $I_3 \nsubseteq I_1$. □

Theorem 3.2. *Let S be a unique strongly prime ideal of X and an ideal I of X such that $I \subset S$. Then I is not a $\mathscr{C}$-ideal of X.*

Proof: Let $I \subset S$. Then there exists a $x \in S \backslash I$. Since I is a unique strongly prime ideal of X, we have $\psi(x) = \psi(i)$ for all $i \in I$ which gives I is not a $\mathscr{C}$-ideal of X. □

Example 3.2. *Consider $X = \{0,a,b,c,d,e\}$ and define a relation $\leq$ on X as follows.*



Then $(X,\leq)$ is a poset and $I_1 = \{0,a,b,c\}$ is the only strongly prime ideal of X and if we take any proper ideal I_1 like $K = \{0,b\} \subset I$ which is not $\mathscr{C}$-ideal. □

Theorem 3.3. *Let X be a poset and $a,b \in X$. Then the following statements hold.*

(i) $SP((a,b)^l) = SP(a) \cap SP(b)$.

(ii) If $\psi(b) \subseteq \psi(a)$, then $\psi((b,c)^l) \subseteq \psi((a,c)^l)$ for any $c \in X$.

Proof: (**i**) Let $t \in SP((a,b)^l)$ and $t \notin SP(a) \cap SP(b)$. Without loss of generality, assume that $t \notin Q_1$ for a strongly prime ideal Q_1 containing a. Since $t \in SP((a,b)^l) \subseteq Q_1$, a contradiction. Hence $SP((a,b)^l) \subseteq SP(a) \cap SP(b)$.

Now, let $r \in SP(a) \cap SP(b)$ and $r \notin SP((a,b)^l)$. Then there exists a strongly prime ideal Q_2 containing $(a,b)^l$and $r \notin Q_2$. Since Q_2 is strongly prime ideal and $((a]^*,(b]^*)^l \subseteq (a,b)^l \subseteq Q_2$, we have $(a] \subseteq Q_2$ or $(b] \subseteq Q_2$. Without loss of generality, assume that $a \in Q_2$. As $r \in SP(a) \subseteq Q_2$, a contradiction. Hence $SP((a,b)^l) = SP(a) \cap SP(b)$.

(**ii**) Let $\psi(b) \subseteq \psi(a)$ for $a,b \in X$ and S be a strongly prime ideal of X containing $(b,c)^l$. Then $S \in \psi((b,c)^l)$ which implies $((b]^*,(c]^*)^l \subseteq S$. Since S is a strongly prime ideal of X, we have $(b] \subseteq S$ or $(c] \subseteq S$.

Case 1: If $(c] \subseteq S$, then $(a,c)^l \subseteq S$ which implies $S \in \psi((a,c)^l)$.

Case 2: If $(b] \subseteq S$, then $S \in \psi(b) \subseteq \psi(a)$ which gives $a \in S$ and $(a,c)^l \subseteq S$. Hence $S \in \psi((a,c)^l)$. □

Theorem 3.4. *Let X be a poset and $a,b \in X$. Then $a \in SP(b)$ if and only if $SP(a) \subseteq SP(b)$ if and only if $\Psi(b) \subseteq \psi(a)$.*

Proof: Let $SP(a) \subseteq SP(b)$. Since $a \in SP(a)$, we have $a \in SP(b)$.

Now, suppose that $a \in SP(b) = \bigcap_{b \in Q \in \Psi} Q$ and $t \in SP(a)$.

Then $t \in \bigcap_{a \in Q \in \Psi} Q$.

Let Q_1 be any strongly prime ideal of X and $b \in Q_1$.

As $a \in SP(b)$, we have $a \in Q_1$ which implies $t \in Q_1$ for all strongly prime ideals containing b. Hence $t \in SP(b)$ and $SP(a) \subseteq SP(b)$.

Let $SP(a) \subseteq SP(b) \Leftrightarrow \bigcap_{a \in Q_1} Q_1 \subseteq \bigcap_{b \in Q_2} Q_2$

$\Leftrightarrow \{Q_2 : b \in Q_2\} \subseteq \{Q_1 : a \in Q_1\}$

$\Leftrightarrow \psi(b) \subseteq \psi(a)$ □

Theorem 3.5. *Let J be an ideal of X.Then the following statements are equivalent*

(i) J is a $\mathscr{C}$-ideal of X.

(ii) If $\psi(a) = \psi(b)$ and $b \in J$ implies $a \in J$.

(iii) $SP(a) \subseteq J$ for all $a \in J$.

(iv) If $SP(b) \subseteq SP(a)$ and $a \in J$ implies $b \in J$.

Proof: (**i**) $\Rightarrow$ (**ii**) It is Obvious.

(**ii**) $\Rightarrow$ (**iii**) Let $t \in SP(a)$. Then by Theorem 3.4, $SP(t) \subseteq SP(a)$. Hence $SP(t) = SP(t) \cap SP(a)$ and by Theorem 3.3, $SP(t) = SP(L(a,t))$ which implies $\psi(t) = \psi(L(a,t))$. If $a \in J$, then $L(a,t) \subseteq J$. By (ii), $t \in J$.

(**iii**) $\Rightarrow$ (**iv**) Let $a \in J$. Then by (iii), $SP(a) \subseteq J$. Suppose $SP(b) \subseteq SP(a)$, then $b \in SP(b) \subseteq J$

(**i**) $\Rightarrow$ (**ii**) It follows from Theorem 3.4. □

Theorem 3.6. *Let X be a poset. If $I \cap M = \Phi$ for a $\mathscr{C}$-ideal I and a strongly m-system M of X. Then there exists a $\mathscr{C}$-ideal K of X containing I and disjoint from M and K is a strongly prime ideal of X.*

Proof: Let $\mathscr{F} = \{J : J \text{ is an } \mathscr{C}\text{-ideal containing } I \text{ and } J \cap M = \phi\}$. Since $I \in \mathscr{F}$, $\mathscr{F} \neq \Phi$.

Let $\mathscr{X}$ be a chain $\mathscr{F}$ and $R = \bigcup_{J \in \mathscr{X}} J$.

To show that R is a $\mathscr{C}$-ideal of X, let $\psi(a) \subseteq \psi(b)$ and $a \in R$. Then $a \in J_i$ for some i. Since J_i is a $\mathscr{C}$-ideal of X, we have $b \in J_i$ and $b \in R$. Thus R is a $\mathscr{C}$-ideal of X.

By Zorn's Lemma, there exists a maximal $\mathscr{C}$-ideal K such that $K \cap M = \Phi$.

Let $(A^*, B^*)^l \subseteq K$ and $A, B \nsubseteq K$. Then $[K \cup A] \cap M \neq \Phi$ and $[K \cup B] \cap M \neq \Phi$. Since M is strongly m-system we have $([K \cup A], [K \cup B])^l \cap M \neq \Phi$ which implies $K \cap M \neq \Phi$, a contradiction. So $A \subseteq K$ or $B \subseteq K$. Hence K is a strongly prime ideal of X. □

Theorem 3.7. *Every $\mathscr{C}$-ideal is a z^J - ideal of X.*

Proof: Let I be a $\mathscr{C}$-ideal of X. To prove I is z^J - ideal, for all $a \in I$ and $J \subseteq I$, let $x \in X_a(J)$. Then $x \in \cap\{Q \in Sspec(X) : Q \in \psi'(J) \cap \psi'(a)\}$

$\Rightarrow x \in Q$ for all $Q \in \psi'(J) \cap \psi'(a) \subseteq \psi'(a)$.

$\Rightarrow x \in \psi(a)$ for all $a \in I$

$\Rightarrow x \in SP(a)$ for all $a \in I$.

By Theorem 3.4, $SP(x) \subseteq SP(a)$ which gives $\psi(a) \subseteq \psi(x)$. Since I is a $\mathscr{C}$-ideal of X and $a \in I$, we have $x \in I$. Hence $X_a(J) \subseteq I$ for all $a \in I$. So I is z^J - ideal. □

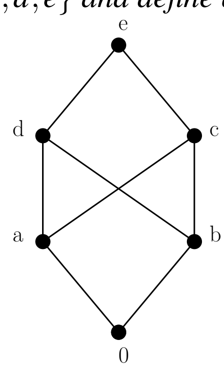
Remark 3.1.

(1) *In the above Example 3.1, $I_1 = \{0,1\}$ is both $\mathscr{C}$-ideal and z^J-ideal of X if we take $J = \{0\}$.*

(2) *In Example 3.2, $I_1 = \{0,a,d\}$ is neither $\mathscr{C}$-ideal nor z^J-ideal of X.* □

The converse of the Theorem 3.7 need not be true in general. The below example gives a z^J-ideal of X which is not $\mathscr{C}$-ideal.

Example 3.3. *Consider $X = \{0,a,b,c,d,e\}$ and define a relation $\leq$ on X as follows.*



Then $(X,\leq)$ is a poset and $I_1 = \{0,a,b,c\}$ and $I_2 = \{0,a,b,d\}$ are the strongly prime ideals of X. $I = \{0,b\}$ is a z^J-ideal of X for $J = \{0\}$. But I is not a $\mathscr{C}$-ideal of X as $\psi(b) \subseteq \psi(a)$ with $b \in I$ and $a \notin I$. □

Remark 3.2. *For any ideal J of X, $J_{\mathscr{C}} = \bigcap\{K : K$ is a $\mathscr{C}$-ideal of X and $K \supseteq J\}$.*

Theorem 3.8. *For an ideal J of X, $J_{\mathscr{C}}$ is the least $\mathscr{C}$-ideal Containing J.*

Proof: Let $\psi(b) \subseteq \psi(a)$ and $b \in J_{\mathscr{C}}$. Then any arbitrary $\mathscr{C}$-ideal Q_1 containing J and $b \in Q_1$ which implies $a \in Q_1$. So $a \in J_{\mathscr{C}}$. Hence $J_{\mathscr{C}}$ is a $\mathscr{C}$-ideal of X.

Let R be any $\mathscr{C}$-ideal of X such that $R \subset J_{\mathscr{C}}$ and $x \in J_{\mathscr{C}}$. Then $x \in R$. So $J_{\mathscr{C}} \subseteq R$ for all R. Hence $J_{\mathscr{C}}$ is the least $\mathscr{C}$-ideal of X. □

Theorem 3.9. *Let A and B be any two ideals of X, then the following statements hold*

(i) if $A \subseteq B$*, then* $A_{\mathscr{C}} \subseteq B_{\mathscr{C}}$.

(ii) $(A_{\mathscr{C}})_{\mathscr{C}} = A_{\mathscr{C}}$.

(iii) $(A \cup B)_{\mathscr{C}} \subseteq A_{\mathscr{C}} \cap B_{\mathscr{C}} \subseteq (A \cap B)_{\mathscr{C}}$

Proof: (**i**) Let $A \subseteq B$ and $t \in A_{\mathscr{C}} = \bigcap_{K \supseteq I} K$, where K is a $\mathscr{C}$-ideal of X. If $t \notin B_{\mathscr{C}}$, then there exists a $\mathscr{C}$-ideal J_1 such that $t \notin J_1$ and $B \subseteq J_1$ which gives $A \subseteq J_1$. Since $t \in A_{\mathscr{C}}$, we have $t \in J_1$, a contradiction.

(**ii**) Clearly, $A_{\mathscr{C}} \subseteq (A_{\mathscr{C}})_{\mathscr{C}}$. Now, let $r \in (A_{\mathscr{C}})_{\mathscr{C}} = \bigcap_{K \supseteq A_{\mathscr{C}}} K$, where K is a $\mathscr{C}$-ideal containing $A_{\mathscr{C}}$. But $A_{\mathscr{C}}$ is the least $\mathscr{C}$-ideal containing $A_{\mathscr{C}}$. Therefore $r \in A_{\mathscr{C}}$. Hence $(A_{\mathscr{C}})_{\mathscr{C}} = A_{\mathscr{C}}$.

(**ii**) It is trivial. □

Remark 3.3. *For any ideal J of X,* $J^{\mathscr{C}} = \bigcup\{K : K$ *is a* $\mathscr{C}$*-ideal of X and* $K \supseteq J\}$*. If union of any two ideals of X is again an ideal in X, then we can say that X has* ξ *property.*

Theorem 3.10. *Let J be an ideal of X and X has* ξ *property. Then* $J^{\mathscr{C}}$ *is the greatest* $\mathscr{C}$*-ideal Containing J.*

Proof: Let $\psi(b) \subseteq \psi(a)$ and $b \in J^{\mathscr{C}}$. Then there exists a $\mathscr{C}$-ideal Q_1 of X containing J and $b \in Q_1$ which implies $a \in Q_1$. So $a \in \bigcup\{K : K$ is a $\mathscr{C}$-ideal of X and $K \supseteq J\}$ = $J^{\mathscr{C}}$. Hence $J^{\mathscr{C}}$ is a $\mathscr{C}$-ideal of X.

Let A be any $\mathscr{C}$-ideal of X such that $J^{\mathscr{C}} \subset A$ and $l \in A$. Then $l \in \bigcup\{K : K$ is a $\mathscr{C}$-ideal of X and $K \supseteq J\}$. So $x \in J^{\mathscr{C}}$. Hence $J^{\mathscr{C}}$ is the greatest $\mathscr{C}$-ideal of X. □

Theorem 3.11. *Let E and F be any two ideals of X, then the following statements hold*

(i) if $E \subseteq F$*, then* $F^{\mathscr{C}} \subseteq E^{\mathscr{C}}$.

(ii) $(E^{\mathscr{C}})^{\mathscr{C}} = E^{\mathscr{C}}$.

(iii) $E_{\mathscr{C}} \subseteq E^{\mathscr{C}}$.

(iv) $(E \cup F)^{\mathscr{C}} \subseteq E^{\mathscr{C}} \cap F^{\mathscr{C}}$.

Proof: (**i**) Let $E \subseteq F$ and $t \in F^{\mathscr{C}} = \bigcup_{K \supseteq F} K$, where K is a $\mathscr{C}$-ideal of X. Then $t \in K_i$ for some $\mathscr{C}$-ideal K_i of X and $K_i \supseteq F \supseteq E$ which implies $t \in E^{\mathscr{C}}$.

(**ii**) Clearly, $E^{\mathscr{C}} \subseteq (E^{\mathscr{C}})^{\mathscr{C}}$. Now, let $r \in (E^{\mathscr{C}})^{\mathscr{C}} = \bigcup_{K \supseteq E^{\mathscr{C}}} K$, where K is a $\mathscr{C}$-ideal containing $E^{\mathscr{C}}$. But $E^{\mathscr{C}}$ is the greatest $\mathscr{C}$-ideal containing $E^{\mathscr{C}}$. Therefore $r \in E^{\mathscr{C}}$. Hence $(E^{\mathscr{C}})^{\mathscr{C}} = E^{\mathscr{C}}$.

(**iii**) It is follows from Theorem 3.8 and Theorem 3.10.

(**iv**) It is trivial. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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