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$(1+\vartheta)$-CONSTACYCLIC CODES OVER $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$

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Abstract. In this paper, $(1+\vartheta)$-constacyclic codes of arbitrary length m over a non-chain finite local frobenious ring $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$ are introduced. A new Gray map is constructed from $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$ to $\mathbb{Z}_8^8$ and proved that the $\mathbb{Z}_8$ -Gray image of $(1+\vartheta)$-constacyclic codes having prescribed length m over the ring $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$ is a cyclic code of length $8m$ over the ring $\mathbb{Z}_8$. Moreover, it has been obtained that the binary image of the $(1+\vartheta)$-constacyclic code of length m over $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$ is a distance invariant binary quasi-cyclic code of length $32m$ with index 16.

Keywords: constacyclic code; gray map; distance invariant; cyclic code; quasi-cyclic code.

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1. Background

Many optimal binary linear codes have been studied from codes over several new classes of rings via some Gray map. Over the ring $F_2+uF_2+vF_2+uvF_2$, linear codes are discussed in [1], self dual codes in [2], cyclic codes in [3] and $(1+u)$-constacyclic codes are described in [4] alongwith the construction of many optimal binary linear codes. More generally, cyclic codes over the ring R_k were investigated in [12]. The rings mentioned above are not finite chain rings, however have rich algebraic structures and produce binary codes with large automorphism

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groups and new binary self-dual codes. This demonstrates that the linear codes over such non-chain rings have been received increasing attention to the authors (see [10]-[12], [14]). More recently, linear codes over the non-chain ring Z_4+uZ_4, where $u^2=0$, has been explored in [6]. Also, linear codes over the non-chain ring Z_8+uZ_8, with $u^2=0$, were obtained in [14]. $(1+u)$-constacyclic codes over Z_4+uZ_4 and a class of constacyclic codes over F_p+uF_p and its gray image were studied in [7] and [13] respectively. Motivated by the work over the ring presented in [7] and [14], we focus on the construction of the constacyclic codes over the ring $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$, with $\vartheta^2=0$ and intent to establish some good binary codes from such codes.

2. The Ring $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$

Throughout this paper, the ring $\mathbb{Z}_8+\vartheta\mathbb{Z}_8$ with $\vartheta^2=0$ is denoted by $\mathscr{R}$. An arbitrary element $a+\vartheta b$ is a unit in $\mathscr{R}$ if and only if a is a unit in $\mathbb{Z}_8$. The ring $\mathscr{R}$ is a local Frobenius ring and a finite non-chain ring having total of 12 ideals defined as

S.No.	Ideals
1	$\mathscr{I}_0=\{0\}$
2	$\mathscr{I}_1=\mathbb{Z}_8+\vartheta\mathbb{Z}_8$
3	$\mathscr{I}_2=\{a+\vartheta b : a,b\in\{0,2,4,6\}\}$
4	$\mathscr{I}_4=\{0,4,4\vartheta,4+4\vartheta\}$
5	$\mathscr{I}_\vartheta=\{a\vartheta : a\in\mathbb{Z}_8\}$
6	$\mathscr{I}_{2\vartheta}=\{a\vartheta : a\in\{0,2,4,6\}\}$
7	$\mathscr{I}_{4\vartheta}=\{0,4\vartheta\}$
8	$\mathscr{I}_{2+\vartheta}=\{0,2\vartheta,4\vartheta,6\vartheta,2+\vartheta,2+3\vartheta,2+5\vartheta,2+7\vartheta,4,4+2\vartheta,4+4\vartheta,4+6\vartheta,6+\vartheta,6+3\vartheta,6+5\vartheta,6+7\vartheta\}$
9	$\mathscr{I}_{4+\vartheta}=\{0,2\vartheta,4\vartheta,6\vartheta,4+\vartheta,4+3\vartheta,4+5\vartheta,4+7\vartheta\}$
10	$\mathscr{I}_{4+2\vartheta}=\{0,4\vartheta,4+2\vartheta,4+6\vartheta\}$
11	$\mathscr{I}_{4,\vartheta}=\{a+b\vartheta : a\in\{0,4\},b\in\mathbb{Z}_8\}$
12	$\mathscr{I}_{4,2\vartheta}=\{0,4,2\vartheta,4\vartheta,6\vartheta,4+2\vartheta,4+4\vartheta,4+6\vartheta\}$

A non-empty subset Á over $\mathscr{R}^m$ of length m is said to be a linear code, if it is an $\mathscr{R}$-submodule of $\mathscr{R}^m$.

Now, defining the mappings ϖ, γ and ζ from $\mathscr{R}^m$ to $\mathscr{R}^m$ as follows:

$$\varpi(c_0, c_1, ..., c_{m-1}) = (c_{m-1}, c_0, c_1, ..., c_{m-2}),$$

$$\gamma(c_0, c_1, ..., c_{m-1}) = (-c_{m-1}, c_0, c_1, ..., c_{m-2}),$$

$$\zeta(c_0, c_1, ..., c_{m-1}) = (\zeta c_{m-1}, c_0, c_1, ..., c_{m-2}).$$

Here, the defined mappings ϖ, γ and ζ are known as the cyclic, negacyclic and constacyclic shift respectively. Moreover, Á is a cyclic code, negacyclic code and ζ-constacyclic code if $\varpi($ Á$)$ = Á, $\gamma($ Á$)$ = Á and $\zeta($ Á$)$ = Á respectively.

The polynomial representation of the codeword $c = (c_0, c_1, ..., c_{m-1})$ is $c(x) = c_0 + c_1x + ... + c_{m-1}x^{m-1}$ and $xc(x)$ corresponds to a ζ-constacyclic shift of $c(x)$ in the ring $\mathscr{R}[x]/ < x^m - \zeta >$. Thus, ζ-constacyclic codes of length m over $\mathscr{R}$ can be identified as ideals in the ring $\mathscr{R}[x]/ < x^m - \zeta >$. Thus, we have the following proposition.

Proposition 2.1. *A subset $\mathscr{C}$ of $\mathscr{R}^m$ is a linear cyclic code of length m if and only if $\mathscr{C}$ is an ideal of $\mathscr{A}_m = \mathscr{R}[x]/ < x^m - 1 >$. A subset $\mathscr{C}$ of $\mathscr{R}^m$ is a linear $(1+\vartheta)$-constacyclic code of length m over $\mathscr{R}$ if and only if $\mathscr{C}$ is an ideal of $\mathscr{B}_m = \mathscr{R}[x]/ < x^m - 1 - \vartheta >$.*

A unique set of generators for cyclic codes over $\mathbb{Z}_8$ are discussed in the next lemma.

Lemma 2.2. *Let $\mathscr{C}$ be a cyclic code of length m over $\mathbb{Z}_8$. Then,*

(1) If m is odd then, $\mathscr{C} =< g(x), 4a(x) > = < g(x) + 4a(x) >$, where $g(x)$, $a(x)$ are binary polynomials with $a(x)|g(x)|(x^m - 1)$ mod 2.

(2) If m is even, then

- **(i):** *If $g(x) = a(x)$ then, $\mathscr{C} = < g(x), 4a(x) > = < g(x) + 4a(x) >$, where $g(x), a(x)$ are the binary polynomials with $g(x)|(x^m - 1)$mod 2, and $g(x)|p(x)\frac{(x^m-1)}{g(x)}$.*
- **(ii):** *$\mathscr{C} = < g(x) + 4p(x), 4a(x) >$, where $g(x), a(x)$ and $p(x)$ are the binary polynomials with $a(x)|g(x)|(x^m - 1)$ mod 2, $a(x)|p(x)\frac{(x^m-1)}{g(x)}$ and deg($g(x)$) > deg($a(x)$)> deg($p(x)$).*

For a linear code $\mathscr{C}$ of length m over $\mathscr{R}$, the two linear codes: Torsion code, Tor($\mathscr{C}$) and Residue code, Res($\mathscr{C}$) of length m over $\mathbb{Z}_8$ are defined as:

$$Tor(\mathscr{C}) = \{x \in \mathbb{Z}_8^m \mid \vartheta x \in \mathscr{C}\},$$

$$Res(\mathscr{C}) = \{x \in \mathbb{Z}_8^m \mid \exists\, y \in Z_8^m : x+\vartheta y \in \mathscr{C}\}.$$

The homomorphism $\varphi : \mathscr{R} \to \mathbb{Z}_8$ as $\varphi(a+\vartheta b) = a$, extends naturally to a ring homomorphism $\varphi : \mathscr{R}^m \to \frac{Z_8[x]}{<x^m-1>}$ defined as

$$\varphi(c_0+c_1x+\ldots+c_{m-1}x^{m-1}) = \varphi(c_0)+\varphi(c_1)x+\ldots+\varphi(c_{m-1})x^{m-1}.$$

Acting φ on $\mathscr{C}$ over $\mathscr{R}$, define a ring homomorphism $\varphi : \mathscr{C} \to Res(\mathscr{C})$ as $\varphi(a+\vartheta b) = a$, where $a,b \in \mathbb{Z}_8$ with $Ker\varphi \cong Tor(\mathscr{C})$ and $\varphi(\mathscr{C}) = Res(\mathscr{C})$.

By the application of first isomorphism theorem of finite groups, $|C| = |Tor(\mathscr{C})||Res(\mathscr{C})|$. Also, the image of $\mathscr{C}$ under the map φ is a cyclic code of length m over $\mathbb{Z}_8$.

Combining the above result with lemma 2.2, the set of generators for cyclic code of length m over $\mathscr{R}$ can be obtained as provided in following theorem.

Theorem 2.3. *Let $\mathscr{C}$ be a $(1+\vartheta)$-constacyclic code of length m over $\mathscr{R}$.Then*

(1) If m is odd then, $\mathscr{C} = <g_1(x), 4a_1(x)+\vartheta b(x), \vartheta(g_2(x)+4a_2(x))>$, where $b(x)$ is a polynomial in $\mathbb{Z}_8[x]$ and for $i=1,2$, $g_i(x), a_i(x)$ are the binary polynomials with $a_i(x) \mid g_i(x) \mid (x^m-1)$ mod 2.

(2) If m is even then,

(i): *If $g_i(x) = a_i(x)$ then, $\mathscr{C} = <g_1(x)+4p_1(x)+\vartheta d_x, \vartheta(g_2(x)+4p_2(x))>$, where $b(x)$ is a polynomial in $\mathbb{Z}_8[x]$, and for $i=1,2$, $g_i(x), a_i(x)$ are the binary polynomial with $g_i(x)|(x^m-1)$mod 2, and $g_i(x)|p_i(x)\frac{(x^m-1)}{g_i(x)}$.*

(ii): *$\mathscr{C} = <g_1(x)+4p_1(x)+\vartheta e_1(x), 4a_1(x)+\vartheta e_2(x), \vartheta g_2(x)+4\vartheta p_2(x), 4a_2(x)>$, where $g(x), a(x)$ and $p(x)$ are the binary polynomials with $a(x)|g(x)|(x^m-1)$ mod 2, $a(x)|p(x)\frac{(x^m-1)}{g(x)}$ and deg(g(x))>deg(a(x))>deg(p(x)).*

3. Gray Maps

Gray images of $(1+\vartheta)$-constacyclic codes over $\mathscr{R}$

The gray map ρ_1 from $\mathbb{Z}_8$ to $\mathbb{Z}_2^4$ defined as

$$\rho_1(z) = (q+r, r, p+r, q+r),$$

where $z = p+2q+4r$ with $p,q,r \in \mathbb{Z}_2$, is a distance preserving map from $\mathbb{Z}_8^m$ (Lee distance) to $\mathbb{Z}_2^{4m}$ (Hamming distance) and can be extended to $\mathbb{Z}_8^m$ as: $\rho_1 : \mathbb{Z}_8^m \to \mathbb{Z}_2^{4m}$ as

$$\rho_1(z_0,z_1,...,z_{m-1}) = (q_0+r_0,...,q_{m-1}+r_{m-1}, r_0,...,r_{m-1}, p_0+r_0,...,p_{m-1}+r_{m-1}, q_0+r_0,...,q_{m-1}+r_{m-1}).$$

Now, defining a new gray map ρ_2 from $\mathscr{R}^m$ to $\mathbb{Z}_8^{8n}$ as

$$\rho_2(c) = (b+7a, b+6a, b+5a, b+4a, b+3a, b+2a, b+a, b),$$

where $c = a+ub$ and $a,b \in \mathbb{Z}_8$ and can also be extended from $\mathscr{R}^m$ to $\mathbb{Z}_8$ as

$$\rho_2(c_0,c_1,...,c_{m-1}) = (b_0+7a_0,...,b_{m-1}+7a_{m-1}, b_0+6a_0,...,b_{m-1}+6a_{m-1}, b_0+5a_0, ..., b_{m-1}+5a_{m-1}, b_0+4a_0,...,b_{m-1}+4a_{m-1}, b_0+3a_0,...,b_{m-1} +3a_{m-1}, b_0+2a_0,...,b_{m-1}+2a_{m-1}, b_0+a_0,...,b_{m-1}+a_{m-1}, b_0, ..., b_{m-1})$$

where $c_i = a_i + \vartheta b_i$ and $a_i,\, b_i \in \mathbb{Z}_8$.

It is well known that the homogeneous weight has many applications for codes over finite rings and provides a good metric for the underlying ring in constructing superior codes. Next, a homogeneous weight on $\mathscr{R}$ is defined after defining of the homogeneous weight on arbitrary finite ring $\mathscr{K}$.

Definition 3.1. A real valued function w on the finite ring $\mathscr{K}$ is called a left homogeneous weight if $w(0) = 0$ and the following holds:

(1) For all $x,y \in \mathscr{K}$, $\mathscr{K}x = \mathscr{K}y$ implies $w(x) = w(y)$.

(2) There exists a real number γ such that

$$\sum_{y\in\mathscr{K}(x)} w(y)=\gamma|\mathscr{K}x| \text{ for all } x\in\mathscr{K}\setminus\{0\}.$$

The Right homogeneous weight can be defined in a similar manner and if weight is both left homogeneous and right homogeneous, it is known as a homogeneous weight. For any element $c = a+\vartheta b\in\mathscr{R}$; the homogeneous weight denoted by $w_{hom}(c)$, as $w_L(b+7a,b+6a,...,b+a,b)$. By simple calculations the weight of any element $c = a+\vartheta b\in\mathscr{R}$ is:

$$w_{hom}(x)=\begin{cases}0 & \text{if } c=0,\\ 8 & \text{if } c=\vartheta,7\vartheta,\\ 24 & \text{if } c=3\vartheta,5\vartheta,\\ 32 & \text{if } c=4\vartheta,\\ 16 & \text{if otherwise.}\end{cases}$$

It is easy to verify that, the above defined weight meets the conditions of the Definition 3.1, hence it is actually a homogeneous weight on $\mathscr{R}$. The homogeneous distance of a linear code $\mathscr{C}$ over $\mathscr{R}$, denoted by $d_{hom}(\mathscr{C})$, is defined as the minimum homogeneous weight of the non-zero codewords of $\mathscr{C}$.

The map ρ_2 is a distance preserving map from $\mathscr{R}^m$(homogeneous distance) to $\mathbb{Z}_8^{8m}$(Lee distance). Thus, we have the following three distance preserving maps:

$$\rho_1:(\mathbb{Z}_8^m, Lee\ Distance)\to(\mathbb{Z}_2^{4m}, Hamming\ Distance)$$

$$\rho_2:(\mathscr{R}^m, Homogeneous\ Distance)\to(\mathbb{Z}_8^{8m}, Lee\ Distance)$$

$$\rho=\rho_1\rho_2:(\mathscr{R}^m, Homogeneous\ Distance)\to(\mathbb{Z}_2^{32m}, Hamming\ Distance)$$

4. $(1+\vartheta)$-Constacyclic Codes

The following theorem defined a result on the above defined map ρ_2.

Theorem 4.1. *If ζ is a $(1+\vartheta)$ - constacyclic shift on $\mathscr{R}^m$, ϖ is a cyclic shift on $\mathbb{Z}_8^{8m}$ and ρ_2 be a map defined as above, then $\rho_2\zeta = \varpi\rho_2$.*

Proof. If $c = (c_0, c_1, ..., c_{m-1}) \in \mathscr{R}^m$ where $c_i = a_i + \vartheta b_i$ and $a_i, b_i \in \mathbb{Z}_8$ for $0 \leq i \leq m-1$. The definition of the map ρ_2, implies

$$\begin{aligned}\rho_2(c) =& (b_0 + 7a_0, ..., b_{m-1} + 7a_{m-1},\ b_0 + 6a_0, ..., b_{m-1} + 6a_{m-1}, b_0 + 5a_0, ..., b_{m-1}\\ &+ 5a_{m-1}, b_0 + 4a_0, ..., b_{m-1} + 4a_{m-1}, b_0 + 3a_0, ..., b_{m-1} + 3a_{m-1}, b_0 + 2a_0,\\ &..., b_{m-1} + 2a_{m-1}, b_0 + a_0, ..., b_{m-1} + a_{m-1}, b_0, ..., b_{m-1}),\end{aligned}$$

and

$$\begin{aligned}\varpi\rho_2(c) =& (b_{m-1}, b_0 + 7a_0, ..., b_{m-1} + 7a_{m-1}, b_0 + 6a_0, ..., b_{m-1} + 6a_{m-1}, b_0 + 5a_0, ...,\\ &b_{m-1} + 5a_{m-1}, b_0 + 4a_0, ..., b_{m-1} + 4a_{m-1}, b_0 + 3a_0, ..., b_{m-1} + 3a_{m-1}, b_0\\ &+ 2a_0, ..., b_{m-1} + 2a_{m-1}, b_0 + a_0, ..., b_{m-1} + a_{m-1}, b_0, ..., b_{m-2}).\end{aligned}$$

On the other hand,

$$\begin{aligned}\zeta(c) &= ((1+\vartheta)c_{m-1}, c_0, c_1, ..., c_{m-2})\\ &= ((1+\vartheta)(a_{m-1} + \vartheta b_{m-1}), a_0 + \vartheta b_0, a_1 + \vartheta b_1, ..., a_{m-2} + \vartheta b_{m-2}),\end{aligned}$$

and therefore,

$$\begin{aligned}\rho_2\zeta(c) =& (b_{m-1} + a_{m-1} + 7a_{m-1}, b_0 + 7a_0, ..., b_{m-1} + 7a_{m-1}, b_0 + 6a_0, ..., b_{m-1} + 6a_{m-1},\\ &b_0 + 5a_0, ..., b_{m-1} + 5a_{m-1}, b_0 + 4a_0, ..., b_{m-1} + 4a_{m-1}, b_0 + 3a_0, ..., b_{m-1}\\ &+ 3a_{m-1}, b_0 + 2a_0, ..., b_{m-1} + 2a_{m-1}, b_0 + a_0, ..., b_{m-1} + a_{m-1}, b_0, ..., b_{m-2})\\ =& (b_{m-1}, b_0 + 7a_0, ..., b_{m-1} + 7a_{m-1}, b_0 + 6a_0, ..., b_{m-1} + 6a_{m-1}, b_0 + 5a_0, ...,\\ &b_{m-1} + 5a_{m-1}, b_0 + 4a_0, ..., b_{m-1} + 4a_{m-1}, b_0 + 3a_0, ..., b_{m-1} + 3a_{m-1}, b_0 + 2a_0,\\ &..., b_{m-1} + 2a_{m-1}, b_0 + a_0, ..., b_{m-1} + a_{m-1}, b_0, ..., b_{m-2}).\end{aligned}$$

Hence, the result follows. □

Theorem 4.2. *A linear code $\mathscr{C}$ of length m over $\mathscr{R}$ is a $(1+\vartheta)$-constacyclic code if and only if $\rho_2(\mathscr{C})$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$.*

Proof. If $\mathscr{C}$ is a $(1+\vartheta)$-constacyclic code, then Theorem 4.1 implies

$$\varpi(\rho_2(\mathscr{C})) = \rho_2(\zeta(\mathscr{C})) = \rho_2(\mathscr{C}).$$

Hence, $\rho_2(\mathscr{C})$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$. Further, if $\rho_2(c)$ is a cyclic code of length $8m$ over $\mathbb{Z}_8$, then use Theorem 4.1 to obtain

$$\rho_2(\zeta(\mathscr{C})) = \varpi(\rho_2(\mathscr{C})) = \rho_2(\mathscr{C})$$

Since, ρ_2 is an injective mapping, therefore $\zeta(\mathscr{C}) = \mathscr{C}$ and hence, the result holds. □

The following corollary is an immediate consequence of above theorem.

Corollary 4.3. *The image of $(1+\vartheta)$-constacyclic code of length m over $\mathscr{R}$ under the map ρ_2 is a distance invariant cyclic code of length $8m$ over $\mathbb{Z}_8$.*

If ϖ is a cyclic shift, then for a positive integer s, the quasi-shift ϖ_s is given by

$$\varpi_s(a^{(1)}|a^{(2)}|...|a^{(s)}) = (\varpi(a^{(1)})|\varpi(a^{(2)})|...|\varpi(a^{(s)})),$$

where $a^{(1)}$, $a^{(2)}$,..., $a^{(s)} \in F_2^{(2m)}$ and " | " represents the usual vector concatenation. A binary quasi-cyclic code $\mathscr{C}$ of index s and length $2ms$ is a subset of $(\mathbb{Z}_2^{2m})^s$ such that $\varpi_s(\mathscr{C}) = \mathscr{C}$.

Lemma 4.4. *If ζ is a $(1+\vartheta)$-constacyclic shift on $\mathscr{R}^m$ and ρ be a mapping defined as above, then $\rho\zeta = \varpi_{16}\rho$.*

Proof. For $r = (r_0, r_1, ..., r_{m-1}) \in \mathscr{R}^m$, where $r_i = a_i + 2b_i + 4c_i + \vartheta d_i + 2\vartheta e_i + 4\vartheta f_i$, a_i, b_i, c_i, d_i, e_i, $f_i \in \mathbb{Z}_2$, for $0 \le i \le m-1$. Then,

$$
\begin{aligned}
\rho(r) =&(c_0+e_0+f_0,...,c_{m-1}+e_{m-1}+f_{m-1},b_0+e_0+f_0,...,b_{m-1}+e_{m-1}+f_{m-1},a_0\\
&+b_0+c_0+e_0+f_0,...,a_{m-1}+b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1},a_0+e_0+f_0,...,\\
&a_{m-1}+e_{m-1}+f_{m-1},a_0+c_0+e_0+f_0,...,a_{m-1}+c_{m-1}+e_{m-1}+f_{m-1},a_0\\
&+b_0+e_0+f_0,...,a_{m-1}+b_{m-1}+e_{m-1}+f_{m-1},b_0+c_0+e_0+f_0,...,b_{m-1}\\
&+c_{m-1}+e_{m-1}+f_{m-1},e_0+f_0,...,e_{m-1}+f_{m-1},a_0+b_0+c_0+f_0,...,a_{m-1}\\
&+b_{m-1}+c_{m-1}+f_{m-1},a_0+b_0+f_0,...,a_{m-1}+b_{m-1}+f_{m-1},a_0+c_0+f_0,\\
&...,a_{m-1}+c_{m-1}+f_{m-1},a_0+f_0,...,a_{m-1}+f_{m-1},b_0+c_0+f_0,...,b_{m-1}+c_{m-1}\\
&+f_{m-1},b_0+f_0,...,b_{m-1}+f_{m-1},c_0+f_0,...,c_{m-1}+f_{m-1},f_0,...,f_{m-1},b_0+c_0\\
&+d_0+f_0,...,b_{m-1}+c_{m-1}+d_{m-1}+f_{m-1},a_0+b_0+d_0+f_0,...,a_{m-1}+b_{m-1}\\
&+d_{m-1}+f_{m-1},c_0+d_0+f_0,...,c_{m-1}+d_{m-1}+f_{m-1},a_0+d_0+f_0,...,a_{m-1}\\
&+d_{m-1}+f_{m-1},a_0+b_0+c_0+d_0+f_0,...,a_{m-1}+b_{m-1}+c_{m-1}+d_{m-1}+f_{m-1},\\
&b_0+d_0+f_0,...,b_{m-1}+d_{m-1}+f_{m-1},a_0+c_0+d_0+f_0,...,a_{m-1}+c_{m-1}+d_{m-1}\\
&+f_{m-1},d_0+f_0,...,d_{m-1}+f_{m-1},c_0+e_0+f_0,...,c_{m-1}+e_{m-1}+f_{m-1},b_0+e_0\\
&+f_0,...,b_{m-1}+e_{m-1}+f_{m-1},a_0+b_0+c_0+e_0+f_0,...,a_{m-1}+b_{m-1}+c_{m-1}\\
&+e_{m-1}+f_{m-1},a_0+e_0+f_0,...,a_{m-1}+e_{m-1}+f_{m-1},a_0+c_0+e_0+f_0,...,a_{m-1}\\
&+c_{m-1}+e_{m-1}+f_{m-1},a_0+b_0+e_0+f_0,...,a_{m-1}+b_{m-1}+e_{m-1}+f_{m-1},b_0\\
&+c_0+e_0+f_0,...,b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1},e_0+f_0,...,e_{m-1}+f_{m-1})
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
\varpi_{16}\rho(r) =& (b_{m-1}+e_{m-1}+f_{m-1}, c_0+e_0+f_0, \ldots, c_{m-1}+e_{m-1}+f_{m-1}, b_0+e_0+f_0, \ldots, \\
& b_{m-2}+e_{m-2}+f_{m-2}, a_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+c_0+e_0+f_0, \ldots, a_{m-1} \\
& +b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, a_0+e_0+f_0, \ldots, a_{m-2}+e_{m-2}+f_{m-2}, a_{m-1} \\
& +b_{m-1}+e_{m-1}+f_{m-1}, a_0+c_0+e_0+f_0, \ldots, a_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, \\
& a_0+b_0+e_0+f_0, \ldots, a_{m-2}+b_{m-2}+e_{m-2}+f_{m-2}, e_{m-1}+f_{m-1}, b_0+c_0+e_0 \\
& +f_0, \ldots, b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, e_0+f_0, \ldots, e_{m-2}+f_{m-2}, a_{m-1}+b_{m-1} \\
& +f_{m-1}, a_0+b_0+c_0+f_0, \ldots, a_{m-1}+b_{m-1}+c_{m-1}+f_{m-1}, a_0+b_0+f_0, \ldots, \\
& a_{m-2}+b_{m-2}+f_{m-2}, a_{m-1}+f_{m-1}, a_0+c_0+f_0, \ldots, a_{m-1}+c_{m-1}+f_{m-1}, \\
& a_0+f_0, \ldots, a_{m-2}+f_{m-2}, b_{m-1}+f_{m-1}, b_0+c_0+f_0, \ldots, b_{m-1}+c_{m-1}+f_{m-1}, \\
& b_0+f_0, \ldots, b_{m-2}+f_{m-2}, f_{m-1}, c_0+f_0, \ldots, c_{m-1}+f_{m-1}, f_0, \ldots, f_{m-2}, a_{m-1} \\
& +b_{m-1}+d_{m-1}+f_{m-1}, b_0+c_0+d_0+f_0, \ldots, b_{m-1}+c_{m-1}+d_{m-1}+f_{m-1}, \\
& a_0+b_0+d_0+f_0, \ldots, a_{m-2}+b_{m-2}+d_{m-2}+f_{m-2}, a_{m-1}+d_{m-1}+f_{m-1}, c_0 \\
& +d_0+f_0, \ldots, c_{m-1}+d_{m-1}+f_{m-1}, a_0+d_0+f_0, \ldots, a_{m-2}+d_{m-2}+f_{m-2}, \\
& b_{m-1}+d_{m-1}+f_{m-1}, a_0+b_0+c_0+d_0+f_0, \ldots, a_{m-1}+b_{m-1}+c_{m-1}+d_{m-1} \\
& +f_{m-1}, b_0+d_0+f_0, \ldots, b_{m-2}+d_{m-2}+f_{m-2}, d_{m-1}+f_{m-1}, a_0+c_0+d_0 \\
& +f_0, \ldots, a_{m-1}+c_{m-1}+d_{m-1}+f_{m-1}, d_0+f_0, \ldots, d_{m-2}+f_{m-2}, b_{m-1}+e_{m-1} \\
& +f_{m-1}, c_0+e_0+f_0, \ldots, c_{m-1}+e_{m-1}+f_{m-1}, b_0+e_0+f_0, \ldots, b_{m-2}+e_{m-2} \\
& +f_{m-2}, a_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+c_0+e_0+f_0, \ldots, a_{m-1}+b_{m-1}+c_{m-1} \\
& +e_{m-1}+f_{m-1}, a_0+e_0+f_0, \ldots, a_{m-2}+e_{m-2}+f_{m-2}, a_{m-1}+b_{m-1}+e_{m-1} \\
& +f_{m-1}, a_0+c_0+e_0+f_0, \ldots, a_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+e_0+f_0, \\
& \ldots, a_{m-2}+b_{m-2}+e_{m-2}+f_{m-2}, e_{m-1}+f_{m-1}, b_0+c_0+e_0+f_0, \ldots, b_{m-1} \\
& +c_{m-1}+e_{m-1}+f_{m-1}, e_0+f_0, \ldots, e_{m-2}+f_{m-2})
\end{aligned}
$$

On the other hand,

$$\begin{aligned}\zeta(r) =&((1+\vartheta)r_{m-1}, r_0, r_1, \dots, r_{m-2})\\ =&((1+\vartheta)(a_{m-1}+2b_{m-1}+4c_{m-1}+\vartheta d_{m-1}+2\vartheta e_{m-1}+4\vartheta f_{m-1}), r_0, \dots, r_{m-2})\\ =&(a_{m-1}+2b_{m-1}+4c_{m-1}+\vartheta(a_{m-1}+d_{m-1})+2\vartheta(b_{m-1}+e_{m-1})+4\vartheta(c_{m-1}\\ &+f_{m-1}), r_0, \dots, r_{m-2})\end{aligned}$$

and therefore,

$$\begin{aligned}\rho(\zeta(r)) =&(b_{m-1}+e_{m-1}+f_{m-1}, c_0+e_0+f_0, \dots, c_{m-1}+e_{m-1}+f_{m-1}, b_0+e_0+f_0, \dots,\\ &b_{m-2}+e_{m-2}+f_{m-2}, a_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+c_0+e_0+f_0, \dots, a_{m-1}\\ &+b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, a_0+e_0+f_0, \dots, a_{m-2}+e_{m-2}+f_{m-2}, a_{m-1}\\ &+b_{m-1}+e_{m-1}+f_{m-1}, a_0+c_0+e_0+f_0, \dots, a_{m-1}+c_{m-1}+e_{m-1}+f_{m-1},\\ &a_0+b_0+e_0+f_0, \dots, a_{m-2}+b_{m-2}+e_{m-2}+f_{m-2}, e_{m-1}+f_{m-1}, b_0+c_0+e_0\\ &+f_0, \dots, b_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, e_0+f_0, \dots, e_{m-2}+f_{m-2}, a_{m-1}+b_{m-1}\\ &+f_{m-1}, a_0+b_0+c_0+f_0, \dots, a_{m-1}+b_{m-1}+c_{m-1}+f_{m-1}, a_0+b_0+f_0, \dots,\\ &a_{m-2}+b_{m-2}+f_{m-2}, a_{m-1}+f_{m-1}, a_0+c_0+f_0, \dots, a_{m-1}+c_{m-1}+f_{m-1}, a_0\\ &+f_0, \dots, a_{m-2}+f_{m-2}, b_{m-1}+f_{m-1}, b_0+c_0+f_0, \dots, b_{m-1}+c_{m-1}+f_{m-1}, b_0\\ &+f_0, \dots, b_{m-2}+f_{m-2}, f_{m-1}, c_0+f_0, \dots, c_{m-1}+f_{m-1}, f_0, \dots, f_{m-2}, a_{m-1}\\ &+b_{m-1}+d_{m-1}+f_{m-1}, b_0+c_0+d_0+f_0, \dots, b_{m-1}+c_{m-1}+d_{m-1}+f_{m-1},\\ &a_0+b_0+d_0+f_0, \dots, a_{m-2}+b_{m-2}+d_{m-2}+f_{m-2}, a_{m-1}+d_{m-1}+f_{m-1}, c_0\\ &+d_0+f_0, \dots, c_{m-1}+d_{m-1}+f_{m-1}, a_0+d_0+f_0, \dots, a_{m-2}+d_{m-2}+f_{m-2},\\ &b_{m-1}+d_{m-1}+f_{m-1}, a_0+b_0+c_0+d_0+f_0, \dots, a_{m-1}+b_{m-1}+c_{m-1}+d_{m-1}\\ &+f_{m-1}, b_0+d_0+f_0, \dots, b_{m-2}+d_{m-2}+f_{m-2}, d_{m-1}+f_{m-1}, a_0+c_0+d_0+f_0,\\ &\dots, a_{m-1}+c_{m-1}+d_{m-1}+f_{m-1}, d_0+f_0, \dots, d_{m-2}+f_{m-2}, b_{m-1}+e_{m-1}+f_{m-1},\\ &c_0+e_0+f_0, \dots, c_{m-1}+e_{m-1}+f_{m-1}, b_0+e_0+f_0, \dots, b_{m-2}+e_{m-2}+f_{m-2},\\ &a_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+c_0+e_0+f_0, \dots, a_{m-1}+b_{m-1}+c_{m-1}+e_{m-1}\\ &+f_{m-1}, a_0+e_0+f_0, \dots, a_{m-2}+e_{m-2}+f_{m-2}, a_{m-1}+b_{m-1}+e_{m-1}+f_{m-1},\\ &a_0+c_0+e_0+f_0, \dots, a_{m-1}+c_{m-1}+e_{m-1}+f_{m-1}, a_0+b_0+e_0+f_0, \dots, a_{m-2}\\ &+b_{m-2}+e_{m-2}+f_{m-2}, e_{m-1}+f_{m-1}, b_0+c_0+e_0+f_0, \dots, b_{m-1}+c_{m-1}\\ &+e_{m-1}+f_{m-1}, e_0+f_0, \dots, e_{m-2}+f_{m-2}).\end{aligned}$$

Hence the result. □

Theorem 4.5. *A linear code $\mathscr{C}$ of length m over $\mathscr{R}$ is a $(1+\vartheta)$-constacyclic code if and only if $\rho(\mathscr{C})$ is a binary quasi-cyclic code of length* $32m$ *with index* 16.

Proof. If $\mathscr{C}$ is a $(1+\vartheta)$-constacyclic code, then use of Theorem 4.4 gives,

$$\varpi_{16}(\rho(\mathscr{C}))=\rho(\zeta(\mathscr{C}))=\rho(\mathscr{C}),$$

which implies $\rho(\mathscr{C})$ is a binary quasi-cyclic code of length $32m$ with index 16, and again applying Theorem 4.4 to obtain

$$\rho(\zeta(\mathscr{C}))=\varpi_{16}(\rho(\mathscr{C}))=\rho(\mathscr{C}).$$

Further, ρ is an injective mapping and therefore, $\zeta(\mathscr{C})=\mathscr{C}$. □

From Theorem 4.5 and the definition of the map ρ, the following result holds immediately.

Corollary 4.6. *The image of a $(1+\vartheta)$-constacyclic code of length m over $\mathscr{R}$ under the map ρ is a distance invariant binary quasi-cyclic code of length* $32m$ *with index* 16.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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