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η-RICCI-YAMABE SOLITONS ON SUBMANIFOLDS OF SOME INDEFINITE ALMOST CONTACT MANIFOLDS

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Abstract. In this paper, we study the η-Ricci-Yamabe soliton on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds, indefinite Kenmotsu manifolds and indefinite trans-Sasakian manifolds concerning Riemannian connection and quarter symmetric metric connection.

Keywords: Ricci-Yamabe soliton; η-Ricci-Yamabe soliton; Einstein manifold; η-Einstein manifold; invariant submanifold; anti-invariant submanifold; quarter symmetric metric connection; indefinite Sasakian manifold; indefinite Kenmotsu manifold; indefinite trans-Sasakian manifold.

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1. Introduction

Hamilton [8] first time introduced the concept of Ricci flow and Yamabe flow simultaneously in 1988. Ricci soliton and Yamabe soliton emerges as the limit of the solutions of the Ricci flow and Yamabe flow respectively. In dimension $n = 2$ the Yamabe soliton is equivalent to Ricci

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soliton. However, in dimension $n > 2$, the Yamabe and Ricci solitons do not agree as the first preserves the conformal class of the metric but the Ricci soliton does not in general.

Over the past twenty years the theory of geometric flows, such as Ricci flow and Yamabe flow has been the focus of attraction of many geometers. Recently, in 2019, Guler and Crasmareanu [7] introduced the study of a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. This is also called Ricci-Yamabe flow of the type (p,q). The Ricci-Yamabe flow is an evolution for the metrics on the Riemannian or semi-Riemannian manifolds defined as [7].

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qr(t)g(t), go = g(0). \tag{1}$$

Due to the sign of involved scalars p and q the Ricci-Yamabe flow can be also a Riemannian or semi-Riemannian or singular Riemannian flow. This kind of multiple choices can be useful in some geometrical or physical model for example relativistic theories.Therefore naturally Ricci-Yamabe soliton emerges as the limit of the soliton of Ricci Yamabe flow. This is a strong inspiration for initiated the study of Ricci-Yamabe solitons is the fact that although Ricci solitons and Yamabe solitons are same in two dimensional study, they are essentially different in higher dimensions. An interpolation solitons between Ricci and Yamabe soliton is consider in [3] where the name Ricci-Bourguignon soliton coresponding to Ricci-Bourguignon flow but its depend on a single scalar.

A soliton to the Ricci Yamabe flow is said Ricci Yamabe soliton on the off chance that it moves just by one boundary gathering of diffeomorphism and scaling. It becomes exactly a Ricci Yamabe soliton on Riemannain complex (M,g) is an information (g,V,λ,p,q) fulfilling

$$L_V g + 2p\,S + (2\lambda - qr)g = 0, \tag{2}$$

here the scalar curvature is r, the Ricci tensor is S along with the vector field L_V is the Lie-derivative along the vector field. In the event that $\lambda = 0, \lambda > 0$ or $\lambda < 0$, at that point (M,g) is called as Ricci Yamabe steady, shrinker or expander soliton separately. In this way, condition (2) is called Ricci Yamabe soliton of (p,q)-type, which is a speculation of Yamabe and Ricci solitons. It notes us that Ricci Yamabe soliton of type $(0,q)$ and $(p,0)$-type are q-Yamabe soliton and p-Ricci soliton separately.

An advance extension of Ricci soliton is the concept of η-Ricci soliton defined by Cho and Kimura [4]. Therefore analogously we can define the new notion by perturbing the equation (2) that define the type of soliton by a multiple of a certain $(0,2)$-tensor field $\eta \otimes \eta$, we obtain a slightly more general notion, namely, η-Ricci -Yamabe soliton of type (p,q) defined as:

$$L_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0. \tag{3}$$

S. Golab [6] characterized and examined quarter symmetric linear connection on a differentiable manifold. A straight association $\overline{\nabla}$ is a n-dimensional Reimannian manifold is known as a quarter symmetric connection if twist tensor T is of the structure

$$T(U_1, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = B(Y)K(X) - B(X)K(Y), \tag{4}$$

where B is a 1-form and K is a tensor of type $(1,1)$. If a quarter symmetric linear connection $\overline{\nabla}$ fulfils the condition $(\overline{\nabla}_X g)(Y,Z) = 0$, for all $X,Y,Z \in \chi(M)$, where $\chi(M)$ is a Lie algebra of vector fields on the manifold M, at that point $\overline{\nabla}$ is known as a quarter symmetric metric connection. To prove a contact metric manifold conceding quarter symmetric connection, here we can take $B = \eta$ and $K = \varphi$ and henceforth (4) becomes

$$T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y. \tag{5}$$

The connection between quarter symmetric metric connection $\overline{\nabla}$ and Levi-Civita connection ∇ of a contact metric manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\varphi Y. \tag{6}$$

A $(2n+1)$-dimensional semi-Riemannian manifold $(\check{M}, \check{g})$ is called an indefinite almost contact manifold if it reaches an indefinite almost contact structure (φ, ξ, η),where φ is a tensor field of type (1,1), ξ is a vector field and η is a 1-form fulfilling for all vector fields X,Y on $\check{M}$ [2].

$\varphi^2 X = -X + \eta(X)\xi, \eta \circ \varphi = 0, \varphi\xi = 0, \eta(\xi) = 1,$

$\breve{g}(\varphi X,\varphi Y)=\breve{g}(X,Y)-\varepsilon\eta(X)\eta(Y),$

$\breve{g}(X,\xi)=\varepsilon\eta(X),\breve{g}(\varphi X,Y)=-\breve{g}(X,\varphi Y).$

Here $\varepsilon=\breve{g}(\xi,\xi)=\pm 1$ and $\check{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\breve{g}$.

In light of the structure conditions of manifolds are as per the following [2]: An indefinite almost contact metric structure $(\varphi,\xi,\eta,\breve{g})$ is called an indefinite Sasakian structure is for all vector fields Z,W on $\overline{M}$,

$$(\overline{\nabla}_Z\varphi)W=\varepsilon\eta(W)Z-\breve{g}(Z,W)\xi,\quad \overline{\nabla}_Z\xi=-\varepsilon\varphi Z. \tag{7}$$

An indefinite almost contact metric structure $(\varphi,\xi,\eta,\breve{g})$ is called an indefinite trans-Sasakian structure of type (α,β) if

$$(\check{\nabla}_Z\varphi)W_1=\alpha[\breve{g}(Z,W_1)\xi-\varepsilon\eta(W_1)Z]+\beta[\breve{g}(\varphi Z,W_1)\xi-\varepsilon\eta(W_1)\varphi Z],\quad \check{\nabla}_Z\xi=-\varepsilon\alpha\varphi Z+\varepsilon\beta[Z-\eta(Z)\xi], \tag{8}$$

for smooth functions α,β on $\check{M}$ and for all vector fields Z,W_1 on $\check{M}$.

Sarkar and De [5], presented and considered the thought of ε-Kenmotsu manifolds with indefinite metric by giving a case of $\alpha=0,\beta=1$, at that point indefinite almost contact metric structure $(\varphi,\xi,\eta,\breve{g})$ is said an indefinite Kenmotsu structure. The structure conditions hence become

$$(\overline{\nabla}_Z\varphi)W_1=[\breve{g}(\varphi Z,W_1)\xi-\varepsilon\eta(W_1)\varphi Z],\quad \check{\nabla}_Z\xi=\varepsilon Z-\varepsilon\eta(Z)\xi. \tag{9}$$

Consider M as a submanifold of dimension m of a manifold $\check{M}(m<n)$ with actuated metric g. Likewise let ∇ and $\nabla^{\perp}$ be the incited connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M individually. At that point the Weingarten and Gauss formulae are started

$$\check{\nabla}_XY=\nabla_XY+h(X,Y), \tag{10}$$

$$\check{\nabla}_XV=-A_VX+\nabla^{\perp}_XV, \tag{11}$$

for all $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. Where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V) respectively for the immersion of M into $\check{M}$. The second fundamental form h and the shape operator A_V are related by [12]

$$g(h(X,Y),V) = g(A_V X,Y), \tag{12}$$

for any $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. The mean curvature vector L on M is given by $L = \frac{1}{m}\Sigma_{i=1}^{m} g(e_i,e_i), \{e_i\}_{i=1}^{m}$ is a local orthonormal frame of vector fields on M.

A submanifold M of a manifold $\check{M}$ is called totally umbilical if

$$h(X,Y) = g(X,Y)L, \tag{13}$$

for $X,Y \in TM$. Moreover if $h(X,Y) = 0$. Also M is called totally geodesic and if $L = 0$, then M is minimal in $\check{M}$.

A submanifold M of a manifold $\tilde{M}$ is called invariant(anti-invariant) if ϕX is tangent(normal) to M for every vector field X tangent to M, that is: $\varphi(TM) \subset TM(\varphi(TM) \subset T^{\perp}M)$ at each pointed M.

Let $\check{\bar{\bar{\nabla}}}$ and $\check{\nabla}$ be the Levi-Civita connection on $\check{M}$ such that

$$\check{\bar{\bar{\nabla}}}_X Y = \check{\nabla}_X Y + H(X,Y), \tag{14}$$

where H is a (1,1) type tensor and $X,Y \in \Gamma(T\check{M})$.

For $\check{\bar{\bar{\nabla}}}$ to be a quarter symmetric metric connection on $\check{M}$, we have

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{15}$$

where

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{16}$$

From (5) and (16) we acquire

$$T'(X,Y) = \eta(X)\varphi Y - g(Y,\varphi X)\xi, \tag{17}$$

$$H(X,Y) = \eta(Y)\varphi X - g(Y,\varphi X)\xi. \tag{18}$$

Thus, a quarter symmetric metric connection $\bar{\check{\nabla}}$ in a manifold $\check{M}$ is specified by

$$\bar{\check{\nabla}}_X Y = \check{\nabla}_X Y + \eta(Y)\varphi X - g(\varphi X,Y)\xi. \tag{19}$$

Somashekhara et al. [9], proved some results on invariant sub-manifolds of *LP*-Sasakian manifolds endowed with semi-symmetric metric conection and shown that the *LP*-Sasakian manifold is totally geodesic. In [10], the authors studied the *C*-Bochner curvature tensor under *D*-homothetic deformation in *LP*-Sasakian manifolds. Angadi et al. [1], studied the Ricci-Yamabe soliton on invariant and anti-invariant submanifolds of indefinite Sasakian manifolds, indefinite Kenmotsu manifolds and indefinite trans-Sasakian manifolds concerning Riemannian connection and quarter symmetric metric connection. In [11], the authors studied some results on indefinite Sasakian manifold admitting quarter-symmetric metric connection and η-Ricci solitons of some curvature tensors.

2. η-Ricci Yamabe Soliton on Submanifolds of Indefinite Sasakian Manifolds in respect of Riemannian Connection

Suppose (g,ξ,λ,μ,p,q) be a η-Ricci-Yamabe soliton on a submanifold M of an indefinite Sasakian manifold $\check{M}$ we have

$$(L_\xi g)(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{20}$$

From (7) and (10) it becomes

$$-\varepsilon\varphi X = \check{\nabla}_X\xi = \nabla_X\xi + h(X,\xi). \tag{21}$$

Whether M is invariant in $\check{M}$, in that case $\varphi X \in TM$, hence equating tangential as well as normal component of (21) we get

$$\nabla_X\xi = -\varepsilon\varphi X, h(X,\xi) = 0. \tag{22}$$

Using (22) we get

$$
\begin{aligned}
(L_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\
&= -\varepsilon[g(\varphi X, Y) + g(X, \varphi Y)] \qquad (23)\\
&= 0.
\end{aligned}
$$

In view of (20) and (23) yields

$$S(X,Y) = (\frac{qr-2\lambda}{2p})g(X,Y) + (\frac{-\mu}{p})\eta(X)\eta(Y). \qquad (24)$$

It suggests M is η-Einstein. Additionally from (13) and (22) it obtains $\eta(X)L = 0$, that is $L = 0$, where as $\eta(X) \neq 0$. Consequently, M is minimal in $\check{M}$. Thus, we have the following:

Theorem 2.1. If $(g, \xi, \lambda, \mu, p, q)$ is a η-Ricci-Yamabe soliton on an Invariant submanifold M of an indefinite Sasakian manifold $\check{M}$,then M is minimal in $\check{M}$ and also M is η-Einstein.

Also from (22), we go through

$$S(X,\xi) = -\varphi X + (n-1)\eta(X)\xi. \qquad (25)$$

In view of (24) and (25), we come into $\lambda = \frac{qr-2\mu-2p(n-1)\xi}{2}$. Thus we express that

Theorem 2.2. An η-Ricci-Yamabe soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is shrinking or expanding or steady accordingly as $\frac{qr-2\mu-2p(n-1)\xi}{2} < 0$ or $\frac{qr-2\mu-2p(n-1)\xi}{2} > 0$ or $\frac{qr-2\mu-2p(n-1)\xi}{2} = 0$.

If $p = 0$ then $\lambda = \frac{qr-2\mu}{2}$. Thus we can state

Corollary 2.1. A q-η-Yamabe soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is shrinking or expanding or steady accordingly as $qr - 2\mu < 0$ or $qr - 2\mu > 0$ or $qr = 2\mu$.

If $q = 0$ then $\lambda = p(1-n)\xi - \mu$.

Corollary 2.2. A p-η-Ricci soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is shrinking or expanding or steady accordingly as $p(1-n) < 0$ or $p(1-n) > 0$ or $p(1-n) = 0$.

Again, If M is anti-invariant in $\check{M}$, then for any $X \in TM$, $\varphi X \in T^{\perp}M$ and hence from (21), it becomes

$$\nabla_X \xi = 0, h(X,\xi) = -\varepsilon\varphi X. \tag{26}$$

Using (23) it gives $(L_\xi g)(X,Y) = 0$. It suggests this ξ is a Killing vector field and consequently (20) capitulates $S(X,Y) = (\frac{qr-2\lambda}{2p})g(X,Y) + \frac{-\mu}{p}\eta(X)\eta(Y)$. Hence, M is an η-Einstein. Thus we can state that:

Theorem 2.3. If (g,ξ,λ,μ,p,q) is an η-Ricci Yamabe soliton on an anti-invariant submanifold M of an indefinite Sasakian manifold $\check{M}$,then ξ is a Killing vector field and M is η-Einstein.

As well, $\nabla_X \xi = 0 \Rightarrow R(X,Y)\xi = 0 \Rightarrow S(X,\xi) = 0 \Rightarrow \lambda = \dfrac{qr-2\mu}{2}$. Hence, we have:

Theorem 2.4. An η-Ricci-Yamabe soliton (g,ξ,λ,μ,p,q) on an anti-invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as $qr-2\mu > 0$ or $qr-2\mu < 0$ or $qr = 2\mu$.

3. η-Ricci Yamabe Soliton on Submanifolds of Indefinite Sasakian Manifolds with respect to Quarter Symmetric Metric Connection

Make us contemplate that (g,ξ,λ,μ,p,q) is an η-Ricci Yamabe soliton on a submanifold M of an indefinite Sasakian manifold $\check{M}$ with respect to quarter symmetric metric connection, where $\overline{\nabla}$ is the actuated connection on M from the connection$\check{\overline{\overline{\nabla}}}$, then we obtain

$$(L_\xi g)(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{27}$$

Let $\overline{h}$ be the second fundamental form $\overline{M}$ regarding prompted connection $\check{\overline{\overline{\nabla}}}$. At that point we have

$$\check{\overline{\overline{\nabla}}}_X = \overline{\nabla}_X Y + \overline{h}(X,Y), \tag{28}$$

and hence by virtue of (10), (19) we get,

$$\overline{\nabla}_X Y + \overline{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\varphi X - g(\varphi X,Y)\xi. \tag{29}$$

If M is an invariant sub manifold of $\check{M}$, then $\varphi X \in TM$ for any $X \in TM$ along with consequently collating tangential parts from (28), it becomes

$$\overline{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \tag{30}$$

which express that M admits quarter symmetric metric connection.

Also from (30), we obtain

$$\overline{\nabla}_X \xi = (-\varepsilon + 1)\varphi X, \tag{31}$$

and hence

$$\begin{aligned}(\overline{L}_\xi g)(X,Y) &= g(\overline{\nabla}_X \xi, Y) + g(X, \overline{\nabla}_Y \xi)\\ &= (-\varepsilon+1)g(\varphi X, Y) + g(X, \varphi Y)\\ &= 0.\end{aligned} \tag{32}$$

Hence from (27), we get

$$\overline{S}(X,Y) = (\frac{qr-2\lambda}{2p})g(X,Y) + (\frac{-\mu}{p})\eta(X)\eta(Y). \tag{33}$$

Thus we state that:

Theorem 3.1. Let $(g, \xi, \lambda, \mu, p, q)$ be an η-Ricci Yamabe soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$, with respect to quarter symmetric metric connection $\ddot{\overline{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.

Also from (31), it becomes

$$\overline{S}(X,\xi) = -\varphi X + (n-1)(\varepsilon-1)\eta(X). \tag{34}$$

Compare (33) and (34), we obtain $\lambda = \frac{qr-2\mu-(n-1)(\varepsilon-1)2p}{2}$.

Theorem 3.2. An η-Ricci Yamabe soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as $qr - 2\mu - (n-1)(\varepsilon - 1)2p > 0$ or $qr - 2\mu - (n-1)(\varepsilon-1)2p < 0$ or $qr = 2\mu + (n-1)(\varepsilon-1)2p$.

If $p = 0$ then $\lambda = \dfrac{qr-2\mu}{2}$. Thus we can state

Corollary 3.1. A q-η-Yamabe soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as $qr-2\mu>0$ or $qr-2\mu<0$ or $qr=2\mu$.

If $q=0$ then $\lambda=(1-n)(\varepsilon-1)p+2\mu$

Corollary 3.2. A p-η-Ricci soliton on an invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as $(1-n)(\varepsilon-1)p+2\mu>0$ or $(1-n)(\varepsilon-1)p+2\mu<0$ or $(1-n)(\varepsilon-1)p+2\mu=0$.

Again if M is an anti-invariant submanifold of $\check{M}$ with respect to quarter symmetric metric connection then from (30) we have, $\overline{\nabla}_X\xi=0$ hence $(\overline{L}_\xi)(X,Y)=0$. We obtain

$$\overline{S}(X,Y)=(\frac{qr-2\lambda}{2p})g(X,Y)+(\frac{-\mu}{p})\eta(X)\eta(Y). \tag{35}$$

Thus we can state:

Theorem 3.3. Let (g,ξ,λ,μ,p,q) be a η-Ricci Yamabe soliton on an anti-invariant submanifold M of an indefinite Sasakian manifold $\check{M}$, with respect to quarter symmetric metric connection $\bar{\bar{\nabla}}$. Then M is η-Einstein with respect to induced Riemannian connection.

Also, $\overline{\nabla}_X\xi=0$. it implies $R(X,Y)\xi=0\Rightarrow S(X,\xi)=0$ then $\lambda=\dfrac{qr-2\mu}{2}$.

Theorem 3.4. An η-Ricci Yamabe soliton (g,ξ,λ,μ,p,q) on an anti-invariant submanifold M of an indefinite Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as $qr-2\mu>0$ or $qr-2\mu<0$ or $qr=2\mu$.

4. η-Ricci Yamabe Soliton on Submanifolds of Indefinite Kenmotsu Manifolds with respect to Riemannian Connection

Let (g,ξ,λ,μ,p,q) be an η-Ricci Yamabe soliton on submanifold M of an indefinite Kenmotsu manifold $\check{M}$ then we have

$$(L_\xi g)(X,Y)+2pS(X,Y)+(2\lambda-qr)g(X,Y)+2\mu\eta(X)\eta(Y)=0. \tag{36}$$

From (9) and (10) we obtain,

$$\varepsilon[X-\eta(X)\xi]=\check{\nabla}_X\xi=\nabla_X\xi+h(X,\xi). \tag{37}$$

Comparing normal and tangential components of (37) we get,

$$\nabla_X \xi = \varepsilon[X - \eta(X)\xi], h(X,\xi) = 0, \tag{38}$$

using (38) we get

$$\begin{aligned}(L_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 2\varepsilon[g(X,Y) - \eta(X)\eta(Y)].\end{aligned} \tag{39}$$

In view of (39) and (36) we obtain

$$S(X,Y) = (\frac{qr - 2\lambda - 2\varepsilon}{2p})g(X,Y) + (\frac{\varepsilon - \mu}{p})\eta(X)\eta(Y). \tag{40}$$

Also from (13) and (38) it gives $L = 0$ as $\eta(X) \neq 0$.

Hence we can state

Theorem 4.1. If $(g, \xi, \lambda, \mu, p, q)$ is η-Ricci Yamabe soliton on a submanifold M of an indefinite Kenmotsu manifold $\check{M}$. Then M is minimal in $\check{M}$ and also M is η-Einstein.

As well as from (38) it becomes

$$S(X,\xi) = [-\varepsilon(n-1) - \xi]\eta(X). \tag{41}$$

Assimilating (40) and (41), it yields $\lambda = \frac{qr - 2\mu + 2p\varepsilon(n-1) + \xi}{2}$.

Theorem 4.2. An η-Ricci Yamabe soliton on an invariant submanifold M of an indefinite Kenmotsu manifold $\check{M}$ is expanding or shrnking or steady accordingly as $qr > 2\mu + 2p\varepsilon(n-1) + \xi$ or $qr < 2\mu + 2p\varepsilon(n-1) + \xi$ or $qr = 2\mu + 2p\varepsilon(n-1) + \xi$.

If $p = 0$ then $\lambda = \frac{qr - 2\mu}{2}$. Thus we have

Corollary 4.1. A q-η-Yamabe soliton on an invariant submanifold M of an indefinite Kenmotsu manifold $\check{M}$ is expanding or shrinking or steady accordingly as $qr > 0$ or $qr < 0$ or $qr = 0$.

If $q = 0$ then $\lambda = p[\varepsilon(n-1) + \xi] - \mu$. Thus we have

Corollary 4.2. A p-η-Ricci soliton on an invariant submanifold M of an indefinite Kenmotsu manifold $\check{M}$ is expanding or shrinking or steady accordingly as $p[\varepsilon(n-1) + \xi] - \mu > 0$ or $p[\varepsilon(n-1) + \xi] - \mu < 0$ or $p[\varepsilon(n-1) + \xi] = \mu$.

5. η-Ricci Yamabe Soliton on Submanifolds of Indefinite Kenmotsu Manifolds with respect to Quarter Symmetric Metric Connection

Let us suggest that (g,ξ,λ,μ,p,q) is an η-Ricci Yamabe soliton on a submanifold M of an indefinite Kenmotsu manifold $\check{M}$ with respect to quarter symmetric metric connection, where $\overline{\nabla}$ is the persuade connection on M from the connection $\check{\bar{\bar{\nabla}}}$. Also let $\bar{h}$ be the second fundamental form of M with respect to induced connection $\overline{\nabla}$. Then we can consider the equations (27), (28), (29).

If M is an Invariant submanifold $\check{M}$, then we have the equation (30) which implies that M accord quarter symmetric metric connection.

Else from $\overline{\nabla}_X\xi = \varepsilon[X-\eta(X)\xi]+\varphi X$ and hence

$$(\overline{L}_\xi g)(X,Y) = g(\overline{\nabla}_X\xi,Y)+g(X,\overline{\nabla}_Y\xi) \\ = 2\varepsilon[g(X,Y)-\eta(X)\eta(Y)]. \tag{42}$$

Using (42) in (27) we get

$\overline{S}(X,Y) = (\frac{qr-2\lambda-2\varepsilon}{2p})g(X,Y)+(\frac{\varepsilon-\mu}{p})\eta(X)\eta(Y).$

Hence it follows that

Theorem 5.1. Let (g,ξ,λ,μ,p,q) be a η-Ricci Yamabe soliton on an Invariant submanifold M of an indefinite Kenmotsu manifold $\check{M}$ with respect to quarter symmetric metric connection $\check{\bar{\bar{\nabla}}}$. Then M is η-Einstein in respect of induced Riemannian connection.

Again, If M is an anti-invariant submanifold of $\check{M}$ in respect of quarter symmetric metric connection then from (30) we obtain

$$\overline{\nabla}_X\xi = \varepsilon[X-\eta(X)\xi], \tag{43}$$

$$(\overline{L}_\xi g)(X,Y) = 2\varepsilon[g(X,Y)-\eta(X)\eta(Y)]. \tag{44}$$

Hence using (44) in (27) we have $\overline{S}(X,Y) = (\frac{qr-2\lambda-2\varepsilon}{2p})g(X,Y)+(\frac{\varepsilon-\mu}{p})\eta(X)\eta(Y)$.

Therefore, we state that

Theorem 5.2. Let $(g, \xi, \lambda, \mu, p, q)$ be a η-Ricci Yamabe soliton on an anti-invariant submanifold M of an indefinite Kenmotsu manifold $\check{M}$ with respect to quarter symmetric metric connection $\bar{\bar{\check{\nabla}}}$. Then M is η-Einstein with respect to induced Riemannian connection.

6. η-Ricci Yamabe Soliton on Submanifolds of Indefinite Trans-Sasakian Manifolds with respect to Riemannian Connection

Let us consider that $(g, \xi, \lambda, \mu, p, q)$ is an η-Ricci Yamabe soliton on a submanifold M of an indefinite trans-Sasakian manifold $\check{\nabla}$. Then we get

$$(L_\xi g)(X,Y) + 2pS(X,Y) + (2\lambda - qr)g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \tag{45}$$

From (8) and (10) we have

$$-\varepsilon\alpha\varphi X + \beta[X - \eta(X)\xi] = \check{\nabla}_X \xi = \nabla_X \xi + h(X, \xi). \tag{46}$$

In the event that M is invariant in $\check{M}$, at that point $\varphi X \in TM$ consequently likening normal and tangential components of (46) we acquire

$$\nabla_X \xi = -\varepsilon\alpha\varphi X + \delta\beta\varphi^2 X, h(X, \xi) = 0, \tag{47}$$

using (47) we have,

$$\begin{aligned}(L_\xi g)(X,Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 2\beta\delta[g(X,Y) + \varepsilon\eta(X)\eta(Y)].\end{aligned} \tag{48}$$

Using this to (45) we get

$$S(X,Y) = (\frac{qr - 2\lambda - 2\beta\delta}{2p})g(X,Y) + (\frac{-\beta - \mu}{p})\eta(X)\eta(Y). \tag{49}$$

It implicit that M is η-Einstein. As well from (13) along with (47) it acquires $\eta(X)L = 0$, that is, $L = 0$, $\eta(X) \neq 0$. Consequently, M is minimal in $\check{M}$ and therefore it follows that:

Theorem 6.1. If $(g, \xi, \lambda, \mu, p, q)$ is η-Ricci Yamabe soliton on an invariant submanifold M of an indefinite trans-Sasakian manifold $\check{M}$. Then M is η-Einstein and also M is minimal in $\check{M}$.

Again if M is anti invariant in $\check{M}$ then for any $X \in TM$, $\varphi X \in T^{\perp}M$ and hence from (46) we have $\nabla_X \xi = \varepsilon[\beta X - \eta(X)\xi], h(X,\xi) = -\varepsilon\alpha\varphi X$.

Hence $(L_\xi g)(X,Y) = 2\beta\delta[g(X,Y) + \varepsilon\eta(X)\eta(Y)]$, so we obtain

$$S(X,Y) = (\frac{qr - 2\lambda - 2\beta\delta}{2p})g(X,Y) + (\frac{-\beta-\mu}{p})\eta(X)\eta(Y).$$

Hence it can be expressed

Theorem 6.2. If (g,ξ,λ,μ,p,q) is η-Ricci Yamabe soliton on an anti-invariant submanifold M of an indefinite trans-Sasakian manifold $\check{M}$. Then M is an η-Einstein.

Also from (47), we obtain

$$S(\xi,\xi) = (n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta. \tag{50}$$

Comparing (49) and (50) we get $\lambda = \frac{qr - 2\beta\delta - 2(\beta+\mu) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2}$.

Theorem 6.3. An η-Ricci Yamabe soliton on an invariant submanifold or anti-invariant submanifold M of an indefinite trans-Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as:

$\frac{qr - 2\beta\delta - 2(\beta+\mu) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} > 0$

or $\frac{qr - 2\beta\delta - 2(\beta+\mu) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} < 0$

or $\frac{qr - 2\beta\delta - 2(\beta+\mu) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} = 0$.

If $q = 0$ then $\lambda = -\beta\delta - (\beta+\mu) - p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]$. Thus we have

Corollary 6.1. A p-η-Ricci soliton on an invariant submanifold or anti-Invariant submanifold M of an indefinite trans-Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as:

$\frac{-2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} > 0$

or $\frac{-2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta]}{2} < 0$

or $-2\beta(\delta-1) - 2p[(n-1)\varepsilon\alpha^2 - \beta^2\delta - 2n\xi\beta] = 0$.

If $p = 0$ then $\lambda = \frac{qr-2\beta\delta-2(\beta+\mu)}{2}$. Thus we have

Corollary 6.2. A q-η-Yamabe soliton on an invariant submanifold or anti-Invariant sub manifold M of an indefinite trans-Sasakian manifold $\check{M}$ is expanding or shrinking or steady accordingly as
$\frac{qr-2\beta\delta-2(\beta+\mu)}{2} > 0$ or $\frac{qr-2\beta\delta-2(\beta+\mu)}{2} < 0$ or $\frac{qr-2\beta\delta-2(\beta+\mu)}{2} = 0$.

7. η-RICCI YAMABE SOLITON ON SUBMANIFOLDS OF INDEFINITE TRANS-SASAKIAN MANIFOLDS WITH RESPECT TO QUARTER SYMMETRIC METRIC CONNECTION

Let us consider that (g,ξ,λ,μ,p,q) is a η-Ricci Yamabe soliton on a submanifold M of an indefinite trans-Sasakian manifold $\check{M}$ in respect of quarter symmetric metric connection, where $\overline{\nabla}$ is the induced connection on M from the connection $\check{\bar{\bar{\nabla}}}$. Further let $\overline{h}$ be the second fundamental form of M with respect to induced connection $\overline{\nabla}$. Then we can consider the equations (27), (28), (29).

If M is an Invariant submanifold $\check{M}$ then we have the equation (30) which implies that M concur quarter symmetric metric connection. As well as from $\overline{\nabla}_X\xi = (-\varepsilon\alpha+1)\varphi X+\beta\delta[X-\eta(X)\xi]$, and hence

$$(51) \qquad \begin{aligned}(\overline{L}_\xi g)(X,Y) &= g(\overline{\nabla}_X\xi,Y)+g(X,\overline{\nabla}_Y\xi)\\ &= 2\beta\delta[g(X,Y)-\varepsilon\eta(X)\eta(Y)].\end{aligned}$$

In view of (27) and (51) we get
$\overline{S}(X,Y) = (\frac{qr-2\lambda-2\beta\delta}{2p})g(X,Y)+(\frac{(\beta-\mu)}{p})\eta(X)\eta(Y)$.
Hence it can be declared as

Theorem 7.1. Let (g,ξ,λ,μ,p,q) be a η-Ricci Yamabe soliton on an invariant submanifold M of an indefinite trans-Sasakian manifold $\check{M}$ with respect to quarter symmetric metric connection $\check{\bar{\bar{\nabla}}}$. Then M is η-Einstein with respect to induced Riemannian connection.

Again, If M is an anti-invariant submanifold of $\check{\bar{\bar{\nabla}}}$ with respect to quarter symmetric metric connection then from (30) we obtain
$\overline{\nabla}_X\xi = \varepsilon[\beta X-\eta(X)\xi]$,

it implies that

$$(\overline{L}_\xi g)(X,Y) = 2\beta\delta[g(X,Y) - \varepsilon\eta(X)\eta(Y)]. \tag{52}$$

Hence from (27), we have $\overline{S}(X,Y) = (\frac{qr-2\lambda-2\beta\delta}{2p})g(X,Y) + (\frac{(\beta-\mu)}{p})\eta(X)\eta(Y)$.

We know have:

Theorem 7.2. Let (g,ξ,λ,μ,p,q) be η-Ricci Yamabe soliton on an anti-invariant submanifold M of an indefinite trans-Sasakian manifold $\bar{\bar{\nabla}}$ with respect to quarter symmetric metric connection $\bar{\bar{\nabla}}$. Then M is an η-Einstein with respect to induced Riemannian connection.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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