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FIXED POINT RESULTS OF GENERALIZED REICH CONTRACTION FOR α-ADMISSIBLE MAPPINGS ON GENERALIZED c-DISTANCE IN CONE b-METRIC SPACES OVER BANACH ALGEBRA

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Abstract. In this work, we introduce the nations of generalized Banach type contraction, generalized Kannan type contraction and generalized Reich type contraction for α-admissible mappings on generalized c-distance in cone b-metric spaces over Banach algebra. Then, we study the Reich fixed point results, Banach fixed point results and Kannan fixed point results for these type of mappings on generalized c-distance in such spaces. The results obtained extend and generalize several well-known results in literature. We present an example to support our work.

Keywords: fixed point; Reich contraction; Banach algebra.

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1. INTRODUCTION

Fixed point theory has a massively application in all fields of quantitative science. Consequently, it is entirely natural to consider various generalizations of metric space in order to address the needs in various fields of quantitative science. So, we consider different generalization of metric space. There is a lot of extension of the notions of metric space. Among which one of the most important generalization is the concept of cone metric space

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over Banach algebra and cone b-metric space over Banach algebra of Liu and Xu [1] and Huang and Radenović [3, 4] respectively. Then, many authors have been studied the problems of common fixed point and fixed point for mappings by considering several types of contractive conditions in such spaces (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]).

As the important of generalizations of a metric space in the fixed point theory, many authors extended and generalized the famous Banach contraction principle [30]. There is a lot of extension of the notions of contractive conditions, among which one of the most important generalization is the Reich contraction [32] which is the generalizations of the famous Banach contraction principle [30] and Kannan contraction [31].

In 2012, Samet et al. [15] introduced the notion of α-admissible mappings which is one of the important generalization of contractive conditions for mappings and proved some fixed point results that generalized several known results of metric spaces. After that, by this idea, some authors presented fixed point results for single and multivalued mappings (see, e.g., [16, 17, 18, 19, 20]).

Later, in 2015, Malhottra et al. [21] used the concept of α-admissibility of mappings and defined the generalized Lipschitz contraction on the cone metric spaces with a Banach algebra and proved Banach fixed point results in such spaces. Next in 2017. Malhottra et al. [22] introduced the generalized Kannan type α-admissible contraction mappings in the setting of cone metric spaces equipped with Banach algebra and proved Kannan fixed point results in such spaces. Also, Han et al. [14] proved fixed point theorems for α-admissible mappings on c-distance in cone metric spaces over Banach algebras. In 2017, Hussaini et al. [23] extended and generalized the results of Malhottra et al. [21] to cone b-metric spaces over Banach algebra , they introduced and studied and the generalized Lipschitz contraction mappings and proved Banach fixed point results in such space by using the concept of α-admissible mappings. In 2018, Vujaković et al. [24] defined the generalized Reich type α-admissible contraction mappings in the setting of cone b-metric spaces with Banach algebra and proved Reich fixed point results in such spaces. After that in 2020, Shatanawi et al. [25] generalized the results of Vujaković et al. [24], they introduced the generalized Hardy-Rogers type α-admissible contraction mappings in such space which is more general than Reich type α-admissible contraction mappings and proved Hardy-Rogers fixed point results in such spaces.

Recently in 2020, Firozjah et al. [26] initiated generalized c-distance on cone b-metric space over Banach algebras which is the generalization of a c-distance of Huang et al. [12] in cone metric space over Banach algebra and proved some fixed point results (for more information about c-distance see [12, 13, 14, 28]).

In this work, we define the nations of generalized Banach type contraction mapping, generalized Kanan type contraction mapping and generalized Reich type contraction mappings with the notion of α-admissibility for mappings on generalized c-distance in cone b-metric spaces over Banach algebra. Then, we study the Reich fixed point results for these type of mappings on generalized c-distance in such spaces. The Banach fixed point results and Kannan fixed point results will be special cases of Reich fixed point results. Our results obtained extend and generalize the main results in [14, 21, 22, 23, 24, 29].

2. PRELIMINARIES

In this section, we recall some basic definitions and results about cone b-metric space over Banach algebras.

Definition 2.1. A real Banach space $\mathscr{A}$ is called a real Banach algebra if for all $x, y, z \in \mathscr{A}$, $\alpha \in \mathbb{R}$, following properties holds:

(1) $x(yz) = (xy)z$,

(2) If $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$,

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

(4) $\|xy\| \leq \|x\|\|y\|$.

Let $\mathscr{A}$ be a real Banach algebra with a unit e, i.e., multiplicative identity e such that for all $x \in \mathscr{A}$, $ex = xe$. An element $x \in \mathscr{A}$ is said to be invertible if there exists an element $y \in \mathscr{A}$ such that $xy = yx = e$, y is called inverse of x and denoted by x^{-1}.

Definition 2.2. A subset P of $\mathscr{A}$ is called a cone if:

(1) P is nonempty set closed and $P \neq \{\theta\}$,
(2) If a, b are nonnegative real numbers and $x, y \in P$ then $ax + by \in P$,
(3) $x \in P$ and $-x \in P$ implies $x = \theta$.

where θ denote to the zero element in $\mathscr{A}$. For any cone $P \subset \mathscr{A}$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $\prec$ stand for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \operatorname{int}P$, where $\operatorname{int}P$ denotes the interior of P. If $\operatorname{int}P \neq \theta$, then P is called a solid. A cone P is called normal if there exists a number K such that for all $x, y \in \mathscr{A}$:

$$\theta \preceq x \preceq y \Longrightarrow \|x\| \leq K\|y\|. \tag{2.1}$$

Equivalently, the cone P is normal if

$$(\forall n)\ x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{2.2}$$

The least positive number K satisfying Condition 2.1 is called the normal constant of P.

Example 2.3. ([27]) Let $\mathscr{A} = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for all $x \in \mathscr{A}$. Define multiplication in $\mathscr{A}$ as just pointwise multiplication. Then $\mathscr{A}$ is a real unital Banach algebra with unit $e = 1$. Let $P = \{x \in E : x(t) \geq 0\}$, this cone is nonnormal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $\theta \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but $\|x_n\| = \max_{t\in[0,1]} |\frac{t^n}{n}| + \max_{t\in[0,1]} |t^{n-1}| = \frac{1}{n} + 1 > 1$; hence x_n does not converge to zero. It follows by Condition 2.2 that P is a nonnormal cone.

During this paper, $\mathscr{A}$ is a real Banach algebra with a unit e and the cone P is solid, that is, $\operatorname{int}P \neq \emptyset$.

Definition 2.4. ([3, 4]) Let X be a nonempty set and $\mathscr{A}$ be a real Banach Algebra with the constant $s \geq 1$. Suppose the mapping $d : X \times X \longrightarrow \mathscr{A}$ satisfy the following conditions:

(1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(3) $d(x,y) \preceq s(d(x,y) + d(y,z))$ for all $x,y,z \in X$.

Then d is called a cone b-metric on X and (X,d) is called a cone b-metric space over Banach algebra $\mathscr{A}$.

Definition 2.5. ([3, 4]) Let (X,d) be a cone b-metric space over Banach algebra $\mathscr{A}$, $\{x_n\}$ be a sequence in X and $x \in X$.

(1) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n,x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \longrightarrow x$.

(2) For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n,x_m) \ll c$ for all $n,m > N$, then $\{x_n\}$ is called a Cauchy sequence in X.

(3) A cone b-metric space (X,d) is called complete if every Cauchy sequence in X is convergent.

Definition 2.6. ([3, 4]) Let $\mathscr{A}$ be a real Banach algebra with a solid cone P. A sequence $\{u_n\} \subset P$ is said to be a c-sequence if for each $c \gg \theta$ there exists a positive integer N such that $u_n \ll c$ for all $n > N$.

Definition 2.7. [26] Let (X,d) be a cone b-metric space with the coefficient $s \geq 1$ over Banach algebra $\mathscr{A}$. A function $q : X \times X \longrightarrow \mathscr{A}$ is called a generalized c-distance on X if the following conditions hold:

(q1) $\theta \preceq q(x,y)$ for all $x,y \in X$,

(q2) $q(x,z) \preceq s(q(x,y) + q(y,z))$ for all $x,y,z \in X$,

(q3) for each $x \in X$ and $n \geq 1$, if $q(x,y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

The following lemma is salutary in our work.

Lemma 2.8. [26] Let (X,d) be a cone b-metric space over Banach algebra $\mathscr{A}$ with the coefficient $s \geq 1$ relative to a solid cone P and q is a generalized c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x,y,z \in X$. Suppose that u_n is a c-sequence in P. Then the following hold:

(1) If $q(x_n,y) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $y = z$.
(2) If $q(x_n,y_n) \preceq u_n$ and $q(x_n,z) \preceq u_n$, then $\{y_n\}$ converges to z.
(3) If $q(x_n,x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.
(4) If $q(y,x_n) \preceq u_n$, then $\{x_n\}$ is a Cauchy sequence in X.

The next lemma is supportive to prove our results.

Lemma 2.9. ([3, 4]) Let $\mathscr{A}$ be a real Banach algebra with a unit e and P a solid cone. The following properties holds:

(1) the $\lim_{n\to\infty} \| u^n \|^{\frac{1}{n}}$ is exists where $u \in \mathscr{A}$ and the spectral radius $\rho(u)$ is satisfies

$$\rho(u) = \lim_{n\to\infty} \| u^n \|^{\frac{1}{n}} = \inf \| u^n \|^{\frac{1}{n}} .$$

If $\rho(u) < 1$ then $e-u$ is invertible in $\mathscr{A}$. Moreover,

$$(e-u)^{-1} = \sum_{i=1}^{\infty} u^i,$$

and

$$\rho((e-u))^{-1} < \frac{1}{1-\rho(u)},$$

(2) if $u,v \in \mathscr{A}$ and u commutes with v, then $\rho(u+v) \leq \rho(u)+\rho(v)$ and $\rho(uv) \leq \rho(u)\rho(v)$,
(3) if $u \preceq ku$ where $u,k \in P$ and $\rho(k) < 1$, then $u = \theta$ and
(4) if $\rho(u) < 1$ then, $\{u^n\}$ is a c-sequence. Further, if $\beta \in P$, then $\{\beta u^n\}$ is a c-sequence.

In 2012, Samet et al. [15] introduced the notion of α-admissible mapping and proved some fixed point results in metric space.

Definition 2.10. ([15]) Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty[$ be two mappings. we say that T is α-admissible if

$$x,y \in X, \alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1.$$

Karapinar et al. [33] introduced the notion of a triangular α-admissible mapping.

Definition 2.11. ([15]) Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty[$ be two mappings. we say that T is α-admissible if

(i) T is α-admissible mapping,

(ii) for all $x,y,z \in X$, if $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$ implies $\alpha(x,z) \geq 1$.

3. Fixed Point Results

First, we give the definitions of generalized Banach type contraction, generalized Kannan type contraction and generalized Reich type contraction on generalized c-distance in cone b-metric spaces over Banach algebra.

Definition 3.1. Let (X,d) be a complete cone b-metric space over Banach algebra $\mathscr{A}$ with the coefficient $s \geq 1$ and q is a generalized c-distance on X where P is a solid cone. Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty[$ be two mappings. Then, the mapping T is said to be generalized Banach type contraction mapping if there exist vector $b \in P$ such that $\rho(b) < \frac{1}{s}$ and,

$$q(Tx,Ty) \preceq bq(x,y),$$

for all $x,y \in X$ with $\alpha(x,y) \geq 1$. Here, the vector b is called the generalized Banach-Lipschitz vector of T.

Definition 3.2. Let (X,d) be a complete cone b-metric space over Banach algebra $\mathscr{A}$ with the coefficient $s \geq 1$ and q is a generalized c-distance on X where P is a solid cone. Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty[$ be two mappings. Then, the mapping T is said to be generalized Kannan type contraction mapping if there exist vectors $k_1, k_2 \in P$ such that the following conditions hold:

$$\rho(sk_1) + \rho(k_2) < 1,$$

and,

$$q(Tx,Ty) \preceq k_1 q(x,Tx) + k_2 q(y,Ty),$$

for all $x,y \in X$ with $\alpha(x,y) \geq 1$. Here, the vectors k_1, k_2 are called the generalized Kannan-Lipschitz vectors of T.

Definition 3.3. Let (X,d) be a complete cone b-metric space over Banach algebra $\mathscr{A}$ with the coefficient $s \geq 1$ and q is a generalized c-distance on X where P is a solid cone. Let $T : X \longrightarrow X$ and $\alpha : X \times X \longrightarrow [0,\infty[$ be two mappings. Then, the mapping T is said to be generalized Reich type contraction mapping if there exist vectors $r_1, r_2, r_3 \in P$ such that the following conditions hold

$$\rho(r_3) + \rho(sr_1 + sr_2) < 1, \tag{3.1}$$

and,

$$q(Tx,Ty) \preceq r_1 q(x,y) + r_2 q(x,Tx) + r_3 q(y,Ty), \tag{3.2}$$

for all $x, y \in X$ with $\alpha(x,y) \geq 1$. Here, the vectors r_1, r_2, r_3 are called the generalized Reich-Lipschitz vectors of T.

Now, we study the Reich fixed point results for α-admissible mappings in cone b-metric spaces over Banach algebra with generalized c-distance.

Theorem 3.4. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone, $\alpha : X \times X \longrightarrow [0,\infty[$ be a funcation and q is a generalized c-distance on X. Suppose the mapping $f : X \longrightarrow X$ is a generalized Reich type contraction mapping with generalized Reich-Lipschitz vectors $r_1, r_2, r_3 \in P$ such that r_3 commutes with $(r_1 + r_2)$ and the following conditions are satisfied:

(i) f is α-admissible mapping,

(ii) there is $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) f is continuous.

Then, f has a fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$.

Proof. Let x_0 be an arbitrary point in X such that $(x_0, fx_0) \geq 1$. Define a sequence x_n in X such that $x_{n+1} = fx_n$ with $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since f α-admissible, we have

$$\alpha(x_0,x_1) = \alpha(x_0, fx_0) \geq 1 \Longrightarrow \alpha(fx_0, f^2x_0) = \alpha(x_1,x_2) \geq 1.$$

Continuing this process, one can assert that

$$\alpha(x_n, x_{n+1}) \geq 1, \forall n. \tag{3.3}$$

Apply Condition 3.2, then we have

$$\begin{aligned} q(x_n,x_{n+1}) &= q(fx_{n-1},fx_n) \\ &\preceq r_1q(x_{n-1},x_n)+r_2q(x_{n-1},fx_{n-1})+r_3q(x_n,fx_n) \\ &= r_1q(x_{n-1},x_n)+r_2q(x_{n-1},x_n)+r_3q(x_n,x_{n+1}) \\ &\preceq r_1q(x_{n-1},x_n)+r_2q(x_{n-1},x_n)+r_3q(x_n,x_{n+1}) \end{aligned}$$

Hence

$$(e-r_3)q(x_n,x_{n+1}) \preceq (r_1+r_2)q(x_{n-1},x_n). \tag{3.4}$$

Since Condition 3.1 shows that $\rho(r_3)<1$, then Lemma 2.9 (1) shows that $(e-r_3)$ is invertible. Furthermore, $(e-r_3)^{-1}=\sum_{i=1}^{\infty}(r_3)^i$ and $\rho(e-r_3)^{-1}<\frac{1}{1-\rho(r_3)}$.

Multiple both sides of (3.4) by $(e-r_3)^{-1}$, we arrive at

$$q(x_n,x_{n+1}) \preceq (e-r_3)^{-1}(r_1+r_2)q(x_{n-1},x_n).$$

Put

$$h=(e-r_3)^{-1}(r_1+r_2).$$

Then we have

$$\begin{aligned} q(x_n,x_{n+1}) &\preceq hq(x_{n-1},x_n) \\ &\preceq h^2q(x_{n-2},x_{n-1}) \\ &\;. \\ &\;. \\ &\;. \\ &\preceq h^nq(x_0,x_1). \end{aligned} \tag{3.5}$$

Since r_3 commutes with (r_1+r_2), one can see that

$$(e-r_3)^{-1}(r_1+r_2)=(r_1+r_2)(e-r_3)^{-1},$$

which means that $(e-r_3)^{-1}$ commutes with (r_1+r_2). Now, aplying 3.1 Lemma 2.9(1) and 2.9 (2) we obtain:

$$\begin{aligned}\rho(h) &= \rho((e-r_3)^{-1})(r_1+r_2)\\ &\leq \rho(e-r_3^{-1}\rho(a_1+r_2)\\ &\leq \frac{1}{1-\rho(r_3)}\rho(r_1+r_2)\\ &< \frac{1}{s}.\end{aligned}$$

Thus $\rho(sh)<1$, which means that $e-sh$ is invertible.

Let $m,n\in\mathbb{N}$ such that $m>n\geq 1$. Then we have

$$\begin{aligned}q(x_n,x_m) &\preceq sq(x_n,x_{n+1})+s^2q(x_{n+1},x_{n+2})+\ldots+s^{m-n}q(x_{m-1},x_m)\\ &\preceq sh^nq(x_0,x_1)+s^2h^{n+1}q(x_0,x_1)+\ldots+s^{m-n}h^{m-1}q(x_0,x_1)\\ &\preceq (sh^n+s^2h^{n+1}+\ldots+s^{m-n}h^{m-1})q(x_0,x_1)\\ &= sh^n(1+sh+(sh)^2+\ldots+(sh)^{m-n-1})q(x_0,x_1)\\ &\preceq sh^n(1-sh)^{-1}q(x_0,x_1). \qquad (3.6)\end{aligned}$$

Since $\rho(h)<\frac{1}{s}<1$ and $\|h^n\|\longrightarrow\theta$ (as $n\longrightarrow\infty$), Lemma 2.9 (4) shows that $\{h^n\}$ is a c-sequence. Thus, Lemma 2.8 (3) shows that x_n is a Cauchy sequence in X. Since X is complete, there exists $t\in X$ such that $x_n\longrightarrow t$ as $n\longrightarrow\infty$. As f is a continuous thus, $t=\lim x_{n+1}=\lim fx_n=f\lim x_n=ft$. The uniqueness of limit shows that t is a fixed point of f.

Finally, Suppose that t is the fixed point of f, that is, $ft=t$. Since f is α-admissible mapping, $ft=t$, inequality (3.3) and $x_n\longrightarrow t$ (as $n\longrightarrow\infty$), obviously we have:

$$\alpha(x_n,x_{n+1})\geq 1\Longrightarrow\alpha(t,t)\geq 1 \quad (\text{as} \quad n\longrightarrow\infty).$$

Applying Condition 3.2, we get:

$$\begin{aligned} q(t,t) &= q(ft,ft) \\ &\preceq r_1 q(t,t) + r_2 q(t,ft) + r_3 q(t,ft) \\ &= (r_1+r_2+r_3) q(t,t). \end{aligned}$$

Note that, $\rho(r_1+r_2+r_3) < \rho(r_3) + \rho(sr_1+sr_2) < 1$. Thus, Lemma 2.9 (3) shows that $q(t,t) = \theta$. □

By omitting the continuity of f, we state the following theorem.

Theorem 3.5. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone, $\alpha : X \times X \longrightarrow [0,\infty[$ be a funcation and q is a generalized c-distance on X. Suppose the mapping $f : X \longrightarrow X$ is a generalized Reich type contraction mapping with generalized Reich-Lipschitz vectors $r_1, r_2, r_3 \in P$ such that r_3 commutes with (r_1+r_2) and the following conditions are satisfied:

(i) f is α-admissible mapping,

(ii) there is $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n,t) \geq 1$ for all $n \in \mathbb{N}$.

Then, f has a fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$.

Proof. By the same process in the proof of Theorem 3.4, we obtain a sequence x_n in X such that x_n is a Cauchy in X. Since X is complete, there exists $t \in X$ such that $x_n \longrightarrow t$ as $n \longrightarrow \infty$.

By q3 we have:

$$q(x_n,t) \preceq \frac{s^2 h^n}{1-sh} q(x_0,x_1). \tag{3.7}$$

On the other hand, by 3.3 and Condition (iii) we get

$$\alpha(x_n,t) \geq 1 \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Note that, from (3.5)

$$q(fx_{n-1}, fx_n) \preceq hq(x_{n-1},x_n) \preceq h^n q(x_0,x_1). \tag{3.9}$$

Now, apply (3.7) and (3.9) with $x_n = t$, we get

$$
\begin{aligned}
q(x_n, ft) &= q(fx_{n-1}, ft) \\
&\preceq hq(x_{n-1}, t) \\
&\preceq h\frac{s^2h^{n-1}}{1-sh}q(x_0, x_1) \\
&= \frac{s^2h^n}{1-sh}q(x_0, x_1). \qquad (3.10)
\end{aligned}
$$

Observe that, the right sides of (3.7) and (3.10) are c-sequences. Thus, Lemma 2.8 (1) shows that $t = ft$. Therefore, t is the fixed point of f.

Finally, Suppose that t is the fixed point of f, that is, $ft = t$. Since f is α-admissible mapping, $ft = t$, inequality (3.3) and $x_n \longrightarrow t$ (as $n \longrightarrow \infty$), obviously we have:

$$\alpha(x_n, x_{n+1}) \geq 1 \Longrightarrow \alpha(t,t) \geq 1 \quad (\text{as} \quad n \longrightarrow \infty).$$

Applying Condition 3.2, we get:

$$
\begin{aligned}
q(t,t) &= q(ft, ft) \\
&\preceq r_1q(t,t) + r_2q(t, ft) + r_3q(t, ft) \\
&= (r_1 + r_2 + r_3)q(t,t).
\end{aligned}
$$

Note that, $\rho(r_1 + r_2 + r_3) < \rho(r_3) + \rho(sr_1 + sr_2) < 1$. Thus, Lemma 2.9 (3) shows that $q(t,t) = \theta$. □

The above Theorems 3.4 and 3.5 guarantee the existences of the fixed point only. To assure the uniqueness of the fixed point. Then we have the following theorem.

Theorem 3.6. Let (X, d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone, $\alpha : X \times X \longrightarrow [0, \infty[$ be a funcation and q is a generalized c-distance on X. Suppose the mapping $f : X \longrightarrow X$ is a generalized Reich type contraction mapping with generalized Reich-Lipschitz vectors $r_1, r_2, r_3 \in P$ such that r_3 commutes with $(r_1 + r_2)$ and the following conditions are satisfied:

(i) f is a triangular α-admissible mapping,

(ii) there is $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) f is continuous or if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n, t) \geq 1$ for all $n \in \mathbb{N}$,

(iv) for all $x, y \in X$ with $x \neq y$, there exists $u \in X$ such that $\alpha(x, u) \geq 1$ and $\alpha(u, y) \geq 1$.

Then, f has a unique fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$.

Proof. Theorem 3.5 (resp. Theorem 3.4) guarantee that, f has a fixed point $t \in X$ and $q(t,t) = \theta$. Suppose that f has another fixed point $p \in X, p = fp$ such that $p \neq t$. Now, by Condition (iv) there there exists $u \in X$ such that $\alpha(t, u) \geq 1$ and $\alpha(u, p) \geq 1$. Since f is a triangular α-admissible mapping, we have $\alpha(t, p) \geq 1$. Now, applying Condition 3.2, we get

$$\begin{aligned} q(t,p) &= q(ft, fp) \\ &\preceq r_1 q(t,p) + r_2 q(t, ft) + r_3 q(p, fp) \\ &= r_1 q(t,p) + r_2 q(t,t) + r_3 q(p,p) \\ &= r_1 q(t,p). \end{aligned}$$

Note that, $\rho(r_1 \rho(r_3) + \rho(sr_1 + sr_2) < 1$. Thus, Lemma 2.9 (3) shows that $q(t,p) = \theta$. Also, $q(t,t) = \theta$. Thus, Lemma 2.8 (1) shows that $t = p$. Therefore, the fixed point is unique. □

Example 3.7. Let $X = [0,1]$ and $\mathscr{A} = C^1_{\mathbb{R}}[0,1] \times C^1_{\mathbb{R}}[0,1]$ with the norm

$$\|(x_1, x_2)\| = \|x_1\|_\infty + \|x_2\|_\infty + \left\|x_1'\right\|_\infty + \left\|x_2'\right\|_\infty, (x_1, x_2) \in \mathscr{A}.$$

Define multiplication in $\mathscr{A}$ by $xy = (x_1 y_1, x_1 y_2 + y_1 x_2)$ where $x = (x_1, x_2), y = (y_1, y_2) \in \mathscr{A}$. Then $\mathscr{A}$ is a Banach algebra with a unit $e(t) = (1,0), t \in [0,1]$. Let $P = \{x = (x_1(t), x_2(t)) \in \mathscr{A} : x_1(t), x_2(t) \geq 0 \text{ for all } t \in [0,1]\}$. Define a cone b-metric $d : X \times X \to \mathscr{A}$ by $d(u,v)(t) = (|u-v|^2, |u-v|^2)e^t$ for all $u, v \in X$ and $t \in [0,1]$. Then (X, d) is a complete cone b-metric space with the coefficient $s = 2$ over Banach algebra $\mathscr{A}$. Define a mapping $q : X \times X \longrightarrow \mathscr{A}$ by $q(u,v)(t) = (v^2, v^2) \cdot e^t$ for all $u, v \in X$ and $t \in [0,1]$. Then q is a generalized c-distance on X. In fact, (q1), (q2) and (q3) are immediate. Let $\varepsilon \in intP$ with $\theta \ll \varepsilon$ be given and put $\delta = \frac{\varepsilon}{4}$.

Suppose that $q(z,u) \ll \delta$ and $q(z,v) \ll \delta$, then we have

$$
\begin{aligned}
d(u,v) &= \left(|u-v|^2, |u-v|^2\right)^2 e^t \\
&\preceq \left(2u^2+2v^2, 2u^2+2v^2\right)^2 e^t \\
&= \left(2u^2, 2u^2\right)^2 e^t + \left(2v^2, 2v^2\right)^2 e^t \\
&= 2q(z,u) + 2q(z,v) \\
&\ll 2\frac{\delta}{4} + 2\frac{\delta}{4} \\
&= \varepsilon.
\end{aligned}
$$

This shows that q satisfies (q4) and hence q is a generalized c-distance. Furthermore,

$$
\begin{aligned}
\rho(x) &= \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \|(x_1,x_2)^n\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left\|(x_1^n, nx_1^{n-1}x_2\right\|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\|x_1^n\|_\infty + \left\|nx_1^{n-1}x_2\right\|_\infty + \left\|(x_1^n)'\right\|_\infty + \left\|(nx_1^{n-1}x_2)'\right\|_\infty\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left(\max_{t\in[0,1]} |x_1^n| + n\max_{t\in[0,1]} |x_1^{n-1}x_2| + \max_{t\in[0,1]} |(x_1^n)'| + \max_{t\in[0,1]} |(nx_1^{n-1}x_2)'|\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} [\max_{t\in[0,1]} |x_1^{n-2}|(\max_{t\in[0,1]} |x_1^2| + n\max_{t\in[0,1]} |x_1x_2| + n\max_{t\in[0,1]} |x_1x_1'| \\
&\qquad + n\max_{t\in[0,1]} |x_1x_2' + (n-1)x_1'x_2|)]^{\frac{1}{n}} \\
&= \lim_{n\to\infty} [(\max_{t\in[0,1]} |x_1|)^{n-2}]^{\frac{1}{n}} \cdot \lim_{n\to\infty} [\max_{t\in[0,1]} |x_1^2| + n\max_{t\in[0,1]} |x_1x_2| + n\max_{t\in[0,1]} |x_1x_1'| \\
&\qquad + n\max_{t\in[0,1]} |x_1x_2' + (n-1)x_1'x_2|]^{\frac{1}{n}} \\
&= \max_{t\in[0,1]} |x_1|.
\end{aligned}
$$

Define the mappings $f : X \longrightarrow X$ and $\alpha : X\times X \longrightarrow [0,\infty[$ by $fx = \frac{x}{\sqrt{2}}$ and $\alpha(x,y) = e^{x+y}$ for all $x,y \in X$. Clearly that

$$x,y\in X, \alpha(x,y) \geq 1 \Longrightarrow \alpha(fx,fy) = \alpha(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}) = e^{\frac{x+y}{\sqrt{2}}} \geq 1.$$

Hence f is a triangular α-admissible mapping and the Condition (i) of Theorem 3.6 is satisfied. On the other hand, take $r_1 = r_1(t) = (\frac{1}{8} + \frac{1}{64}t, \frac{3}{32}t)$, $r_2 = r_2(t) = (\frac{1}{32}t, \frac{1}{32})$ and $r_3 r_3(t) = (\frac{1}{4} + \frac{1}{16}t, \frac{3}{16}t)$. Then, it is easy to see that $\rho(r_3) + \rho(sr_1 + sr_2) = \frac{11}{16} < 1$. Note that,

$$\begin{aligned}
(\frac{y^2}{2}, \frac{y^2}{2}) &\preceq ((\frac{1}{4} + \frac{3}{64}t)y^2, (\frac{1}{4} + \frac{3}{64}t)y^2 + (\frac{3}{16}t)y^2) \\
&= (\frac{1}{4} + \frac{3}{64}t, \frac{3}{16}t)(y^2, y^2) \\
&= (\frac{1}{8} + \frac{1}{64}t, \frac{3}{32}t)(y^2, y^2) + (\frac{1}{8} + \frac{1}{32}t, \frac{3}{32}t)(y^2, y^2) \\
&\preceq (\frac{1}{8} + \frac{1}{64}t, \frac{3}{32}t)(y^2, y^2) + (\frac{1}{64}t, \frac{1}{64}t)(x^2, x^2) + (\frac{1}{8} + \frac{1}{32}t, \frac{3}{32}t)(y^2, y^2) \\
&\preceq (\frac{1}{8} + \frac{1}{64}t, \frac{3}{32}t)(y^2, y^2) + (\frac{1}{32}t, \frac{1}{32}t)(\frac{x^2}{2}, \frac{x^2}{2}) + (\frac{1}{4} + \frac{1}{16}t, \frac{3}{16}t)(\frac{y^2}{2}, \frac{y^2}{2}) \\
&= r_1(y^2, y^2) + r_2(\frac{x^2}{2}, \frac{x^2}{2}) + r_3(\frac{y^2}{2}, \frac{y^2}{2}).
\end{aligned}$$

Now,

$$\begin{aligned}
q(fx, fy)(t) &= ((fy)^2, (fy)^2) \cdot e^t \\
&= (\frac{y^2}{2}, \frac{y^2}{2}) \cdot e^t \\
&\preceq r_1(y^2, y^2) \cdot e^t + r_2(\frac{x^2}{2}, \frac{x^2}{2}) \cdot e^t + r_3(\frac{y^2}{2}, \frac{y^2}{2}) \cdot e^t \\
&= r_1 q(x, y) \cdot e^t + r_2 q(x, fx) \cdot e^t + r_3(y, fy) \cdot e^t.
\end{aligned}$$

Hence, f is a generalized Reich type contraction mapping with generalized Reich-Lipschitz vectors $r_1, r_2, r_3 \in P$. Indeed,

$$\alpha(x_0, fx_0) = \alpha(\frac{1}{\sqrt{2}}, f(\frac{1}{\sqrt{2}})) = \alpha(\frac{1}{\sqrt{2}}, \frac{1}{2}) = e^{\frac{1}{\sqrt{2}} + \frac{1}{2}} \geq 1.$$

Hence, the Condition (ii) of Theorem 3.6 is satisfied.

Finally, let x_n be a sequence in $[0,1]$ such that $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$. Clearly, $\alpha(x_n, x_{n+1}) = e^{x_n + x_{n+1}} \geq 1$ also $t \in [0,1]$ since $[0,1]$ is complete and $\alpha(x_n, t) = e^{x_n + t} \geq 1$. Hence, the Condition (iii) of Theorem 3.6 is satisfied. Thus, all conditions of Theorem 3.6 are satisfied and f has a fixed point, that is $t = 0$ is the fixed point of f. Note that, $\alpha(0, u) = e^u \geq 1$ for all $u \in X$. Therefore, $t = 0$ is the unique fixed point of f with $q(0,0) = \theta$.

As consequence results of Theorem 3.6, we have the Banach fixed point result and Kannan fixed point result for a triangular α-admissible mappings in cone b-metric spaces over Banach algebra with generalized c-distance.

Theorem 3.8. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone, $\alpha : X \times X \longrightarrow [0,\infty[$ be a funcation and q is a generalized c-distance on X. Suppose the mapping $f : X \longrightarrow X$ is a generalized Banach type contraction mapping with generalized Banach-Lipschitz vectors $b \in P$ such that the following conditions are satisfied:

(i) f is a triangular α-admissible mapping,

(ii) there is $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) f is continuous or if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n, t) \geq 1$ for all $n \in \mathbb{N}$,

(iv) for all $x, y \in X$ with $x \neq y$, there exists $u \in X$ such that $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$.

Then, f has a unique fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$.

Theorem 3.9. Let (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone, $\alpha : X \times X \longrightarrow [0,\infty[$ be a funcation and q is a generalized c-distance on X. Suppose the mapping $f : X \longrightarrow X$ is a generalized Kannan type contraction mapping with generalized Kannan-Lipschitz vectors $k_1, k_2 \in P$ such that k_2 commutes with k_2 and the following conditions are satisfied:

(i) f is a triangular α-admissible mapping,

(ii) there is $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) f is continuous or if x_n is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$, then $\alpha(x_n, t) \geq 1$ for all $n \in \mathbb{N}$.

(iv) for all $x, y \in X$ with $x \neq y$, there exists $u \in X$ such that $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$.

Then, f has a unique fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$.

The next theorem is an ordered version of generalized Riech type contraction on partially ordered cone b-metric space with Banach algebra.

Theorem 3.10. Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) be a complete cone b-metric space over Banach algebra with the coefficient $s \geq 1$, P is a solid cone and q is a generalized c-distance on X. Let $f : X \longrightarrow X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose the following conditions hold:

(i) there exist $r_1, r_2, r_3 \in P$ such that r_3 commutes with $(r_1 + r_2)$, $\rho(r_3) + \rho(sr_1 + sr_2) < 1$ and

$$q(fx, fy) \preceq r_1 q(x,y) + r_2 q(x, fx) + r_3 q(y, fy),$$

for all $x, y \in X$ with $x \sqsubseteq y$.

(ii) there is $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$,

(iii) f is continuous or if x_n is nondecreasing sequence in X such that $xn \longrightarrow t \in X$ as $n \longrightarrow \infty$, then $x_n \sqsubseteq t$ for all $n \in \mathbb{N}$.

Then, f has a fixed point t in X. Furthermore, if t is a fixed point of f, then $q(t,t) = \theta$. The fixed point is unique.

Proof. Define the mapping $\alpha_j : X \times X \longrightarrow [0, \infty[$ by

$$\alpha_j = \begin{cases} 1, & \text{if } x \sqsubseteq y, \\ 0, & \text{otherwise.} \end{cases}$$

Note that, the condition (i) implies that the mapping f is a generalized Reich type contraction mapping with generalized Reich-Lipschitz vectors $r_1, r_2, r_3 \in P$. Since f is nondecreasing then, it is an α_j-admissible mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) = 1$. Since x_n is nondecreasing sequence in X then The condition (iii) means that, $\alpha_j(x_n, x_{n+1}) = 1$ implies $\alpha_j(x_n, t) = 1$. Hence, all the conditions of Theorem 3.6 are satisfied. Therefore, the mapping f has a unique fixed point t in X and $q(t,t) = \theta$. $\square$

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5. Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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