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NUMERICAL COMPUTATION OF SUCCESSIVE APPROXIMATIONS METHOD AND VARIATIONAL ITERATION METHOD FOR SOLVING KLEIN-GORDON SCHRÖDINGER EQUATION

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Abstract: This paper is devoted to investigating and comparing the Successive Approximations Method (SAM) and Variational Iteration Method (VIM) for solving Klein-Gordon Schrödinger (KGS) Equation. Furthermore, the approximate solutions that obtained by both methods have been represented numerically and graphically.

Keywords: successive approximations method; variational iteration method; Klein-Gordon Schrödinger.

2010 AMS Subject Classification: 5810.

1. INTRODUCTION

Nowadays, many authors have their attention toward studying the solution of linear and nonlinear partial differential equations. Because, partial differential equations can be used as a proper tool for describing most of the natural phenomena of engineering and science models. Moreover, a wide range of significant phenomena arising in physics, biology, mathematics, mathematical

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physics and another fields are modeled through partial differential equations. While, finding the exact solutions of partial differential equations has become one of the most significant and challenging problems in engineering and physics, biology, mathematics and other science[1]-[2]. Therefore, numerical methods were applied to overcome such problems. Currently, the numerical methods are in competition with each other to find the best and more accurate approximate solutions [3].

In this paper, the Successive Approximations Method (SAM) [3]-[6] and Variational Iteration Method (VIM) [6]-[11] are applied to, the Klein-Gordon Schrödinger (KGS) Equation.

2. MATHEMATICAL MODEL

Consider the Klein-Gordon Schrödinger (KGS) Equation [12]

$$\begin{cases} iq_t = -aq_{xx} - b\varphi q \\ \varphi_{tt} = c^2\varphi_{xx} - \beta^2\varphi + \lambda|q|^2 \end{cases} \quad \in \mathbb{R}, t \geq 0, \quad i = \sqrt{-1} \quad (1)$$

Where a, b, c, β and λ are considered as arbitrary constants. While, q is the complex nucleon field and φ is the neutral real meson fields. Moreover, the system (1) has wide-range applications in many fields such as quantum physics and modern physics [12]-[13].

We can separate equation (1) into real and imaginary parts. Therefore, one can obtain a tripled system (a system of three real equations) in the following form:

$$\begin{cases} u_t = -av_{xx} - bv\varphi \\ v_t = au_{xx} + bu\varphi \\ \varphi_{tt} = c^2\varphi_{xx} - \beta^2\varphi + \lambda u^2 + \lambda v^2 \end{cases} \quad x \in \mathbb{R}, t \geq 0, \quad i = \sqrt{-1} \quad (2)$$

$$\text{For} \quad q = u + iv \quad (3)$$

Where u and v and φ are real functions of x and t .

3. DESCRIPTION OF THE METHODS

3.1 BASIC IDEA OF SUCCESSIVE APPROXIMATIONS METHOD [3]-[6]:

Consider the following general nonlinear partial differential equation:

$$LU(x, t) + R[U(x, t)] + N[U(x, t)] = g(x, t) \quad (4)$$

Subject to the initial condition:

$$U(x, 0) = f(x), \quad C_i = \left. \frac{\partial^i U(x, t)}{\partial t^i} \right|_{t=0} \quad U_0 = \sum_{i=0}^{m-1} \left(C_i \frac{t^i}{i!} \right) \quad (5)$$

For $L = \frac{\partial^m}{\partial t^m}$, $m \in \mathbb{N}$, is the highest order partial derivative with respect to time t . Moreover, the reminder linear term is R , the nonlinear operator is N and the inhomogeneous source term is $g(x, t)$.

The SAM considers the approximate solution of an integral equation as a sequence, which is usually, convergent to an accurate solution. For solving equation (4) by using SAM, we apply $L^{-1}[\cdot]$, which is:

$$L^{-1}[\cdot] = \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} [\cdot] ds \quad (6)$$

on both sides of equation (4), we get:

$$U(x, t) = \sum_{i=0}^{m-1} \left(C_i \frac{t^i}{i!} \right) + L^{-1}(g(x, s) - R[U(x, s)] - N[U(x, s)]) \quad (7)$$

The solution of (7) by SAM is a sequence as follows:

$$\{U_n(x, t)\}_{n=0}^{\infty} \quad (8)$$

Now, for $g(x, t) = 0$, then SAM introduces the recurrence relation of the form:

$$U_{n+1} = \sum_{i=0}^{m-1} \left(C_i \frac{t^i}{i!} \right) - L^{-1}(R[U(x, s)]) + L^{-1}(N[U(x, s)]) \quad (9)$$

Therefore, The solution is computed as:

$$\lim_{n \rightarrow \infty} U_n = U(x, t) \quad \text{and} \quad U(x, t) = \sum_{n=0}^{\infty} U_n(x, t) \quad (10)$$

The Successive Approximations Method is simple in its principles. While the difficulties appear in proving the convergence of the introduced series [3].

3.2 BASIC IDEA OF VARIATIONAL ITERATION METHOD [6]-[11]

Applying VIM on eq. (4), then we write the correction functionals of equation (4) as:

$$U_{n+1}(x, t) = U_n(x, t) + \int_0^t \lambda(s) \left[\left(LU_n(x, s) + R\tilde{U}_n(x, s) + N\tilde{U}_n(x, s) - g(x, s) \right) \right] ds$$

Which implies:

$$U_{n+1}(x, t) = U_n(x, t) + \int_0^t \lambda(s) \left[(LU_n(x, s) + R\tilde{U}_n(x, s) + N\tilde{U}_n(x, s)) \right] ds \quad (11)$$

For $g(x, s) = 0$,

Where λ is called a Lagrange multiplier, which will be identified optimally by variational iteration method. Now, \tilde{U}_n is a restricted variation, which demonstrates that $\delta\tilde{U}_n = 0$. Creating the correct functional of (4), that yields:

$$\delta U_{n+1}(x, t) = \delta U_n(x, t) + \delta \int_0^t \lambda(s) \left[(LU_n(x, s) + R\tilde{U}_n(x, s) + N\tilde{U}_n(x, s)) \right] ds \quad (12)$$

Therefore, its stationary conditions can be find by applying integration by parts on equation (12).

Then, we get the general form of Lagrange multiplier as follows [11]:

$$\lambda(s) = \frac{(-1)^m}{(m-1)!} (s-t)^{m-1} \quad (13)$$

Substituting equation (12) into equation (13) resulting the following iteration formula:

$$U_{n+1}(x, t) = U_n(x, t) + \int_0^t \frac{(-1)^m (s-t)^{m-1}}{(m-1)!} \left[(LU_n(x, s) + RU_n(x, s) + NU_n(x, s)) \right] ds \quad (14)$$

Then, the approximate solution of equation (4) is given by:

$$U(x, t) = \lim_{n \rightarrow \infty} U_n(x, t) \quad (15)$$

4. DERIVATION OF SAM AND VIM FOR SOLVING KGS-SYSTEM

4.1 DERIVATION OF SAM FOR SOLVING KGS-SYSTEM

Applying L^{-1} [.] on both sides of equation (2) we get:

$$u_{n+1}(x, t) = u_0 - L^{-1} \left[a \frac{\partial^2 v_n}{\partial x^2} + b v_n \varphi_n \right], \quad m = 1 \quad (16)$$

$$v_{n+1}(x, t) = v_0 + L^{-1} \left[a \frac{\partial^2 u_n}{\partial x^2} + b u_n \varphi_n \right], \quad m = 1 \quad (17)$$

$$\varphi_{n+1}(x, t) = \varphi_0 + L^{-1} \left[c^2 \frac{\partial^2 \varphi_n}{\partial x^2} - \beta^2 \varphi_n + \lambda u_n^2 + \lambda v_n^2 \right], \quad m = 2 \quad (18)$$

Using equation (6) and substituting it in equations (16), (17) and (18), to determine the several successive approximations of the system (2) as follow:

$$u_{n+1}(x, t) = u_0 - \int_0^t \left(a \frac{\partial^2 v_n(x, s)}{\partial x^2} + b v_n(x, s) \varphi_n(x, s) \right) ds \quad (19)$$

$$v_{n+1}(x, t) = v_0 + \int_0^t \left(a \frac{\partial^2 u_n(x, s)}{\partial x^2} + b u_n(x, s) \varphi_n(x, s) \right) ds \quad (20)$$

$$\begin{aligned} \varphi_{n+1}(x, t) = \varphi_0 & \\ & + \int_0^t \left((t-s) \left(c^2 \frac{\partial^2 \varphi_n(x, s)}{\partial x^2} - \beta^2 \varphi_n(x, s) + \lambda u_n(x, s)^2 \right. \right. \\ & \left. \left. + \lambda v_n(x, s)^2 \right) \right) ds \end{aligned} \quad (21)$$

For all $n \geq 0$.

To find the initial approximations, we use equation (5), so we get:

$$u_0 = u(x, 0), \quad v_0 = v(x, 0) \quad \text{and}$$

$$\varphi_0 = \varphi(x, 0) + t \varphi_t(x, 0) \quad (22)$$

Then, the approximate solutions will take the forms:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \approx u_n(x, t) \quad (23)$$

$$v(x, t) = \lim_{n \rightarrow \infty} v_n(x, t) \approx v_n(x, t) \quad (24)$$

$$\varphi(x, t) = \lim_{n \rightarrow \infty} \varphi_n(x, t) \approx \varphi_n(x, t) \quad (25)$$

Where n is the closing iteration step.

4.2 DERIVATION OF VIM FOR SOLVING KGS-SYSTEM

The variational iteration formula in equation (14) is used to find the iteration formulas of equation (2) as follow:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda_1(s) \left[\frac{\partial u_n(x, s)}{\partial t} + a \frac{\partial^2 v_n(x, s)}{\partial x^2} + b v_n(x, s) \varphi_n(x, s) \right] ds \quad (26)$$

$$\begin{aligned} v_{n+1}(x, t) = v_n(x, t) + \\ \int_0^t \lambda_2(s) \left[\frac{\partial v_n(x, s)}{\partial t} - a \frac{\partial^2 u_n(x, s)}{\partial x^2} - b u_n(x, s) \varphi_n(x, s) \right] ds \end{aligned} \quad (27)$$

$$\begin{aligned} \varphi_{n+1}(x, t) = \varphi_n(x, t) + \int_0^t \lambda_3(s) \left[\frac{\partial^2 \varphi_n(x, s)}{\partial t^2} - c^2 \frac{\partial^2 \varphi_n(x, s)}{\partial x^2} + \beta^2 \varphi_0(x, s) \right. \\ \left. - \lambda(u_n(x, s))^2 - \lambda(v_n(x, s))^2 \right] ds \end{aligned} \quad (28)$$

Then, we can find the Lagrange multiplier by using equation (13), so we get $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = (s - t)$, and substitute them in equations (26), (27) and (28) respectively, then the iteration formulas will become as follow:

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial u_n(x, s)}{\partial t} + a \frac{\partial^2 v_n(x, s)}{\partial x^2} + b v_n(x, s) \varphi_n(x, s) \right] ds \quad (29)$$

$$v_{n+1}(x, t) = v_n(x, t) + \int_0^t \left[-\frac{\partial v_n(x, s)}{\partial t} + b \frac{\partial^2 u_n(x, s)}{\partial x^2} + b u_n(x, s) \varphi_n(x, s) \right] ds \quad (30)$$

$$\begin{aligned} \varphi_{n+1}(x, t) = \varphi_n(x, t) + \\ \int_0^t (s - t) \left[\frac{\partial^2 \varphi_n(x, s)}{\partial x^2} - c^2 \frac{\partial^2 \varphi_n(x, s)}{\partial x^2} + \beta^2 \varphi_0(x, s) - \lambda(u_n(x, s))^2 \right. \\ \left. - \lambda(v_n(x, s))^2 \right] ds \end{aligned} \quad (31)$$

For all $n \geq 0$. With initial approximations in equation(22).

Then, the approximate solutions will take the forms:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t) \approx u_n(x, t) \quad (32)$$

$$v(x, t) = \lim_{n \rightarrow \infty} v_n(x, t) \approx v_n(x, t) \quad (33)$$

$$\varphi(x, t) = \lim_{n \rightarrow \infty} \varphi_n(x, t) \approx \varphi_n(x, t) \quad (34)$$

Where n is the closing iteration step.

5. APPLICATION WITH NUMERICAL RESULTS (TABLES, FIGURES)

This section will be devoted to find the numerical results (Tables, Figures) of Klein-Gordon Schrödinger (KGS) Equation by using Successive Approximation Method (SAM) and Variationals Iteration Method (VIM).

Example:

If we take the arbitrary constants of equations (1) and (2) to be:

$\left\{ b = c = \lambda = \beta = 1 \text{ and } a = \frac{1}{2} \right\}$ [14]. Then we get:

$$\begin{cases} iq_t = -\frac{1}{2}q_{xx} - \varphi q \\ \varphi_{tt} = \varphi_{xx} - \varphi + |q|^2 \end{cases} \quad x \in \mathbb{R}, t \geq 0, \quad i = \sqrt{-1} \quad (35)$$

and

$$\begin{cases} u_t = -\frac{1}{2}v_{xx} - v\varphi \\ v_t = \frac{1}{2}u_{xx} + u\varphi \\ \varphi_{tt} = \varphi_{xx} - \varphi + u^2 + v^2 \end{cases} \quad x \in \mathbb{R}, t \geq 0, \quad i = \sqrt{-1} \quad (36)$$

The exact solitary wave solution of system (35) as in [13]-[14] are:

$$\begin{cases} q(x, t) = \frac{3\text{Sech}\left[\frac{x-x_0-t\alpha}{2\sqrt{1-\alpha^2}}\right]^2 e^{i(x\alpha+t(-\frac{\alpha^2}{2}+\frac{1}{2-2\alpha^2}))}}{2\sqrt{2-2\alpha^2}} \\ \varphi(x, t) = -\frac{3\text{Sech}\left[\frac{x-x_0-t\alpha}{2\sqrt{1-\alpha^2}}\right]^2}{4(\alpha^2-1)} \end{cases} \quad (37)$$

With the initial conditions:

$$\begin{cases} q(x, 0) = \frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2 e^{ix\alpha}}{2\sqrt{2-2\alpha^2}} \\ \varphi(x, 0) = -\frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2}{4(\alpha^2-1)} \end{cases} \quad (38)$$

Where $|\alpha| > 0$ is the propagating velocity of the wave and x_0 is the initial phase[13].

Therefore, the exact solitary wave solutions the system (36) are:

$$\begin{cases} u(x, t) = \frac{3\text{Sech}\left[\frac{x-x_0-t\alpha}{2\sqrt{1-\alpha^2}}\right]^2 \text{Cos}\left[x\alpha + t\left(-\frac{\alpha^2}{2} + \frac{1}{2-2\alpha^2}\right)\right]}{2\sqrt{2-2\alpha^2}} \\ v(x, t) = \frac{3\text{Sech}\left[\frac{x-x_0-t\alpha}{2\sqrt{1-\alpha^2}}\right]^2 \text{Sin}\left[x\alpha + t\left(-\frac{\alpha^2}{2} + \frac{1}{2-2\alpha^2}\right)\right]}{2\sqrt{2-2\alpha^2}} \\ \varphi(x, t) = -\frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2}{4(\alpha^2-1)} \end{cases} \quad (39)$$

With the initial conditions

$$\begin{cases} u(x, 0) = \frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2 \text{Cos}[x\alpha]}{2\sqrt{2-2\alpha^2}} \\ v(x, 0) = \frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2 \text{Sin}[x\alpha]}{2\sqrt{2-2\alpha^2}} \\ \varphi(x, 0) = -\frac{3\text{Sech}\left[\frac{x-x_0}{2\sqrt{1-\alpha^2}}\right]^2}{4(\alpha^2-1)} \end{cases} \quad (40)$$

Where $|\alpha| > 0$ is the propagating velocity of the wave and x_0 is the initial phase [13].

In this section we considered initial-values ($\alpha = 0.8$, $x_0 = -10$)[14].

Table 1. The absolute errors of the two terms approximate solutions $q(x, t)$ obtained by SAM and VIM of equation (35), $x = 10$ and $t \in [0, 1]$.

Time	$ q_{SAM} - q_{Exact} $	$ q_{VIM} - q_{Exact} $
0	0	0
0.1	$2.030110674 \times 10^{-17}$	$2.030110674 \times 10^{-17}$
0.2	$1.67971031 \times 10^{-16}$	$1.67971031 \times 10^{-16}$
0.3	$5.864410163 \times 10^{-16}$	$5.864410163 \times 10^{-16}$
0.4	$1.4382761 \times 10^{-15}$	$1.4382761 \times 10^{-15}$
0.5	$2.907058086 \times 10^{-15}$	$2.907058086 \times 10^{-15}$
0.6	$5.199395215 \times 10^{-15}$	$5.199395215 \times 10^{-15}$
0.7	$8.547065525 \times 10^{-15}$	$8.547065525 \times 10^{-15}$
0.8	$1.320930239 \times 10^{-14}$	$1.320930239 \times 10^{-14}$
0.9	$1.947523274 \times 10^{-14}$	$1.947523274 \times 10^{-14}$
1	$2.766648102 \times 10^{-14}$	$2.766648102 \times 10^{-14}$

Table 2. The absolute errors of the two terms approximate solutions $\varphi(x, t)$ obtained by SAM and VIM of equation (35), $x = 10$ and $t \in [0, 1]$.

Time	$ \varphi_{SAM} - \varphi_{Exact} $	$ \varphi_{VIM} - \varphi_{Exact} $
0	0	0
0.1	$2.389954647 \times 10^{-17}$	$1.919379919 \times 10^{-18}$
0.2	$2.079472414 \times 10^{-16}$	$3.210590896 \times 10^{-17}$
0.3	$7.630275695 \times 10^{-16}$	$1.695630726 \times 10^{-16}$
0.4	$1.964709902 \times 10^{-15}$	$5.579792421 \times 10^{-16}$
0.5	$4.163329138 \times 10^{-15}$	$1.415808318 \times 10^{-15}$
0.6	$7.793832134 \times 10^{-15}$	$3.046116159 \times 10^{-15}$
0.7	$1.338535651 \times 10^{-14}$	$5.846159384 \times 10^{-15}$
0.8	$2.157050366 \times 10^{-14}$	$1.031665839 \times 10^{-14}$
0.9	$3.309426236 \times 10^{-14}$	$1.707072094 \times 10^{-14}$
1	$4.882253312 \times 10^{-14}$	$2.684236656 \times 10^{-14}$

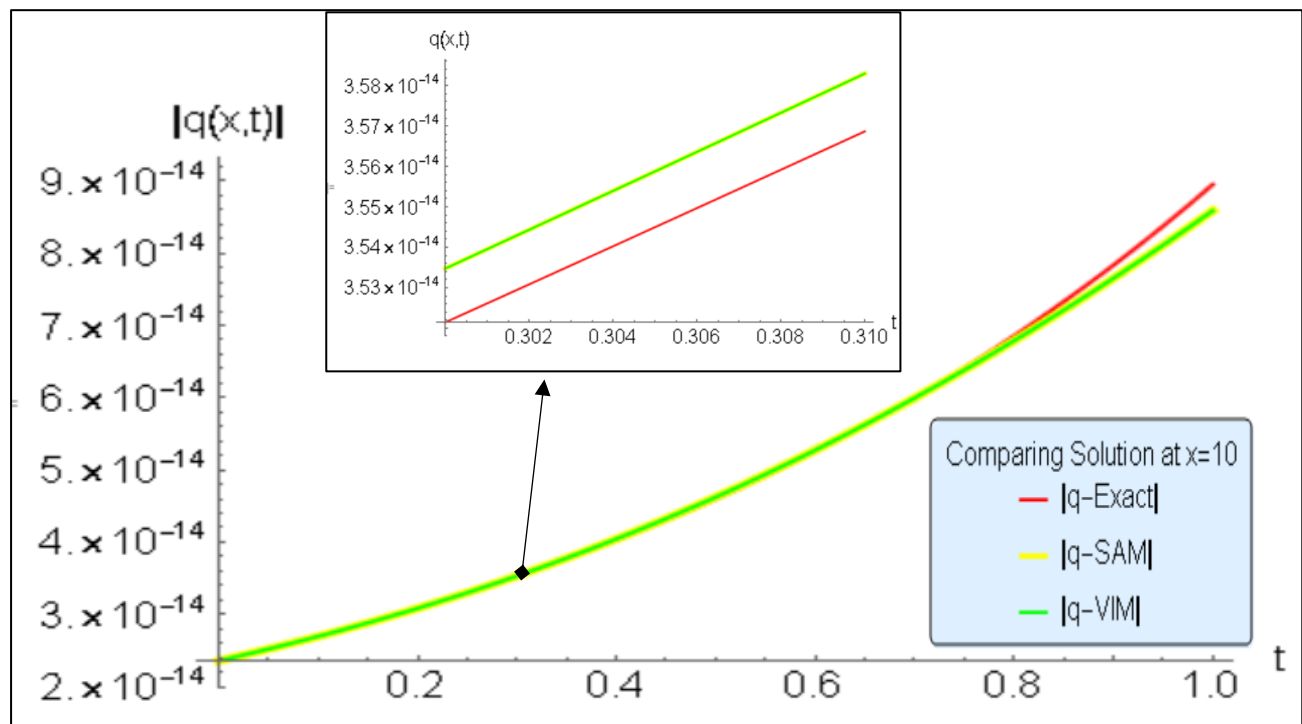


Figure 1: The comparison between SAM and VIM with exact solution for $|q(x, t)|$, when $x = 10$ and $t \in [0, 1]$.

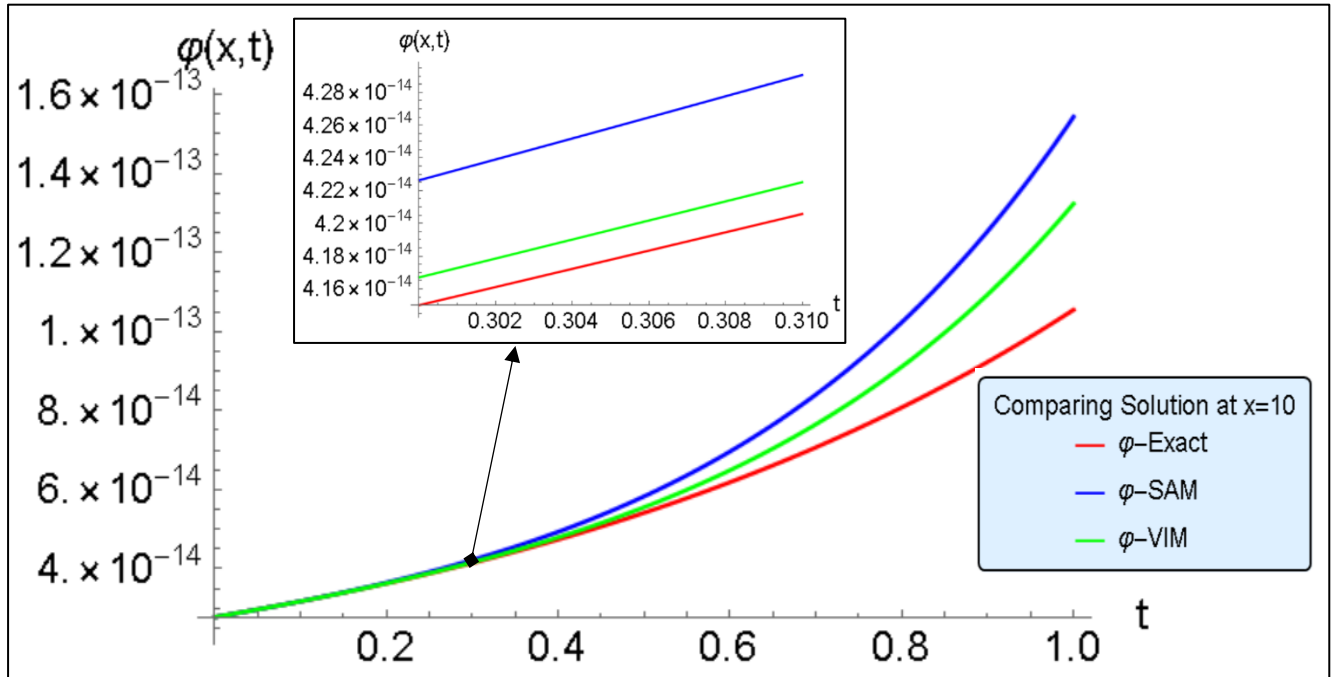


Figure 2: The comparison between SAM and VIM with exact solution for $\varphi(x,t)$, when $x = 10$ and $t \in [0,1]$.

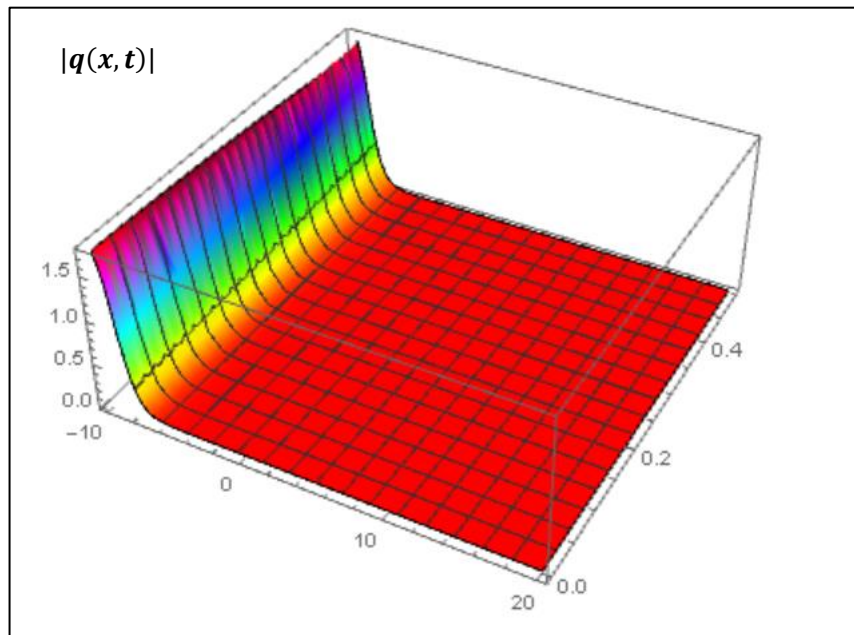


Figure 3: The surfaces of exact solutions $|q(x,t)|$, when $x \in [-10,20]$ and $t \in [0,0.5]$.

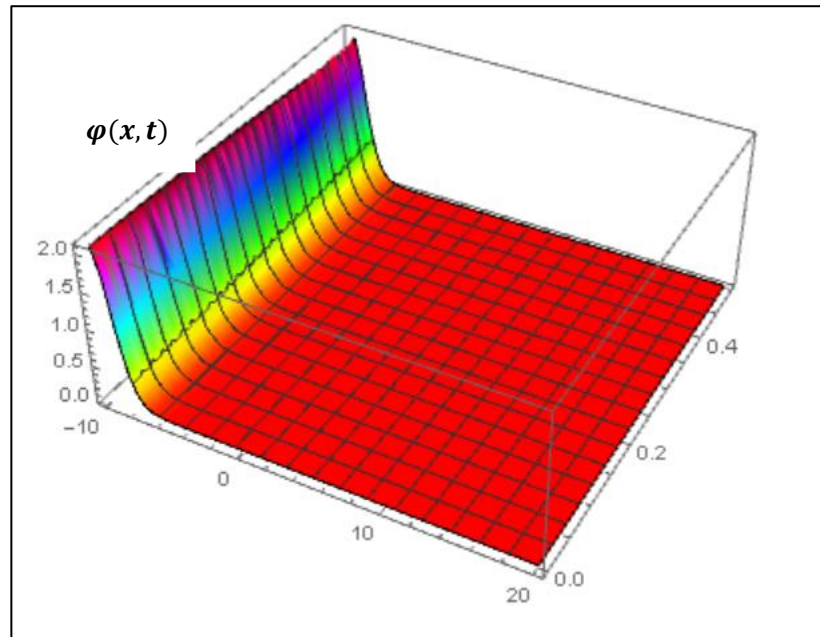


Figure 4: The surfaces of exact solutions $\varphi(x, t)$, when $x \in [-10, 20]$ and $t \in [0, 0.5]$.

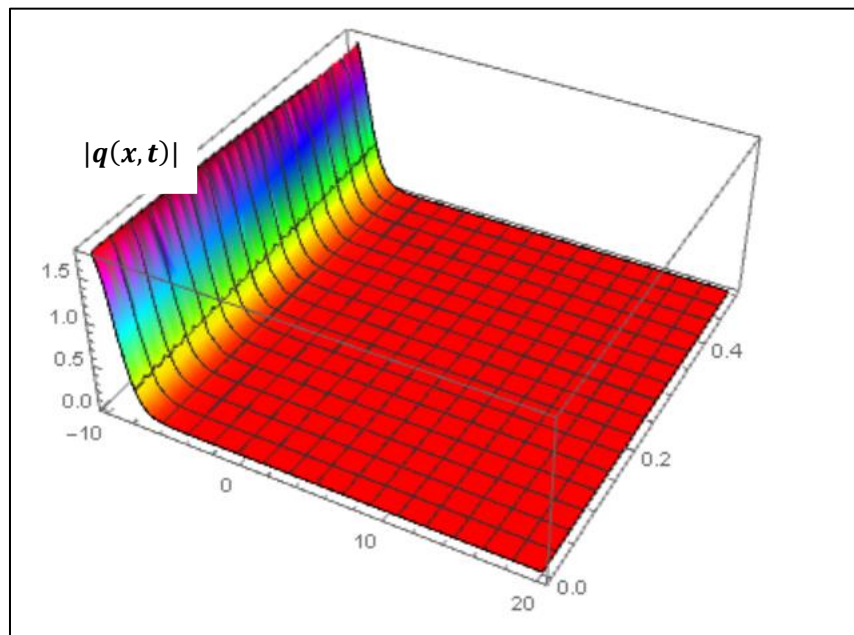


Figure 5: The surfaces of approximate solutions $|q(x, t)|$ by SAM, when $x \in [-10, 20]$ and $t \in [0, 0.5]$.

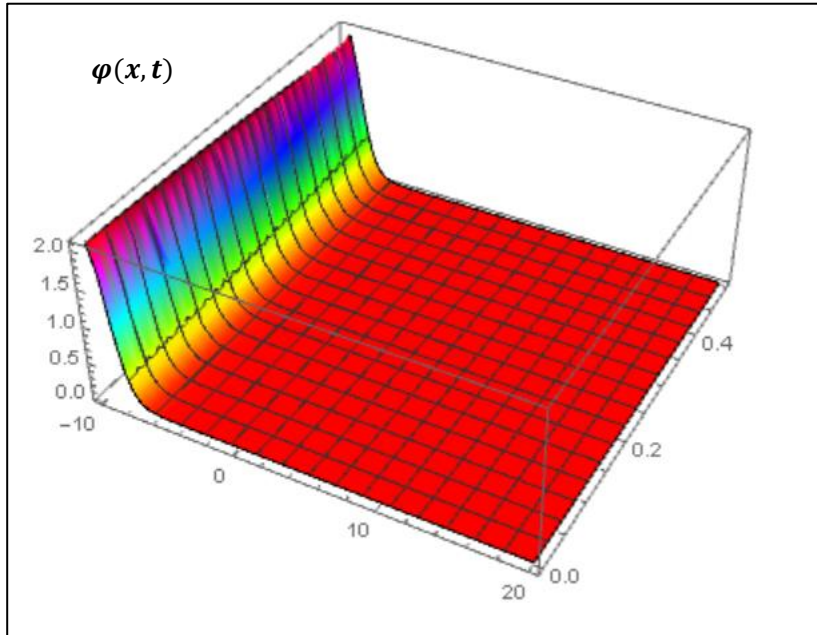


Figure 6: The surfaces of approximate solutions $\varphi(x, t)$, by SAM, when $x \in [-10, 20]$ and $t \in [0, 0.5]$.

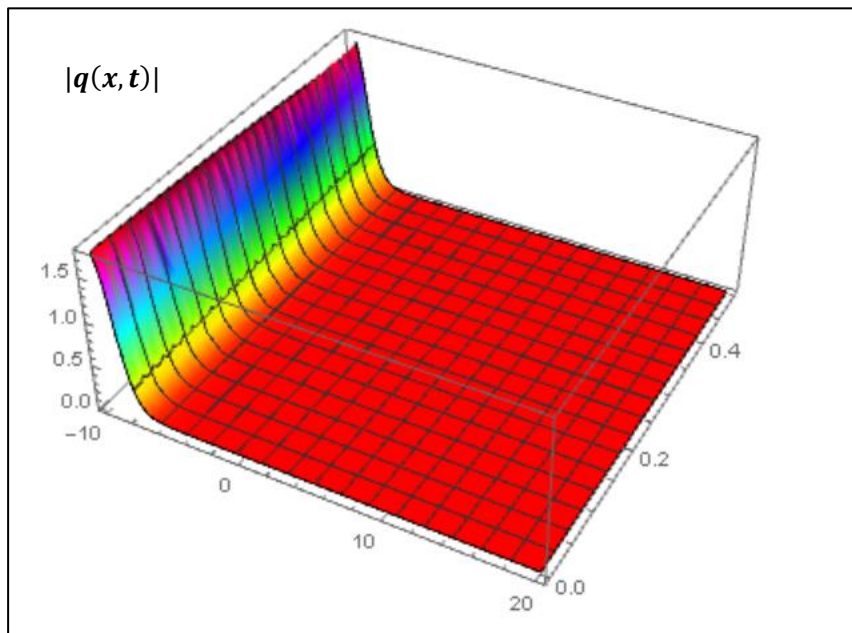


Figure 7: The surfaces of approximate solutions $|q(x, t)|$ by VIM, when $x \in [-10, 20]$ and $t \in [0, 0.5]$.

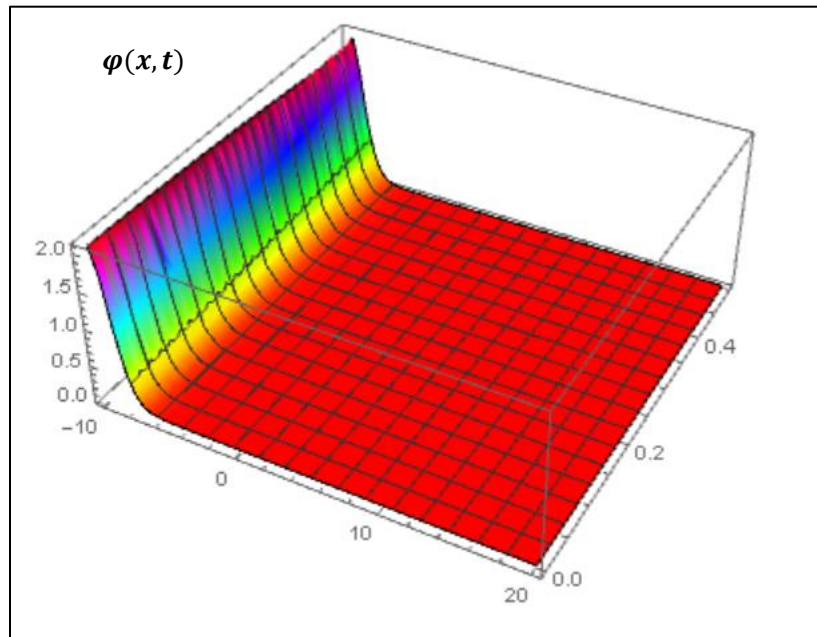


Figure 8: The surfaces of approximate solutions $\varphi(x, t)$, by VIM, when $x \in [-10, 20]$ and $t \in [0, 0.5]$.

6. CONCLUSION

In this paper the Klein-Gordon Schrödinger (KGS) system was solved numerically by using Successive Approximations Method and Variational Iteration Method. We took an example of (KGS) equation to find the comparison between our solutions and the exact solution, and we showed that both methods are very accurate and effective in solving (KGS) equation. However, it is clear from Table1 and Figure1 that the obtained solutions for $q(x, t)$ by both methods are equivalent. While, Table2 and Figure2 showed that the obtained solutions for $\varphi(x, t)$ by VIM is more accurate than SAM.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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