Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 3949-3962
https://doi.org/10.28919/jmcs/5811
ISSN: 1927-5307

CHARACTERIZATIONS OF FUZZY $(\alpha,\beta,\theta,\partial,\ell)$-CONTINUOUS MULTIFUNCTIONS AND THEIR DECOMPOSITION

I.M. TAHA[1,2,*], S.E. ABBAS[2]

[1]Department of Basic Sciences, Higher Institute of Engineering and Technology, Menoufia, Egypt

[2]Department of Mathematics, Faculty of Science, Sohag University, Egypt

Abstract. In this paper we introduce the concept of fuzzy upper and fuzzy lower $(\alpha,\beta,\theta,\partial,\ell)$-continuous multifunctions. Also, we investigate some properties of them and their decomposition. Later, in order to unify several characterizations of some kind of fuzzy continuity, we introduced and studied generalized form of fuzzy continuous multifunctions namely fuzzy upper and fuzzy lower $\kappa\kappa'$-continuous multifunctions. These multifunctions enable us to reduce many generalized forms of continuity to a single theoretical unified framework.

Keywords: fuzzy ideal; fuzzy multifunction; fuzzy upper (lower) $(\alpha,\beta,\theta,\partial,\ell)$-continuous multifunctions.

2010 AMS Subject Classification: 54A40, 54C08, 54C60.

1. Introduction and Preliminaries

The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. Since its inception thirty years ago by Zadeh [21], this theory has found wide applications in information sciences, engineering, medicine, economics, etc.; for details the reader is referred to [11, 22]. A fuzzy multifunction is a fuzzy set valued function [4, 12, 18, 19]. Fuzzy

*Corresponding author
E-mail address: imtaha2010@yahoo.com
Received April 4, 2021

multifunctions arise in many applications, for instance, the budget multifunction occurs in artificial intelligence, economic theory and decision theory. The biggest difference between fuzzy multifunctions and fuzzy functions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse then leads to two definitions of continuity, for more details the reader is referred to [1-3, 7-8, 15-17]. In this paper, we introduce the concept of fuzzy upper (lower) $(\alpha,\beta,\theta,\partial,\ell)$-continuous multifunctions and prove that if α,β are fuzzy operators on the fuzzy topological space (X,τ) in Šostak sense [14] and θ,θ^*,∂ are fuzzy operators on the fuzzy topological space (Y,η) in Šostak sense and ℓ is a proper fuzzy ideal on X [13], then a fuzzy multifunction $F : X \multimap Y$ is fuzzy upper (resp. lower) $(\alpha,\beta,\theta\sqcap\theta^*,\partial,\ell)$-continuous multifunction iff it is both fuzzy upper (resp. lower) $(\alpha,\beta,\theta,\partial,\ell)$-continuous and fuzzy upper (resp. lower) $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous multifunctions. Also, we introduce new generalized notions that cover many of the generalized forms of fuzzy upper (resp. lower) semi-continuous multifunctions.

Throughout this paper, X refers to an initial universe. The family of all fuzzy sets in X is denoted by I^X and for $\lambda\in I^X$, $\lambda^c(x) = 1-\lambda(x)$ for all $x\in X$ (where $I = [0,1]$ and $I_\circ = (0,1]$). For $t\in I$, $\underline{t}(x) = t$ for all $x\in X$. All other notations are standard notations of fuzzy set theory. Also, let us define the fuzzy difference between two fuzzy sets $\lambda,\mu\in I^X$ as follows:

$$\lambda \,\overline{\wedge}\, \mu = \begin{cases} \underline{0}, & \text{if } \quad \lambda\le\mu, \\ \lambda\wedge\mu^c, & \text{otherwise.} \end{cases}$$

An applications α, β, $id_X : I^X\times I_\circ \to I^X$ are fuzzy operators on X and θ, ∂, $id_Y : I^Y\times I_\circ\to I^Y$ are fuzzy operators on Y. Recall that a fuzzy idea ℓ on X [13], is a map $\ell : I^X \longrightarrow I$ that satisfies the following conditions:

(i) $\forall\,\lambda,\mu\in I^X$ and $\lambda\le\mu \Rightarrow \ell(\mu)\le\ell(\lambda)$.

(ii) $\forall\,\lambda,\mu\in I^X \Rightarrow \ell(\lambda\vee\mu)\ge\ell(\lambda)\wedge\ell(\mu)$.

Also, ℓ is called proper if $\ell(\underline{1}) = 0$ and there exists $\mu\in I^X$ such that $\ell(\mu) > 0$. The simplest fuzzy ideal ℓ_0 on X defined as follows:

$$\ell_0(\lambda) = \begin{cases} 1, & \text{if } \quad \lambda=\underline{0}, \\ 0, & \text{otherwise.} \end{cases}$$

Let (X,τ) be a fuzzy topological space in Šostak sense [14]. The closure and the interior of any fuzzy set $\lambda \in I^X$denoted by $C_\tau(\lambda,r)$ and $I_\tau(\lambda,r)$. Any fuzzy set $\lambda \in I^X$ is called r-fuzzy preclosed [10] iff $C_\tau(I_\tau(\lambda,r),r) \le \lambda$, where

$$PC_\tau(\lambda,r) = \bigwedge\{\mu \in I^X \ : \lambda \le \mu,\ \mu \text{ is } r\text{-fuzzy preclosed}\} \text{ [10].}$$

A mapping $F : X \multimap Y$ is called a fuzzy multifunction [5] iff $F(x) \in I^Y$ for each $x \in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y) = G_F(x,y)$ for any $(x,y) \in X \times Y$. Also, F is a Crisp iff $G_F(x,y) = 1$ for each $x \in X$, $y \in Y$ and F is Normalized iff for each $x \in X$, there exists $y_0 \in Y$ such that $G_F(x,y_0) = 1$. The image $F(\lambda)$ of $\lambda \in I^X$, the lower inverse $F^l(\mu)$ and the upper inverse $F^u(\mu)$ of $\mu \in I^Y$ are defined as follows: $F(\lambda)(y) = \bigvee_{x\in X}[G_F(x,y) \wedge \lambda(x)]$, $F^l(\mu)(x) = \bigvee_{y\in Y}[G_F(x,y) \wedge \mu(y)]$ and $F^u(\mu)(x) = \bigwedge_{y\in Y}[G^c_F(x,y) \vee \mu(y)]$. All definitions and properties of image, upper and lower are found in [1].

2. On Fuzzy Upper and Lower $(\alpha,\beta,\theta,\partial,\ell)$-Continuous Multifunctions

Definition 2.1. Let $F : (X,\tau) \multimap (Y,\eta)$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). Then F is said to be fuzzy lower $(\alpha,\beta,\theta,\partial,\ell)$-continuous (resp. fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous) iff for every $\mu \in I^Y$ and $r \in I_o$, $\ell[\alpha(F^l(\partial(\mu,r)),r) \overline{\wedge} \beta(F^l(\theta(\mu,r)),r)]$ $\ge \eta(\mu)$ (resp. $\ell[\alpha(F^u(\partial(\mu,r)),r)\ \overline{\wedge}\ \beta(F^u(\theta(\mu,r)),r)] \ge \eta(\mu)$.)

We can see that the above definition generalizes the concept of fuzzy upper (resp. lower) semi-continuous multifunction [1], when we choose, α = identity operator, β = interior operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

Let us give a historical justification of the above definition:

1. In 2015, Abbas [2] defined the concept of fuzzy lower almost continuous (resp. fuzzy upper almost continuous) multifunction: $\forall\ \ \mu \in I^Y$ with $\eta(\mu) \ge r$, $F^l(\mu) \le I_\tau(F^l(I_\eta(C_\eta(\mu,r),r)),r)$ (resp. $F^u(\mu) \le I_\tau(F^u(I_\eta(C_\eta(\mu,r),r)),r)$). Here α = identity operator, β = interior operator, θ = interior closure operator, ∂ = identity operator and $\ell = \ell_0$.

2. In 2015, Abbas [2] defined the concept of fuzzy lower weakly continuous (resp. fuzzy upper weakly continuous) multifunction: $\forall\ \ \mu \in I^Y$ with $\eta(\mu) \ge r$, $F^l(\mu) \le I_\tau(F^l(C_\eta(\mu,r)),r)$

(resp. $F^u(\mu) \leq I_\tau(F^u(C_\eta(\mu,r)),r)$). Here α = identity operator, β = interior operator, θ = closure operator, ∂ = identity operator and $\ell = \ell_0$.

3. In 2015, Abbas [2] defined the concept of fuzzy lower almost weakly continuous (resp. fuzzy upper almost weakly continuous) multifunction: $\forall \quad \mu \in I^Y$ with $\eta(\mu) \geq r$, $F^l(\mu) \leq I_\tau(C_\tau(F^l(C_\eta(\mu,r)),r),r)$ (resp. $F^u(\mu) \leq I_\tau(C_\tau(F^u(C_\eta(\mu,r)),r),r)$). Here α = identity operator, β = interior closure operator, θ = closure operator, ∂ = identity operator and $\ell = \ell_0$.

4. In 2014, Hebeshi [6] defined the concept of fuzzy lower precontinuous (resp. fuzzy upper precontinuous) multifunction: $\forall \quad \mu \in I^Y$ with $\eta(\mu) \geq r$, $F^l(\mu) \leq I_\tau(C_\tau(F^l(\mu),r),r)$ (resp. $F^u(\mu) \leq I_\tau(C_\tau(F^u(\mu),r),r)$). Here α = identity operator, β = interior closure operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

5. In 2014, Hebeshi [6] defined the concept of fuzzy lower strongly precontinuous (resp. fuzzy upper strongly precontinuous) multifunction: $F^l(\mu) \leq I_\tau(PC_\tau(F^l(\mu),r),r)$ (resp. $F^u(\mu) \leq I_\tau(PC_\tau(F^u(\mu),r),r)$) $\forall \quad \mu \in I^Y$ with $\eta(\mu) \geq r$. Here α = identity operator, β = interior preclosure operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

6. In 2015, Hebeshi [7] defined the concept of fuzzy lower strongly semi-continuous (resp. fuzzy upper strongly semi-continuous) multifunction: $\forall \quad \mu \in I^Y$ with $\eta(\mu) \geq r$, $F^l(\mu) \leq I_\tau(C_\tau(I_\tau(F^l(\mu),\, r),r),r)$ (resp. $F^u(\mu) \leq I_\tau(C_\tau(I_\tau(F^u(\mu),r),r),r)$). Here α = identity operator, β = interior closure interior operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

7. In 2015, Hebeshi [7] defined the concept of fuzzy lower almost strongly semi-continuous (resp. fuzzy upper almost strongly semi-continuous) multifunction: $\forall \quad \mu \in I^Y$ with $\mu = I_\eta(C_\eta(\mu,r),r)$, $F^l(\mu) \leq I_\tau(C_\tau(I_\tau(F^l(\mu),r),r),r)$ (resp. $F^u(\mu) \leq I_\tau(C_\tau(I_\tau(F^u(\mu),r),r),r)$). Here α = identity operator, β = interior closure interior operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

8. In 2015, Hebeshi [7] defined the concept fuzzy lower weakly strongly semi-continuous (resp. fuzzy upper weakly strongly semi-continuous) multifunction: $\forall \quad \mu \in I^Y$ with $\eta(\mu) \geq r$, $F^l(\mu) \leq I_\tau(C_\tau(I_\tau(F^l(C_\eta(\mu,r)),r),r),r)$ (resp. $F^u(\mu) \leq I_\tau(C_\tau(I_\tau(F^u(C_\eta(\mu,r)),r),r),r)$). Here α = identity operator, β = interior closure interior operator, θ = closure operator, ∂ = identity operator and $\ell = \ell_0$.

9. In 2015, Abbas [3] defined the concept of fuzzy lower semi-precontinuous (resp. fuzzy upper semi-precontinuous) multifunction: $\forall\ \mu \in I^Y$ with $\eta(\mu) \geq r$, $F^l(\mu) \leq C_\tau(I_\tau(C_\tau(F^l(\mu),r),r),r)$ (resp. $F^u(\mu) \leq C_\tau(I_\tau(C_\tau(F^u(\mu),r),r),r)$). Here α = identity operator, β = closure interior closure operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

10. In 2016, Hebeshi [8] defined the concept of fuzzy lower almost semi-precontinuous (resp. fuzzy upper almost semi-precontinuous): $F^l(\mu) \leq C_\tau(I_\tau(C_\tau(F^l(\mu),r),r),r)$ (resp. $F^u(\mu) \leq C_\tau(I_\tau(C_\tau(F^u(\mu),r),r),r)$) $\forall\ \mu \in I^Y$ with $\mu = I_\eta(C_\eta(\mu,r),r)$. Here α = identity operator, β = closure interior closure operator, θ = identity operator, ∂ = identity operator and $\ell = \ell_0$.

11. In 2016, Hebeshi [8] defined the concept of fuzzy lower weakly semi-precontinuous (resp. fuzzy upper weakly semi-precontinuous): $F^l(\mu) \leq C_\tau(I_\tau(C_\tau(F^l(C_\eta(\mu,r)),r),r),r)$ (resp. $F^u(\mu) \leq C_\tau(I_\tau(C_\tau(F^u(C_\eta(\mu,r)),r),r),r)$) $\forall\ \mu \in I^Y$ with $\eta(\mu) \geq r$. Here α = identity operator, β = closure interior closure operator, θ = closure operator, ∂ = identity operator and $\ell = \ell_0$.

Definition 2.2. Let $F : (X,\tau) \multimap (Y,\eta)$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). Then F is called fuzzy lower $\mathbb{k}$-continuous (resp. fuzzy upper $\mathbb{k}$-continuous) iff $\tau(F^l(\mu)) \geq \eta(\mu)$ (resp. $\tau(F^u(\mu)) \geq \eta(\mu)$) for each $\mu \in I^Y$ satisfies property $\mathbb{k}$.

Let $\theta_{\mathbb{k}} : I^Y \times I_\circ \to I^Y$ be an operator on (Y,η) defined as follows:

$$\theta_{\mathbb{k}}(\mu,r) = \begin{cases} \mu, & \text{if} \quad \mu \text{ satisfies property } \mathbb{k} \text{ with } \eta(\mu) \geq r, \\ \underline{1}, & \text{otherwise} \end{cases} \quad \text{and } r \in I_\circ.$$

Theorem 2.3 (1) Let $F : (X,\tau) \multimap (Y,\eta)$ be a fuzzy multifunction. Then F is fuzzy lower $\mathbb{k}$-continuous iff it is fuzzy lower $(id, I_\tau, \theta_{\mathbb{k}}, id, \ell_0)$-continuous.

(2) Let $F : (X,\tau) \multimap (Y,\eta)$ be a normalized fuzzy multifunction. Then F is fuzzy upper $\mathbb{k}$-continuous iff it is fuzzy upper $(id, I_\tau, \theta_{\mathbb{k}}, id, \ell_0)$-continuous.

Proof. (1) ($\Rightarrow$) Suppose that F is fuzzy lower $\mathbb{k}$-continuous and $\mu \in I^Y$.

Case 1. If μ satisfies property $\mathbb{k}$ with $\eta(\mu) \geq r$, $\theta_{\mathbb{k}}(\mu,r) = \mu$ and $\tau(F^l(\mu)) \geq r$. Thus, we obtain $F^l(\mu) \leq I_\tau(F^l(\mu),r) = I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r)$. Then $F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r) = \underline{0}$.

Hence $\ell_0[F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r)] \geq \eta(\mu)$.

Case 2. If μ does not satisfies property $\mathbb{k}$, $\theta_{\mathbb{k}}(\mu,r) = \underline{1}$. Thus, we obtain $F^l(\mu) \leq I_\tau(F^l(\underline{1}),r) = I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r)$. Then $F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r) = \underline{0}$ and hence

$$\ell_0[F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\mathbb{k}}(\mu,r)),r)] \geq \eta(\mu).$$

Then F is fuzzy lower $(id, I_\tau, \theta_{\Bbbk}, id, \ell_0)$-continuous.

($\Leftarrow$) Suppose there exists $\mu \in I^Y$ such that $\tau(F^l(\mu)) \not\geq \eta(\mu)$. There exists $r \in I_\circ$ such that $\tau(F^l(\mu)) < r < \eta(\mu)$. Since $\ell_0[F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\Bbbk}(\mu,r)),r)] \geq \eta(\mu)$. Thus, $F^l(\mu) \overline{\wedge} I_\tau(F^l(\theta_{\Bbbk}(\mu,r)),r) = \underline{0}$ and $F^l(\mu) \leq I_\tau(F^l(\theta_{\Bbbk}(\mu,r)),r)$ for each $\mu \in I^Y$. If μ satisfies property $\Bbbk$ with $\eta(\mu) \geq r$, $\theta_{\Bbbk}(\mu,r) = \mu$ and hence $F^l(\mu) \leq I_\tau(F^l(\mu),r)$. Thus $\tau(F^l(\mu)) \geq r$, it is a contradiction. Then $\tau(F^l(\mu)) \geq \eta(\mu)$ and hence F is fuzzy lower $\Bbbk$-continuous.

(2) Similar to the proof in (1).

Definition 2.4 If α and β are operators on (X,τ), the intersection operator $\alpha \sqcap \beta$ is defined as follows: $(\alpha \sqcap \beta)(\lambda,r) = \alpha(\lambda,r) \wedge \beta(\lambda,r)$, $\forall\, \lambda \in I^X$. The operators α and β are said to be mutually dual if $\alpha \sqcap \beta$ is the identity operator.

Theorem 2.5 Let $F : (X,\tau) \multimap (Y,\eta)$ be a normalized fuzzy multifunction and ℓ be a fuzzy ideal on X. Let α, β be operators on (X,τ) and θ, θ^* and ∂ be operators on (Y,η). Then F is fuzzy upper $(\alpha,\beta,\theta \sqcap \theta^*,\partial,\ell)$-continuous iff it is both fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous and fuzzy upper $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

Proof. If F is both fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous and fuzzy upper $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous then for each $\mu \in I^Y$, $\ell[\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta(\mu,r)),r)] \geq \eta(\mu)$ and

$$\ell[\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta^*(\mu,r)),r)] \geq \eta(\mu).$$

Thus, we obtain

$$\ell[[\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta(\mu,r)),r)] \vee [\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta^*(\mu,r)),r)]] \geq \eta(\mu).$$

But

$$\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u((\theta \sqcap \theta^*)(\mu,r)),r)$$

$$= \alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta(\mu,r) \wedge \theta^*(\mu,r)),r)$$

$$= \alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} [\beta(F^u(\theta(\mu,r)),r) \wedge \beta(F^u(\theta^*(\mu,r)),r)]$$

$$= [\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta(\mu,r)),r)] \vee [\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u(\theta^*(\mu,r)),r)].$$

Thus, $\ell[\alpha(F^u(\partial(\mu,r)),r) \overline{\wedge} \beta(F^u((\theta \sqcap \theta^*)(\mu,r)),r)] \geq \eta(\mu)$. Then F is fuzzy upper $(\alpha,\beta,$ $\theta \sqcap \theta^*,\partial,\ell)$-continuous.

Conversely; if F is fuzzy upper $(\alpha,\beta,\theta\sqcap\theta^*,\partial,\ell)$-continuous and $\mu\in I^Y$, then

$$\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u((\theta\sqcap\theta^*)(\mu,r)),r)]\geq\eta(\mu).$$

Now, by the above equalities, we get that

$$\ell[[\alpha(F^u(\partial(\mu,r)),r)\overline{\wedge}\beta(F^u(\theta(\mu,r)),r)]\vee[\alpha(F^u(\partial(\mu,r)),r)\overline{\wedge}\beta(F^u(\theta^*(\mu,r)),r)]]\geq\eta(\mu).$$

Then $\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u(\theta(\mu,r)),r)]\geq\eta(\mu)$ and $\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u(\theta^*(\mu,r)),r)]\geq\eta(\mu)$. Hence F is both fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous and fuzzy upper $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

Theorem 2.6 Let $F:(X,\tau)\multimap(Y,\eta)$ be a fuzzy multifunction and ℓ be a fuzzy ideal on X. Let α, β be operators on (X,τ) and θ, θ^* and ∂ be operators on (Y,η). Then F is fuzzy lower $(\alpha,\beta,\theta\sqcap\theta^*,\partial,\ell)$-continuous if it is both fuzzy lower $(\alpha,\beta,\theta,\partial,\ell)$-continuous and fuzzy lower $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

Definition 2.7 Let α and β be operators on (X,τ). Then $\alpha\sqsubseteq\beta$ iff $\alpha(\lambda,r)\leq\beta(\lambda,r)$, $\forall\,\lambda\in I^X$.

Theorem 2.8 (1) Let $F:(X,\tau)\multimap(Y,\eta)$ be a fuzzy multifunction and ℓ be a fuzzy ideal on X. Let α, β be operators on (X,τ) and θ, θ^* and ∂ be operators on (Y,η) with $\theta\sqsubseteq\theta^*$. If F is fuzzy lower $(\alpha,\beta,\theta,\partial,\ell)$-continuous then it is fuzzy lower $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

(2) Let $F:(X,\tau)\multimap(Y,\eta)$ be a normalized fuzzy multifunction and ℓ be a fuzzy ideal on X. Let α, β be operators on (X,τ) and θ, θ^* and ∂ be operators on (Y,η) with $\theta\sqsubseteq\theta^*$. If F is fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous then it is fuzzy upper $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

Proof. (1) If F is fuzzy lower $(\alpha,\beta,\theta,\partial,\ell)$-continuous. Thus,

$$\ell[\alpha(F^l(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^l(\theta(\mu,r)),r)]\geq\eta(\mu).$$

Now we know that $\theta\sqsubseteq\theta^*$, then for every $\mu\in I^Y$, $\beta(F^l(\theta(\mu,r)),r)\leq\beta(F^l(\theta^*(\mu,r)),r)$. Therefore

$$\alpha(F^l(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^l(\theta^*(\mu,r)),r)\leq\alpha(F^l(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^l(\theta(\mu,r)),r).$$

Thus, $\ell[\alpha(F^l(\partial(\mu,r)),r)\overline{\wedge}\beta(F^l(\theta^*(\mu,r)),r)]\geq\ell[\alpha(F^l(\partial(\mu,r)),r)\overline{\wedge}\beta(F^l(\theta(\mu,r)),r)]\geq\eta(\mu)$. Then F is fuzzy lower $(\alpha,\beta,\theta^*,\partial,\ell)$-continuous.

(2) Similar to the proof in (1).

Definition 2.9 An operator β on (X,τ) induces another operator $I_\tau(\beta)$ defined as follows: $I_\tau(\beta)(\lambda,r)=I_\tau(\beta(\lambda,r),r)$, $\forall\,\lambda\in I^X$. Observe that $I_\tau(\beta,r)\sqsubseteq\beta$.

Theorem 2.10 Let α and β be operators on (X,τ), θ and ∂ be operators on (Y,η) and ℓ a proper ideal on X. If $F:(X,\tau)\multimap(Y,\eta)$ is fuzzy upper (resp. lower) $(\alpha,\beta,\theta,\partial,\ell)$-continuous multifunction and $\beta[F^u(\mu),r]\leq\beta[F^u(I_\eta(\mu,r)),r]$ (resp. $\beta[F^l(\mu),r]\leq\beta[F^l(I_\eta(\mu,r)),r]$) $\forall\,\mu\in I^Y$ and $r\in I_\circ$. Then F is fuzzy upper (resp. lower) $(\alpha,\beta,I_\eta(\theta),\partial,\ell)$-continuous multifunction.

Proof. If F is fuzzy upper $(\alpha,\beta,\theta,\partial,\ell)$-continuous. Thus,

$$\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u(\theta(\mu,r)),r)]\geq\eta(\mu).$$

Since $\beta[F^u(\theta(\mu,r)),r]\leq\beta[F^u(I_\eta(\theta(\mu,r),r)),r]$,

$$\alpha(F^u(\partial(\mu,r)),r)\overline{\wedge}\beta(F^u(I_\eta(\theta(\mu,r),r)),r)\leq\alpha(F^u(\partial(\mu,r))r)\overline{\wedge}\beta(F^u(\theta(\mu,r)),r).$$

Hence, $\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u(I_\eta(\theta(\mu,r),r)),r)]\geq\ell[\alpha(F^u(\partial(\mu,r)),r)\,\overline{\wedge}\,\beta(F^u(\theta(\mu,r)),r)]\geq\eta(\mu)$. Then F is fuzzy upper $(\alpha,\beta,I_\eta(\theta),\partial,\ell)$-continuous.

Definition 2.11 Let (X,τ) be a fts, $\lambda\in I^X$ and $r\in I_\circ$. Then λ is called r-fuzzy θ-compact iff for every family $\{\mu_i\in I^X\mid\tau(\mu_i)\geq r\}_{i\in\Gamma}$ such that $\lambda\leq\bigvee_{i\in\Gamma}\mu_i$, there exists a finite subset $\Gamma_\circ$ of Γ such that $\lambda\leq\bigvee_{i\in\Gamma_\circ}\theta(\mu_i,r)$.

Definition 2.12 ([1]) Let $F:X\multimap Y$ be a fuzzy multifunction between two fts's (X,τ), (Y,η) and $r\in I_\circ$. Then F is called compact-valued iff $F(x_t)$ is r-fuzzy compact for each $x_t\in dom(F)$.

Theorem 2.13 Let $F:X\multimap Y$ be a crisp fuzzy upper $(\alpha,I_\tau,\theta,\partial,\ell_0)$-continuous and compact-valued between two fts's (X,τ), (Y,η) and $\lambda\leq\alpha(\lambda,r)$, $\mu\leq\partial(\mu,r)$ $\forall\,\lambda\in I^X$, $\mu\in I^Y$. Then $F(\lambda)$ is r-fuzzy θ-compact if λ is r-fuzzy compact.

Proof. Let $\{\mu_i\in I^Y\mid\eta(\mu_i)\geq r\}_{i\in\Gamma}$ with $F(\lambda)\leq\bigvee_{i\in\Gamma}\mu_i$. Since $\lambda=\bigvee_{x_t\in\lambda}x_t$, we have

$$F(\lambda)=F(\bigvee_{x_t\in\lambda}x_t)=\bigvee_{x_t\in\lambda}F(x_t)\leq\bigvee_{i\in\Gamma}\mu_i.$$

It follows that for each $x_t\in\lambda$, $F(x_t)\leq\bigvee_{i\in\Gamma}\mu_i$. Since F is compact-valued, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t)\leq\bigvee_{n\in\Gamma_{x_t}}\mu_n=\mu_{x_t}$. Thus, we have $x_t\leq F^u(F(x_t))\leq F^u(\mu_{x_t})$ and $\lambda=\bigvee_{x_t\in\lambda}x_t\leq\bigvee_{x_t\in\lambda}F^u(\mu_{x_t})$. Since F is fuzzy upper $(\alpha,I_\tau,\theta,\partial,\ell_0)$-continuous,

$$F^u(\mu)\leq\alpha(F^u(\partial(\mu,r)),r)\leq I_\tau(F^u(\theta(\mu,r)),r)\leq F^u(\theta(\mu,r)).$$

Then $\lambda \leq \bigvee_{x_t \in \lambda} I_\tau(F^u(\theta(\mu_{x_t}, r)), r)$. Since λ is r-fuzzy compact (see [9]), there exists finite index set N of Γ_{x_t} such that $\lambda \leq \bigvee_{n\in N} I_\tau(F^u(\theta(\mu_{x_{(t_n)}}, r)), r) \leq \bigvee_{n\in N} F^u(\theta(\mu_{x_{(t_n)}}, r))$.

It follows that, $F(\lambda) \leq F(\bigvee_{n\in N} F^u(\theta(\mu_{x_{(t_n)}}, r))) = \bigvee_{n\in N} F(F^u(\theta(\mu_{x_{(t_n)}}, r))) \leq \bigvee_{n\in N} \theta(\mu_{x_{(t_n)}}, r)$. Then $F(\lambda)$ is r-fuzzy θ-compact.

Corollary 2.14 (1) Let $F : X \multimap Y$ be a crisp fuzzy upper semi-continuous and compact-valued between two fts's (X,τ), (Y,η). Then $F(\lambda)$ is r-fuzzy compact if $\lambda \in I^X$ is r-fuzzy compact.

(2) Let $F : X \multimap Y$ be a crisp fuzzy upper almost continuous and compact-valued between two fts's (X,τ), (Y,η). Then $F(\lambda)$ is r-fuzzy nearly compact if $\lambda \in I^X$ is r-fuzzy compact.

(3) Let $F : X \multimap Y$ be a crisp fuzzy upper weakly continuous and compact-valued between two fts's (X,τ), (Y,η). Then $F(\lambda)$ is r-fuzzy almost compact if $\lambda \in I^X$ is r-fuzzy compact.

Proof. (1) Let α = identity operator, β = interior operator, θ = identity operator, ∂ = identity operator, $\ell = \ell_0$. Then the result follows from Theorem 2.13.

(2) Let α = identity operator, β = interior operator, θ = interior closure operator, ∂ = identity operator and $\ell = \ell_0$. Then the result follows from Theorem 2.13.

(3) Let α = identity operator, β = interior operator, θ = closure operator, ∂ = identity operator and $\ell = \ell_0$. Then the result follows from Theorem 2.13.

3. A Unified Theory of Generalized Forms of Fuzzy Continuous Multifunctions

In order to unify several characterizations of some kind of forms of fuzzy continuity, we introduced and study generalized form of fuzzy continuous multifunctions namely fuzzy upper (lower) $\kappa\kappa'$-continuous multifunctions. These multifunctions enable us to reduce many generalized forms of continuity to a single theoretical unified framework. Let X, Y be nonempty sets and $\kappa : I^X \to I$, $\kappa' : I^Y \to I$ be any maps on X and Y, respectively.

Definition 3.1 Let $F : X \multimap Y$ be a fuzzy multifunction (resp. normalized fuzzy multifunction). Then F is said to be fuzzy lower $\kappa\kappa'$-continuous (resp. fuzzy upper $\kappa\kappa'$-continuous) iff $\kappa(F^l(\mu)) \geq \kappa'(\mu)$ (resp. $\kappa(F^u(\mu)) \geq \kappa'(\mu)$) for each $\mu \in I^Y$.

Remark 3.2 1. Observe that if in Definition 3.1, κ and κ' are fuzzy topology on X and Y, respectively, we just obtain the notion of fuzzy lower semi-continuous (resp. fuzzy upper semi-continuous) multifunction introduced in [1].

2. The concept of fuzzy r-minimal structure was introduced by Yoo et al. [20] which is an extension of fuzzy topology introduced by Šostak [14], as a fuzzy mapping $M : I^X \longrightarrow I$ on X is said to be fuzzy r-minimal structure if the family $M_r = \{\lambda \in I^X \mid M(\lambda) \geq r\}$ contains $\underline{0}$ and $\underline{1}$. Now if in Definition 3.1, $\kappa = M_r$ and $\kappa' = M'_r$ are fuzzy r-minimal structures on X and Y, respectively, we just obtain the notion of fuzzy lower $M_rM'_r$-continuous (resp. fuzzy upper $M_rM'_r$-continuous) multifunction.

Let $\theta_\kappa : I^X \times I_\circ \to I^X$ be an operator on X defined as follows:

$$\theta_\kappa(\lambda, r) = \begin{cases} \lambda, & \text{if} \quad \kappa(\lambda) \geq r, \\ \underline{1}, & \text{otherwise} \end{cases} \quad \text{and } r \in I_\circ.$$

In the case that κ is a generalized fuzzy topology (or supra fuzzy topology), we obtain other operators $I_\kappa(\lambda, r)$ and $C_\kappa(\lambda, r)$ of λ, respectively, as follows:

$$I_\kappa(\lambda, r) = \bigvee\{\mu \in I^X : \ \mu \leq \lambda, \kappa(\mu) \geq r\}.$$

$$C_\kappa(\lambda, r) = \bigwedge\{\mu \in I^X : \ \lambda \leq \mu, \kappa(\mu^c) \geq r\}.$$

Similarly, in the case of a fuzzy r-minimal structure M_r on X (see[20]). The fuzzy r-minimal interior and fuzzy r-minimal closure of λ, denoted by $I_m(\lambda, r)$ and $C_m(\lambda, r)$, respectively, are defined as $I_m(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \ \mu \in M_r\}$; $C_m(\lambda, r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu, \ \mu^c \in M_r\}$.

Remark 3.3 Observe that if M_r has the Yoo property (see[20]), then the above operators, respectively, agree. Also, each fuzzy r-minimal structure with the Yoo property is a generalized fuzzy topology.

The following results give the relationship between fuzzy lower $\kappa\kappa'$-continuity (resp. fuzzy upper $\kappa\kappa'$-continuity) and fuzzy lower $(\alpha, \beta, \theta, \partial, \ell)$-continuity (resp. fuzzy upper $(\alpha, \beta, \theta, \partial, \ell)$-continuity). We obtain some interesting properties of fuzzy lower $\kappa\kappa'$-continuous (resp. fuzzy upper $\kappa\kappa'$-continuous) multifunction.

Theorem 3.4 Let $F : X \multimap Y$ be a fuzzy multifunction and $\kappa : I^X \to I$, $\kappa' : I^Y \to I$ be any maps on X and Y, respectively. If $\kappa(\underline{1}) = 1$, then F is fuzzy lower $\kappa\kappa'$-continuous iff F is fuzzy lower $(\theta_\kappa, id, \theta_{\kappa'}, id, \ell_0)$-continuous.

Proof. $(\Rightarrow)$ Suppose that F is fuzzy lower $\kappa\kappa'$-continuous and $\mu \in I^Y$.

Case 1. If $\kappa'(\mu) \geq r$, $\theta_{\kappa'}(\mu,r) = \mu$ and $\kappa(F^l(\mu)) \geq r$. Thus, we obtain $F^l(\theta_{\kappa'}(\mu,r)) = F^l(\mu) = \theta_\kappa(F^l(\mu),r)$. Then $\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r)) = \underline{0}$. Hence

$$\ell_0[\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r))] \geq \kappa'(\mu).$$

Case 2. If $\kappa'(\mu) \not\geq r$, $\theta_{\kappa'}(\mu,r) = \underline{1}$. Thus, we obtain $F^l(\theta_{\kappa'}(\mu,r)) = F^l(\underline{1}) \geq \theta_\kappa(F^l(\mu),r)$. Then $\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r)) = \underline{0}$ and hence $\ell_0[\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r))] \geq \kappa'(\mu)$.

Then F is fuzzy lower $(\theta_\kappa, id, \theta_{\kappa'}, id, \ell_0)$-continuous.

$(\Leftarrow)$ Suppose there exist $\mu \in I^Y$ such that $\kappa(F^l(\mu)) \not\geq \kappa'(\mu)$. There exist $r \in I_\circ$ such that $\kappa(F^l(\mu)) < r < \kappa'(\mu)$. Since $\ell_0[\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r))] \geq \kappa'(\mu)$. Thus, we obtain $\theta_\kappa(F^l(\mu),r) \,\overline{\wedge}\, F^l(\theta_{\kappa'}(\mu,r)) = \underline{0}$ and $\theta_\kappa(F^l(\mu),r) \leq F^l(\theta_{\kappa'}(\mu,r))$ for each $\mu \in I^Y$. If $\kappa'(\mu) \geq r$, $\theta_{\kappa'}(\mu,r) = \mu$ and $\theta_\kappa(F^l(\mu),r) \leq F^l(\mu)$. Thus, we obtain $\theta_\kappa(F^l(\mu),r) = F^l(\mu)$ and hence $\kappa(F^l(\mu)) \geq r$, it is a contradiction. Then $\kappa(F^l(\mu)) \geq \kappa'(\mu)$ and hence F is fuzzy lower $\kappa\kappa'$-continuous.

Theorem 3.5 Let $F : X \multimap Y$ be a normalized fuzzy multifunction and $\kappa : I^X \to I$, $\kappa' : I^Y \to I$ be any maps on X and Y, respectively. If $\kappa(\underline{1}) = 1$, then F is fuzzy upper $\kappa\kappa'$-continuous iff F is fuzzy upper $(\theta_\kappa, id, \theta_{\kappa'}, id, \ell_0)$-continuous.

In the case that κ is a generalized fuzzy topology, the following result is obtained.

Theorem 3.6 Let $F : X \multimap Y$ be a fuzzy multifunction, κ be a generalized fuzzy topology on X and $\kappa' : I^Y \to I$ be any map on Y. Then F is fuzzy lower $\kappa\kappa'$-continuous iff F is fuzzy lower $(id, I_\kappa, \theta_{\kappa'}, id, \ell_0)$-continuous.

Proof. $(\Rightarrow)$ Suppose that F is fuzzy lower $\kappa\kappa'$-continuous and $\mu \in I^Y$.

Case 1. If $\kappa'(\mu) \geq r$, $\theta_{\kappa'}(\mu,r) = \mu$ and $\kappa(F^l(\mu)) \geq r$. Thus, we obtain $F^l(\mu) \leq I_\kappa(F^l(\mu),r) = I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)$. Then $F^l(\mu) \,\overline{\wedge}\, I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r) = \underline{0}$. Hence

$$\ell_0[F^l(\mu) \,\overline{\wedge}\, I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)] \geq \kappa'(\mu).$$

Case 2. If $\kappa'(\mu) \not\geq r$, $\theta_{\kappa'}(\mu,r) = \underline{1}$. Thus, we obtain $F^l(\mu) \leq I_\kappa(F^l(\underline{1}),r) = I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)$. Then $F^l(\mu) \,\overline{\wedge}\, I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r) = \underline{0}$. and hence $\ell_0[F^l(\mu) \,\overline{\wedge}\, I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)] \geq \kappa'(\mu)$.

Then F is fuzzy lower $(id, I_\kappa, \theta_{\kappa'}, id, \ell_0)$-continuous.

($\Leftarrow$) Suppose there exist $\mu \in I^Y$ such that $\kappa(F^l(\mu)) \ngeq \kappa'(\mu)$. There exist $r \in I_\circ$ such that $\kappa(F^l(\mu)) < r < \kappa'(\mu)$. Since $\ell_0[F^l(\mu) \overline{\wedge} I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)] \geq \kappa'(\mu)$. Thus, we obtain $F^l(\mu) \overline{\wedge} I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r) = \underline{0}$ and $F^l(\mu) \leq I_\kappa(F^l(\theta_{\kappa'}(\mu,r)),r)$ for each $\mu \in I^Y$. If $\kappa'(\mu) \geq r$, $\theta_{\kappa'}(\mu,r) = \mu$ and hence $F^l(\mu) \leq I_\kappa(F^l(\mu),r)$. Thus $\kappa(F^l(\mu)) \geq r$, it is a contradiction. Then $\kappa(F^l(\mu)) \geq \kappa'(\mu)$ and hence F is fuzzy lower $\kappa\kappa'$-continuous.

Theorem 3.7 Let $F : X \multimap Y$ be a normalized fuzzy multifunction, κ be a generalized fuzzy topology on X and $\kappa' : I^Y \to I$ be any map on Y. Then F is fuzzy upper $\kappa\kappa'$-continuous iff F is fuzzy upper $(id, I_\kappa, \theta_{\kappa'}, id, \ell_0)$-continuous.

Corollary 3.8 Let $F : X \multimap Y$ be a fuzzy multifunction (resp. normalized fuzzy multifunction), κ be a generalized fuzzy topology on X and $\kappa' : I^Y \to I$ be any map on Y. If F is fuzzy lower $\kappa\kappa'$-continuous (resp. fuzzy upper $\kappa\kappa'$-continuous), then F is fuzzy lower $(id, I_\kappa, \theta_{\kappa'}, id, \ell)$-continuous (resp. fuzzy upper $(id, I_\kappa, \theta_{\kappa'}, id, \ell)$-continuous).

Corollary 3.9 Let $F : X \multimap Y$ be a fuzzy multifunction (resp. normalized fuzzy multifunction), κ be a generalized fuzzy topology on X and $\kappa' : I^Y \to I$ be any map on Y. If F is fuzzy lower $\kappa\kappa'$-continuous (resp. fuzzy upper $\kappa\kappa'$-continuous), then F is fuzzy lower $(id, id, \theta_{\kappa'}, id, \ell)$-continuous (resp. fuzzy upper $(id, id, \theta_{\kappa'}, id, \ell)$-continuous).

4. CONCLUSION

In our theoretical work, we introduce the concept of fuzzy upper (lower) $(\alpha, \beta, \theta, \partial, \ell)$-continuous multifunctions and prove that if α, β are fuzzy operators on the fuzzy topological space (X, τ) based on the sense of Šostak and $\theta, \theta^*, \partial$ are fuzzy operators on the fuzzy topological space (Y, η) based on the sense of Šostak and ℓ is a proper fuzzy ideal on X, then a fuzzy multifunction $F : X \multimap Y$ is fuzzy upper (resp. lower) $(\alpha, \beta, \theta \sqcap \theta^*, \partial, \ell)$-continuous multifunction iff it is both fuzzy upper (resp. lower) $(\alpha, \beta, \theta, \partial, \ell)$-continuous and fuzzy upper (resp. lower) $(\alpha, \beta, \theta^*, \partial, \ell)$-continuous multifunctions. Also, we introduce new generalized notions that cover many of the generalized forms of fuzzy upper (resp. lower) semi-continuous multifunctions.

ACKNOWLEDGEMENTS

The authors would like to thank the referees for their valuable comments and suggestions which have improved this paper.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] S.E. Abbas, M.A. Hebeshi, I.M. Taha, On fuzzy upper and lower semi-continuous multifunctions, J. Fuzzy Math. 22(4) (2014), 951-962.

[2] S.E. Abbas, M.A. Hebeshi, I.M. Taha, On upper and lower almost weakly continuous fuzzy multifunctions, Iran. J. Fuzzy Syst. 12(1) (2015), 101-114.

[3] S.E. Abbas, M.A. Hebeshi, I.M. Taha, On fuzzy upper and lower β-irresolute multifunctions, J. Fuzzy Math. 23(1) (2015), 171-187.

[4] J. Albrycht, M. Maltoka, On fuzzy multivalued functions, Fuzzy Sets Syst. 12 (1984), 61-69.

[5] I. Beg, Continuity of fuzzy multifunctions, J. Appl. Math. Stoch. Anal. 12(1) (1999), 17-22.

[6] M.A. Hebeshi, I.M. Taha, On double fuzzy topological spaces, J. Fuzzy Math. 22(2) (2014), 419-434.

[7] M.A. Hebeshi, I.M. Taha, On upper and lower α-continuous fuzzy multifunctions, J. Intell. Fuzzy Syst. 25 (2015), 2537-2546.

[8] M.A. Hebeshi, I.M. Taha, On fuzzy upper and lower almost β-continuous multifunctions, J. Fuzzy Math. 24(1) (2016) 15-29.

[9] Y.C. Kim, S.E. Abbas, On several types of R-fuzzy compactness, J. Fuzzy Math. 12(4) (2004), 827-844.

[10] Y.C. Kim, A.A. Ramadan, S.E. Abbas, Weaker forms of continuity in Šostak's fuzzy topology, India. J. Pure Appl. Math. 34(2) (2003), 311-333.

[11] H.X. Li, V.C. Yen, Fuzzy sets and fuzzy decision making, CRC Press, London (1995).

[12] N.S. Papageorgiou, Fuzzy topology and fuzzy multifunctions, J. Math. Anal. Appl. 109 (1985), 397-425.

[13] A.A. Ramadan, Y.C. Kim, M.K. El-Gayyar, Smooth ideals, J. Korean Inst. Intell. Syst. 12(1) (2002), 90-95.

[14] A.P. Šostak, On a fuzzy topological structure, Rend. Circ. Mat. Palermo (2). 11 (1985), 89-103.

[15] I.M. Taha, On r-fuzzy ℓ-open sets and continuity of fuzzy multifunctions via fuzzy ideals, J. Math. Comput. Sci. 10(6) (2020), 2613-2633.

[16] I.M. Taha, On fuzzy upper and lower α-ℓ-continuity and their decomposition, J. Math. Comput. Sci. 11(1) (2021), 427-441.

[17] I.M. Taha, r-fuzzy δ-ℓ-open sets and fuzzy upper (lower) δ-ℓ-continuity via fuzzy idealization, J. Math. Computer Sci. 25 (2022), 1-9.

[18] E. Tsiporkova, B. De Baets, E. Kerre, A fuzzy inclusion based approach to upper inverse images under fuzzy multivalued mappings, Fuzzy Sets Syst. 85 (1997), 93-108.

[19] E. Tsiporkova, B. De Baets, E. Kerre, Continuity of fuzzy multivalued mappings, Fuzzy Sets Syst. 94 (1998), 335-348.

[20] Y.H. Yoo, W.K. Min, J.I. Kim, Fuzzy r-minimal structures and fuzzy r-minimal spaces, Far East J. Math. Sci. 33(2) (2009), 193-205.

[21] L. A. Zadeh, Fuzzy sets, Inform. Control. 8 (1965), 338-353.

[22] H.J. Zimmermann, Fuzzy set theory and its applications, Kluwer Academic Publishers, Boston (1991).