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CONTINUITY ON APPROXIMATION SPACES

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Abstract. The present paper investigates some of the properties of the induced topology on generalized approximation spaces. The properties of the induced topology are characterized in terms of the type of the binary relation used. Also, the conditions upon which the induced topology will be an indiscrete or a discrete one are derived. Besides, the separation axioms on the induced topological space are studied. Moreover, characterization theorems for continuity of a function and homeomorphism are explored.

Keywords: approximation space; continuity; homeomorphism; rough set; topology.

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1. Introduction

The theory of rough sets was launched by Zdzislaw Pawlak in the early 1980's [13] in order to provide a mathematical mechanism for tackling the uncertainty present in data occuring due to the incompleteness in the available information. This is achieved by the construction of approximations of concepts when the information at hand is incomplete. That is, a vague concept is expressed in terms of two precise concepts called the lower and upper approximations.

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The mathematical framework of this theory has been enhanced further by many researchers in various directions.

Rough set theory is apparently connected with the theory of topology and their interrelations provide much scope for research [2, 6, 8, 10, 12, 17]. In his pioneering paper, Z. Pawlak [13] mentioned that the family of equivalence classes on an approximation space induce a topology on the set under consideration and the collection of equivalence classes of the rough equality relation produce a topology on the power set of it. Both of them were quasi discrete topologies. A study of the basic properties of the induced topology of generalized rough sets was made by M. Kondo [7]. A comparison of the topologies induced by a reflexive relation and the transmissing expression of it was carried out by Z. Li [9]. J. Mahanta and P. K. Das [11] proposed the idea of transmissing neighbourhood and investigated the properties of Yao's Rough Set in the topological point of view based on a reflexive relation. Z. Pei et al. [15] explored the relations among different topologies determined by a pre-order, a reflexive relation and an inverse serial relation. Q. Qiao [16] described the topological structure of rough sets based on reflexive and transitive relation. The topology defined by a reflexive relation and a quasi order are proved to be the same as the those respectively defined by its transitive closure and equivalence closure by H. Yu and W. R. Zhan [20]. K. Anitha [2] made a generalization of rough sets in the context of topological spaces and also studied the quasi discrete topology and π_0-rough sets.

In this paper, we give a comprehensive exploration of the induced topology on generalized approximation spaces and try to bridge the gaps in the existing studies. We obtain some characterization theorems on the interconnections between the induced topology on (X,θ) and the binary relation θ. The conditions for the induced topology to be discrete or indiscrete are also presented. Then, the concepts of separation axioms, continuity and homeomorphism on generalized approximation spaces are discussed. The paper is organized as follows: section 2 provides some basic concepts of rough set theory and topology, section 3 and 4 respectively discuss the induced topology and continuity on generalized approximation spaces and section 5 concludes the paper.

2. PRELIMINARIES

Definition 2.1. [3] A binary relation θ on X is called (i) *connected (serial)* if $\forall u \in X$, $\exists\, v \in X$ such that $(u,v) \in \theta$, (ii) *reflexive* if it satisfies $(u,u) \in \theta, \forall u \in X$, (iii) *symmetric* if for all $u,v \in X, (u,v) \in \theta \Rightarrow (v,u) \in \theta$, (iv) *transitive* if $(u,v) \in \theta, (v,z) \in \theta \Rightarrow (u,z) \in \theta$ for all $u,v,z \in X$, (v) *quasiorder* if it is reflexive and transitive and (vi) *equivalence* if it is reflexive, symmetric and transitive.

Definition 2.2. [5] The topological space (X,τ) is said to be *regular* at the point $u \in X$, if for a closed set G not containing u, there exist two disjoint open sets H and K such that $u \in H$ and $G \subseteq K$. (X,τ) is called *a regular topological space* if it is regular at every point in X.

Definition 2.3. [5] Let (X,τ) and (X',τ') be topological spaces. A function $f : X \to X'$ is called *continuous* with respect to the topologies τ and τ' if the inverse image of every open set in τ' is open in τ.

Theorem 2.4. [5] *For the function $f : X \to X'$, the following conditions are equivalent.*

(i) f is continuous
(ii) There exists a sub base $S \subseteq \mathscr{P}(X')$ for τ' such that $f^{-1}(V) \in \tau$, $\forall V \in S$
(iii) For every closed subset $A' \subseteq X'$, $f^{-1}(A')$ is a closed subset of X
(iv) For every $A \subseteq X$, $f(cl(A)) \subseteq cl(f(A))$

Definition 2.5. [5] The function $f : X \to X'$ is labeled as a *homeomorphism* if both f and f^{-1} are continuous with respect to τ and τ' and f is a bijective function.

Definition 2.6. [19] The pair (X,θ) is called a *generalized approximation space* if θ is an arbitrary binary relation on X. The *lower approximation* and *upper approximations* of $A \subseteq X$ with respect to θ are defined as

$$\underline{\theta}(A) = \{u \in X : \theta(u) \subseteq A\}$$
$$\overline{\theta}(A) = \{u \in X : \theta(u) \cap A \neq \phi\},$$

respectively, where $\theta(u) = \{v \in X : (u,v) \in \theta\}$.

Proposition 2.7. [3, 19] *Consider a generalized approximation space (X, θ) and A and B be subsets of X. Then,*

(i) $\overline{\theta}(\phi) = \phi$

(ii) $\underline{\theta}(X) = X$

(iii) $\underline{\theta}(A \cap B) = \underline{\theta}(A) \cap \underline{\theta}(B)$

(iv) $\overline{\theta}(A \cap B) \subseteq \overline{\theta}(A) \cap \overline{\theta}(B)$

(v) $\underline{\theta}(A \cup B) \supseteq \underline{\theta}(A) \cup \underline{\theta}(B)$

(vi) $\overline{\theta}(A \cup B) = \overline{\theta}(A) \cup \overline{\theta}(B)$

(vii) $\underline{\theta}(A) = [\overline{\theta}(A^C)]^C$ *and* $\overline{\theta}(A) = [\underline{\theta}(A^C)]^C$

(viii) $A \subseteq B \Rightarrow \underline{\theta}(A) \subseteq \underline{\theta}(B)$ *and* $\overline{\theta}(A) \subseteq \overline{\theta}(B)$.

(ix) $\underline{\theta}(A) \subseteq \overline{\theta}(A)$ *if* θ *is connected.*

(x) $\underline{\theta}(A) \subseteq A$ *and* $A \subseteq \overline{\theta}(A)$*, iff* θ *is reflexive.*

(xi) $\underline{\theta}(A) \subseteq \underline{\theta}(\underline{\theta}(A)$ *iff* θ *is transitive.*

(xii) $\overline{\theta}(\overline{\theta}(A)) \subseteq \overline{\theta}(A)$ *iff* θ *is transitive.*

3. TOPOLOGY ON GENERALIZED APPROXIMATION SPACES

Let (X, θ) be a generalized approximation space. In [7], it is shown that if θ is reflexive, then

$$\tau_\theta = \{A \subseteq X : \underline{\theta}(A) = A\} \tag{1}$$

is a topology on X, which is called the induced topology on (X, θ). But, the following example illustrates that the converse need not be true.

Example 3.1. Let $X = \{p, q, r, s\}$ and $\theta = \{(p,q), (q,r), (r,p), (s,s)\}$. Then, $\theta(p) = \{q\}$, $\theta(q) = \{r\}$, $\theta(r) = \{p\}$ and $\theta(s) = \{s\}$. From equation (1), $\tau_\theta = \{\emptyset, X, \{p,q,r\}, \{s\}\}$. Obviously, τ_θ defines a topology on X but θ is not a reflexive relation.

It is also clear from example 3.1 that τ_θ can form a topology even if θ is neither a symmetric nor a transitive relation. The following theorem presents a necessary and sufficient condition for τ_θ to form a topology on X.

Theorem 3.2. τ_θ *is a topology on* X *if and only if* θ *is connected.*

Proof. Suppose that τ_θ forms a topology on X. Clearly, $\emptyset \in \tau_\theta$. From equation (1), $\underline{\theta}(\emptyset) = \emptyset$.
$\therefore \theta(u) = \emptyset \Rightarrow \theta(u) \subseteq \emptyset \Rightarrow u \in \underline{\theta}(\emptyset) \Rightarrow u \in \emptyset$. This is a contradiction. Hence, $\theta(u) \neq \emptyset, \forall u \in X$.
Therefore, θ is connected.

Now, if θ is a connected relation, then $\underline{\theta}(\emptyset) = \emptyset$.
Again, $\underline{\theta}(X) = X$, $\underline{\theta}(A \cap B) = \underline{\theta}(A) \cap \underline{\theta}(B)$ and $\underline{\theta}(\cup A_i) \supseteq \cup_i \underline{\theta}(A_i)$.
$\therefore A, B \in \tau_\theta \Rightarrow \underline{\theta}(A) = A$, $\underline{\theta}(B) = B \Rightarrow \underline{\theta}(A \cap B) = \underline{\theta}(A) \cap \underline{\theta}(B) = A \cap B. \Rightarrow A \cap B \in \tau_\theta$
For $i \in \Lambda$, $A_i \in \tau_\theta \Rightarrow \underline{\theta}(A_i) = A_i, \forall i \in \Lambda \Rightarrow \underline{\theta}(\cup A_i) \supseteq \cup \underline{\theta}(A_i) = \cup A_i$.
Hence, $\underline{\theta}(\cup A_i) = \cup A_i$, as $\underline{\theta}(\cup A_i) \subseteq \cup_i A_i$ So, $\cup A_i \in \tau_\theta$. Thus, τ_θ is a topology on X. □

From proposition 2.7, the property

$$\underline{\theta}(A) \subseteq A \subseteq \overline{\theta}(A) \tag{2}$$

is satisfied if and only if θ is reflexive. So, it is obvious that the approximations of a set will be a meaningful concept only if the binary relation θ on X is reflexive. Hence, only reflexive approximation spaces are considered in the following discussion.

Lemma 3.3. *On a reflexive approximation space (X, θ), $\theta(u)$ is an open set in $\tau_\theta, \forall u \in X$ if and only if θ is transitive.*

Theorem 3.4. *On a reflexive approximation space (X, θ), $\underline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\}$, $\forall A \subseteq X$ if and only if θ is transitive.*

Proof. Assume that $\underline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\}$, $\forall A \subseteq X$. Then, $\forall u \in X$,

$$\underline{\theta}\,[\theta(u)] = \cup\{\theta(a) : a \in X, \theta(a) \subseteq \theta(u)\} \tag{3}$$

Again, $\theta(u) \in \{\theta(a) : a \in X, \theta(a) \subseteq \theta(u)\}$ because $\theta(u) \subseteq \theta(u)$.
Hence, $\theta(u) \subseteq \underline{\theta}\,[\theta(u)]$, from equation (3).
Using equation (2), $\underline{\theta}\,[\theta(u)] \subseteq \theta(u)$, since θ is reflexive.
Thus $\underline{\theta}\,[\theta(u]) = \theta(u), \forall u \in X$. It follows from lemma 3.3 that θ is transitive.

Conversely, let θ be transitive and $A \subseteq X$. Then, $u \in \underline{\theta}(A) \Rightarrow \theta(u) \subseteq A$

$$\Rightarrow \theta(u) \in \{\theta(a) : a \in X, \theta(a) \subseteq A\} \Rightarrow u \in \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\}.$$

Therefore, $\underline{\theta}(A) \subseteq \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\}$.

Now, $u \in \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\} \Rightarrow u \in \theta(v)$ for some $v \in X$ with $\theta(v) \subseteq A$

$\Rightarrow \theta(u) \subseteq A$, as $\theta(u) \subseteq \theta(v)$, since θ is transitive $\Rightarrow u \in \underline{\theta}(A)$.

Hence, $\cup\{\theta(a) : a \in X, \theta(a) \subseteq A\} \subseteq \underline{\theta}(A)$.

Thus, $\underline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \subseteq A\}, \forall A \subseteq X$. □

Theorem 3.5. *On a reflexive approximation space (X, θ), the family $\mathscr{B} = \{\theta(a) : a \in X\}$ is an open base for τ_θ on X if and only if θ is transitive.*

Proof. A family $\mathscr{B}$ form an open base for τ_θ if and only if it is possible to express every open set in τ_θ as the union of some members of $\mathscr{B}$ and $\mathscr{B} \subseteq \tau_\theta$. Here,

$$\mathscr{B} = \{\theta(a) : a \in X\}. \tag{4}$$

In case $\mathscr{B}$ form a base for τ_θ, then $\mathscr{B} \subseteq \tau_\theta$. Hence, $\theta(a)$ is open for all $a \in X$.

Using lemma 3.3, θ is transitive.

Also, if θ is a transitive relation, using lemma 3.3, $\mathscr{B} \subseteq \tau_\theta$.

From equation (1), a subset $A \subseteq X$ is an open set if and only if $\underline{\theta}(A) = A$.

It follows from theorem 3.4 that every open set can be written as the union of members of $\mathscr{B}$.

Therefore, $\mathscr{B}$ form an open base for τ_θ. □

Theorem 3.6. *On a reflexive approximation space, $\overline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \cap A \neq \emptyset\}$, $\forall A \subseteq X$ if and only if θ is an equivalence relation.*

Proof. If θ is an equivalence, then $\overline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \cap A \neq \emptyset\}$, $\forall A \subseteq X$ [14].

So, consider that

$$\overline{\theta}(A) = \cup\{\theta(a) : a \in X, \theta(a) \cap A \neq \emptyset\},\ \forall A \subseteq X \tag{5}$$

Let $(u, v) \in \theta$. So, $v \in \theta(u)$.

Then, $(v, u) \notin \theta \Rightarrow u \notin \theta(v) \Rightarrow u \in [\theta(v)]^C \Rightarrow \theta(u) \cap [\theta(v)]^C \neq \emptyset$

$\Rightarrow \theta(u) \in \{\theta(a) : a \in X, \theta(a) \cap [\theta(v)\,]^{\,C}) \neq \emptyset\} \Rightarrow \theta(u) \subseteq \overline{\theta}\ (\ [\theta(v)\,]^{\,C})$, from equation 5

$\Rightarrow v \in \overline{\theta}\ (\ [\theta(v)\,]^{\,C})$, since $v \in \theta(u) \Rightarrow \theta(v) \cap [\theta(v)]^C \neq \emptyset$.

This is a contradiction. Therefore, θ is symmetric.

Now, using lemma 3.3, it is enough to prove that $\theta(u)$ is open in τ_θ, for all $u \in X$.

Consider $v \in \theta(u)$. Then, $u \in \theta(v)$, as θ is symmetric.

So, $\theta(v) \nsubseteq \theta(u) \Rightarrow \theta(v) \cap [\theta(u)]^C \neq \emptyset \Rightarrow \theta(v) \subseteq \overline{\theta}\ (\ [\theta(u)\ ]^C)$, from equation 5

$\Rightarrow u \in \overline{\theta}(\ [\theta(u)\ ]^C)$, as $u \in \theta(v) \Rightarrow \theta(u) \cap [\theta(u)]^C \neq \emptyset$, which is a contradiction.

Therefore, $\theta(v) \subseteq \theta(u)$. So, $v \in \underline{\theta}\ [\theta(u)]$. Hence, $\theta(u) \subseteq \underline{\theta}\ [\theta(u)]$.

Thus, $\underline{\theta}\ [\theta(u)] = \theta(u), \forall u \in X$.

Hence, θ is a transitive relation and so, it is an equivalence. □

Theorem 3.7. *On a reflexive approximation space (X, θ), $\theta(u)$ is a closed set in τ_θ, $\forall u \in X$ if and only if θ is an equivalence.*

Proof. Assume that θ is an equivalence. Then all open set in τ_θ are closed and vice versa. Since θ is transitive, for all $u \in X$, $\theta(u)$ is open in τ_θ using lemma 3.3. Hence, for all $u \in X$, $\theta(u)$ is closed for all $u \in X$.

Conversely, consider that for all $u \in X$, $\theta(u)$ is a closed set.

Consider $u, v \in X$. Then, $(u, v) \in \theta \Rightarrow v \in \theta(u) \Rightarrow \theta(u) \cap \theta(v) \neq \emptyset$, because $v \in \theta(v)$.

So, $u \in \overline{\theta}\ [\theta(v)]$. As, $\theta(v)$ is a closed set, $\overline{\theta}\ [\theta(v)] = \theta(v)$.

Therefore, $u \in \theta(v)$ and so $(v, u) \in \theta$. Hence, θ is symmetric.

If $u, v, z \in X$, then $(u, v) \in \theta$ and $(v, z) \in \theta \Rightarrow v \in \theta(u)$ and $z \in \theta(v)$

$\Rightarrow v \in \theta(u)$ and $v \in \theta(z)$, as θ is symmetric $\Rightarrow \theta(u) \cap \theta(z) \neq \emptyset$

$\Rightarrow z \in \overline{\theta}\ [\theta(u)] \Rightarrow z \in \theta(u)$, because $\theta(u)$ is closed.

So, θ is transitive. Hence θ is an equivalence. □

Lemma 3.8. *If θ_1 and θ_2 are reflexive relations on X, then, the following conditions are equivalent.*

(i) $\theta_1 \subseteq \theta_2$

(ii) $\overline{\theta_1}(A) \subseteq \overline{\theta_2}(A)$, $\forall A \subseteq X$

(iii) $\underline{\theta_1}(A) \supseteq \underline{\theta_2}(A)$, $\forall A \subseteq X$

Theorem 3.9. *If θ_1 and θ_2 are reflexive relations on X, then $\theta_1 \subseteq \theta_2 \Rightarrow \tau_{\theta_2} \subseteq \tau_{\theta_1}$.*

Proof. Let $\theta_1 \subseteq \theta_2$, Then $A \in \tau_{\theta_2} \Leftrightarrow \underline{\theta_2}(A) = A \Leftrightarrow \underline{\theta_1}(A) \supseteq A$, from lemma 3.8

$\Leftrightarrow \underline{\theta_1}(A) = A$, from equation 2 $\Leftrightarrow A \in \tau_{\theta_1}$.

Thus, $\tau_{\theta_2} \subseteq \tau_{\theta_1}$. □

Corollary 3.10. *If $\theta_1, \theta_2, ..., \theta_n$ are reflexive relations on the set X and $\theta = \theta_1 \cap \theta_2 \cap ... \cap \theta_n$, then $\tau_{\theta_i} \subseteq \tau_\theta$, for $i = 1, 2, ..., n$.*

It is obvious that the finest possible reflexive relation on the set X is $\theta_D = \{(u,u) : u \in X\}$. Also, the strongest topology on X is the discrete topology on it. At this point is it is interesting to check whether any reflexive relation can induce the discrete topology on X and if yes, whether it can be induced by θ_D. The following theorem presents an answer to this question.

Theorem 3.11. *For a reflexive approximation space (X, θ), τ_θ is the discrete topology on X if and only if $\theta = \theta_D = \{(u,u) : u \in X\}$.*

Proof. τ_θ is the discrete topology on $X \Leftrightarrow A \in \tau_\theta$, for all $A \subseteq X \Leftrightarrow \{u\} \in \tau_\theta$, $\forall u \in X$

$$\Leftrightarrow \underline{\theta}(\{u\}) = \{u\}, \forall u \in X \Leftrightarrow \theta(u) = \{u\}, \forall u \in X \Leftrightarrow \theta = \{(u,u) : u \in X\}$$ □

In a similar way, in the following theorem, the weakest topology, namely, the indiscrete topology is proved to be induced by $\theta_I = X \times X$.

Theorem 3.12. *τ_{θ_I} is the indiscrete topology on X, where $\theta_I = X \times X$.*

Proof. For all $u \in X$, $\theta_I(u) = X$ as $\theta_I = X \times X$. From equation 1, it follows that τ_{θ_I} consists of $\emptyset$ and X only. Therefore, τ_{θ_I} coincides with the indiscrete topology on X. □

Unlike the case with the discrete topology, the indiscrete topology can also be obtained by other reflexive relations also, as can be seen from the following example.

Example 3.13. Take $X = \{p,q,r\}$ and let $\theta = \{(p,p),(p,q),(q,q),(q,r),(r,r),(r,p)\}$. Then, $\theta(a) = \{p,q\}$, $\theta(q) = \{q,r\}$ and $\theta(r) = \{p,r\}$. Hence, $\theta \neq X \times X$.
From equation (1), $\tau_\theta = \{\emptyset, X\}$, the indiscrete topology on X.

Theorem 3.14. *If θ is a quasi order on X, then τ_θ coincides with the indiscrete topology on X if and only if $\theta = X \times X$.*

Proof. From lemma 3.3, $\theta(u) \in \tau_\theta, \forall u \in X$ because θ is a quasi order.
Hence, τ_θ is the indiscrete topology on $X \Leftrightarrow$ the only open sets in τ_θ are $\emptyset$ and X

$$\Leftrightarrow \theta(u) = X, \forall u \in X \Leftrightarrow \theta = X \times X$$ □

Theorem 3.15. *On a reflexive approximation space (X,θ), τ_θ satisfies T_0 separation axiom if and only if $\theta = \{(u,u) : u \in X\}$.*

Proof. Suppose that (X,τ_θ) is T_0. Let $(u,v) \in X$ and $u \neq v$.
As (X,τ_θ) is T_0, there is an open set containing u but not v or an open set containing v but not u. Without any loss of generality, assume that A be an open set such that $u \in A$ and $v \notin A$.
So,$\theta(u) \subseteq A$, but $v \notin A$. Therefore, $v \notin \theta(u)$. ie; $(u,v) \notin \theta$, $\forall\,(u,v)$ with $u \neq v$.
But, since θ is reflexive, $(u,u) \in \theta, \forall u \in X$,
Therefore, $\theta = \{(u,u) : u \in X\}$.

Now if $\theta = \{(u,u) : u \in X\}$, then τ_θ is the discrete topology and hence it is T_0, by theorem 3.11. □

Corollary 3.16. *If θ is a reflexive relation on X, then τ_θ satisfies T_1, T_2, T_3 or T_4 separation axioms if and only if $\theta = \{(u,u) : u \in X\}$.*

Now, the family $\mathscr{B} = \{\theta(a) : a \in X, \theta(a) \subseteq A\}$, constitutes an open base for the τ_θ on X if and only f θ is reflexive and transitive. Also, the lower and upper approximation operators are the interior and closure operators in τ_θ if and only if θ is reflexive and transitive [4]. So, we further study the topology of induced by quasi orders. If θ is an equivalence on X, then τ_θ will be a quasi-discrete topology on X [13]. Also, if θ is a reflexive and symmetric relation, then τ_θ will be a quasi discrete topology [7]. However, the following example demonstrates that symmetry is not a necessary condition for this.

Example 3.17. Take $X = \{p,q,r,s\}$ and let $\theta = \{(p,p),(p,q),(q,q),(q,r),(r,r),(r,p),(s,s)\}$. Then, $\theta(p) = \{p,q\}$, $\theta(q) = \{q,r\}$, $\theta(r) = \{p,r\}$ and $\theta(s) = \{s\}$. It is clear that θ is not symmetric. From equation (1), $\tau_\theta = \{\emptyset, X, \{p,q,r\}, \{s\}\}$. Thus, each open set in τ_θ is closed. Therefore, τ_θ is quasi discrete.

In the next theorem,we prove that in a quasi ordered approximation space, symmetry is a necessary and sufficient condition for τ_θ to be quasi discrete.

Theorem 3.18. *If θ is a quasi order on X, then the topology τ_θ is quasi-discrete if and only if the binary relation θ is symmetric.*

Proof. In case θ is symmetric, then it will be an equivalence and so, then topoloy τ_θ will be quasi-discrete.

Conversely, assume that τ_θ is a quasi-discrete topology. Then, any subset $A \subseteq X$ is open if and only if it is closed. ie; $\underline{\theta}(A) = A$ if and only if $\overline{\theta}(A) = A$, as θ is a quasi order.

Let $(u,v) \in \theta$, $u,v \in X$. Then, $(v,u) \notin \theta \Rightarrow u \notin \theta(v) \Rightarrow u \in [\theta(v)]^C$.

By lemma 1, $\theta(v)$ is an open set and hence, $[\theta(v)]^C$ is closed. Hence, $[\theta(v)]^C$ is an open set containing u. Therefore, $\theta(u) \subseteq [\theta(v)]^C$.

Hence, $\theta(u) \cap \theta(v) = \emptyset$. This is a contradiction since $v \in \theta(u) \cap \theta(v)$.

Hence, $(v,u) \in \theta$ and θ is symmetric. □

Corollary 3.19. *If θ is a partial order on X, then the topology τ_θ is quasi-discrete if and only if $\theta = \{(u,u) : u \in X\}$.*

Theorem 3.20. *If θ is a quasi order on X, then, (X, τ_θ) is regular if and only if θ is symmetric.*

Proof. First, assume that (X, τ_θ) is regular.

Then, $(u,v) \in \theta \Rightarrow v \in \theta(u)$. Also, $(v,u) \notin \theta \Rightarrow u \notin \theta(v) \Rightarrow u \in [\theta(v)]^C$.

From lemma 3.3, the set $\theta(v)$ is open in τ_θ. Thus, $[\theta(v)]^C$ is a closed set and $v \notin [\theta(v)]^C$.

(X, τ_θ) being regular, there exist two disjoint open sets A and B with $v \in A$ and $[\theta(v)]^C \subseteq B$.

As, A is open and $v \in A$, $\theta(v) \subseteq A$. Again, $u \in B$, since $u \in [\theta(v)]^C$.

Hence, $\theta(u) \subseteq B$, since B is open. Therefore, $v \in B$. This is a contradiction to $A \cap B = \emptyset$.

Hence, $(v,u) \in \theta$ and θ is symmetric.

In case θ is an equivalence, θ will be reflexive and symmetric.

Hence (X, τ_θ) is regular [9]. □

4. Continuity on Approximation Spaces

In this section, the continuity of a bijective function defined on quasi ordered approximations is discussed. Further, the largest topology on the range set X' which makes a bijective function $g : X \to X'$ continuous in the context of quasiordered approximation spaces is obtained.

Lemma 4.1. *Let (X,θ) and (X',θ') be two quasi ordered approximation spaces. Then, the function $g:(X,\tau_\theta)\to(X',\tau_{\theta'})$ is continuous if and only if $(u,v)\in\theta \Rightarrow (g(u),g(v))\in\theta'$, for all $u,v\in X$.*

Proof. First, assume that $g:(X,\tau_\theta)\to(X',\tau_{\theta'})$ is continuous. Consider $(u,v)\in\theta$.
As θ' is a transitive relation and $g(u)\in X'$, the set $\theta'[g(u)]$ is open in $\tau_{\theta'}$ using lemma 3.3.
Also, $u\in g^{-1}[\theta'[g(u)]]$. So, $\theta(u)\subseteq g^{-1}[\theta'[g(u)]]$. Thus, $v\in g^{-1}[\theta'[g(u)]]$, since $v\in\theta(u)$.
Hence, $g(v)\in\theta'[g(u)]$. Therefore $(g(u),g(v))\in\theta'$.

Now let $(u,v)\in\theta\Rightarrow(g(u),g(v))\in\theta', \forall u,v\in X$. Take A be an open set in τ_θ.
Then, $u\in g^{-1}(A)\Rightarrow g(u)\in A\Rightarrow\theta'[g(u)]\subseteq A$.
Also, $v\in\theta(u)\Rightarrow(u,v)\in\theta\Rightarrow(g(u),g(v))\in\theta'$, by assumption.

$$\Rightarrow g(v)\in\theta'[g(u)]\subseteq A\Rightarrow v\in g^{-1}(A). \text{ Thus, } \theta(u)\subseteq g^{-1}(A).$$

Therefore, $\underline{\theta}[g^{-1}(A)]=g^{-1}(A)$. Hence, $g^{-1}(A)\in\tau_{\theta'}$. Thus, g is continuous. □

Theorem 4.2. *Let (X,θ) and (X',θ') be quasi ordered approximation spaces. A function $g:(X,\tau_\theta)\to(X',\tau_{\theta'})$ is continuous if and only if $g[\theta(u)]\subseteq\theta'[g(u)]$, $\forall u\in X$.*

Proof. By lemma 4.1, $g:(X,\tau_\theta)\to(X',\tau_{\theta'})$ is continuous $\Leftrightarrow (u,v)\in\theta$

$$\Leftrightarrow (g(u),g(v))\in\theta',\ \forall u,v\in X \Leftrightarrow v\in\theta(u)\Leftrightarrow g(v)\in\theta'[g(u)]\Leftrightarrow g[\theta(u)]\subseteq\theta'[g(u)]$$ □

Theorem 4.3. *Let (X,θ) and (X',θ') be two quasi ordered approximation spaces. Consider a bijective function $g:(X,\tau_\theta)\to(X',\tau_{\theta'})$. Then the following statements are equivalent.*

(i) *g is a homeomorphism*
(ii) $g[\overline{\theta}(A)]=\overline{\theta'}[g(A)], \forall A\subseteq X.$
(iii) $g[\underline{\theta}(A)]=\underline{\theta'}[g(A)], \forall A\subseteq X$

Proof. First let g be a homeomorphism. Then, g and g^{-1} will be continuous.
So, $v\in g[\overline{\theta}(A)]\Rightarrow v=g(u)$, for some $u\in\overline{\theta}(A)$.

$$\Rightarrow v=g(u), \text{ for some } u\in X, \text{ with } \theta(u)\cap A\neq\emptyset.$$

Let $x\in\theta(u)\cap A$. Then, $(u,x)\in\theta$ and $x\in A$. Since g is a continuous bijection, $(g(u),g(x))\in\theta'$ by lemma 4.1. Also, $g(x)\in g(A)$. *ie*; $g(x)\in\theta'[g(u)]$ and $g(x)\in g(A)$.
As, $v=g(u)$, we have, $g(x)\in\theta'(v)\cap g(A)$. Hence, $\theta'(v)\cap g(A)\neq\emptyset$. Therefore, $v\in\overline{\theta'}\,[g(A)]$.

Thus, $g[\overline{\theta}(A)] \subseteq \overline{\theta'}[g(A)]$. Also, $v \in \overline{\theta'}[g(A)] \Rightarrow \theta'(v) \cap g(A) \neq \emptyset$.

Let $z \in \theta'(v) \cap g(A)$. Then, $(v,z) \in \theta'$ and $z \in g(A)$. *ie*; $(v,z) \in \theta'$, $z = g(y)$, for some $y \in A$.

Thus, $(g(u), g(y)) \in \theta'$. So, $(u,y) \in \theta$, as g^{-1} is continuous. So, $y \in \theta(u)$. Also, $y \in A$.

Thus, $y \in \theta(u) \cap A$. Hence, $\theta(u) \cap A \neq \emptyset$. Thus, $u \in \overline{\theta}(A)$. Therefore, $v \in g[\overline{\theta}(A)$.

So, $\overline{\theta'}[g(A)] \subseteq g[\overline{\theta}(A)]$. Therefore, $g[\overline{\theta}(A)] = \overline{\theta'}[g(A)]$.

Thus $(i) \Rightarrow (ii)$.

Now assume (ii). Since g is a bijection, $g(A^C) = [g(A)]^C, \forall A \subseteq X$.

Also, by the property of approximations, $[\overline{\theta}(A^C)]^C = \underline{\theta}(A)$.

Then, $g(\underline{\theta}(A)) = g[\overline{\theta}(A^C)]^C] = [g[\overline{\theta}(A^C)]]^C = [\overline{\theta'}[g(A^C)]]^C = [\overline{\theta'}[g(A)^C)]]^C = \underline{\theta'}[g(A)]$.

Thus $(ii) \Rightarrow (iii)$.

Again, suppose that $g[\underline{\theta}(A)] = \underline{\theta'}\,[g(A)], \forall A \subseteq X$. Take A be an open set in $\tau_{\theta'}$.

Then $\underline{\theta'}(A) = A$. Consider $g^{-1}(A)$. Then, $g[\underline{\theta}[g^{-1}(A)]] = \underline{\theta'}[g[g^{-1}(A)]] = \underline{\theta'}(A)] = A$.

So, $\underline{\theta}[g^{-1}(A)] = g^{-1}(A)$. Hence, $g^{-1}(A) \in \tau_{\theta}$. Therefore g is continuous.

Now consider $B \in \tau_{\theta}$. Then, $\underline{\theta}(B) = B$. Also, $\underline{\theta'}[g(B)] = g[\underline{\theta}(B)] = g(B)$.

Therefore, $g(B)$ is open in $\tau_{\theta'}$. Hence g^{-1} is continuous. So, g is a homeomorphism.

Thus $(iii) \Rightarrow (i)$. This completes the proof. □

Theorem 4.4. *If the function $g : X \to X'$ is bijective and (X, θ) is a quasi ordered approximation space, then g defines a quasi order θ'_g on X' such that $g : (X, \tau_{\theta}) \to (X', \tau_{\theta'_g})$ is a homeomorphism.*

Proof. Consider the relation θ'_g on X' defined by

$$(u,v) \in \theta \Leftrightarrow (g(u), g(v)) \in \theta'_g \tag{6}$$

As g is a bijective function, $\forall u' \in X'$, $\exists u \in X$ such that $g(u) = u'$.

Also, $(u,u) \in \theta$ since θ is reflexive. Using equation 6, $(u',u') = (g(u), g(u)) \in \theta'_g, \forall u' \in X'$.

Also, $(u',v') \in \theta'_g$, $(v',z') \in \theta'_g \Rightarrow (g(u), g(v)) \in \theta'_g$, $(g(v), g(z)) \in \theta'_g$.

From equation 6, $(u,v) \in \theta$ and $(v,z) \in \theta$. Hence, $(u,z) \in \theta$ as θ is transitive.

So, $(g(u), g(z)) \in \theta'_g$. That is, $(u',z') \in \theta'_g$. Hence, θ'_g is transitive.

Therefore, (X', θ'_g) is a quasi ordered approximation space.

From lemma 4.1 and equation (6) it follows that both g and g^{-1} are continuous with respect to (X, τ_θ) and $(X', \tau_{\theta'_g})$. Therefore g is a homeomorphism. $\square$

Corollary 4.5. *If (X, θ) is a quasi ordered approximation space, and $g : X \to X'$ is bijective, then g determines a quasi order θ'_g on X' such that $\tau_{\theta'_g}$ is the strongest topology on X' that makes g continuous.*

Proof. The relation θ'_g on X' defined by the equation (6) makes g continuous with respect to (X, τ_θ) and (X', τ_{θ_g}).

Now, assume that τ is a topology on X' such that $g : (X, \tau_\theta) \to (X, \tau)$ is continuous. If $A \in \tau$, then $g^{-1}(A) \in \tau_\theta$. Consider $\theta_g(u')$ for $u' \in A$.
Then, $v' \in \theta_g(u') \Rightarrow (u', v') \in \theta_g \Rightarrow (g(u), g(v)) \in \theta_g$, for $u, v \in X$, since g is a bijection
$\Rightarrow (u, v) \in \theta$, from equation (8) $\Rightarrow v \in \theta(u)$, where $u \in g^{-1}(A)$
$\Rightarrow v \in g^{-1}(A)$, as $g^{-1}(A) \in \tau_\theta \Rightarrow g(v) \in A \Rightarrow v' \in A$.
So, $\theta_g(u') \subseteq A$. Therefore, $A \subseteq \underline{\theta_g}(A)$. Hence, $\underline{\theta_g}(A) = A$. Therefore, $A \in \tau_{\theta_g}$. So, $\tau \subseteq \tau_{\theta_g}$ $\square$

5. CONCLUSION

The concept of approximation spaces is closely related to the concept of topological spaces and many studies have been conducted on this point of view. In the present paper, an exploratory discussion of the properties of the topology induced by rough set approximations on generalized approximation space with respect to the type of binary relation used is conducted. Also, many characterization theorems are presented on this aspect. The continuity of functions defined on generalized approximation spaces are also investigated and necessary and sufficient conditions for a function to be continuous and homeomorphism are obtained.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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