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ON PRIME ONE-SIDED ORDERED IDEALS OF ORDERED SEMIRINGS

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Abstract. We define the notions of prime, semiprime, weakly prime and weakly semiprime properties of one-sided ordered ideals of ordered semirings, study their connections and use them to characterize several kinds of regularities on ordered semirings.

Keywords: semiring; ordered semiring; ordered ideal; prime ordered ideal.

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1. INTRODUCTION

There are several different notions of prime, weakly prime, semiprime and weakly semiprime properties of ideals of semirings e.g. [1, 2, 3, 5, 6]. However, on an ordered semigroup S , Kehayopulu [7] defined an ideal T of S to be prime (resp. semiprime) if for any $A, B \subseteq S$, $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$ (resp. $A^2 \subseteq T$ implies $A \subseteq T$). In sense of Kehayopulu, an ideal T of an ordered semigroup S is weakly prime if for any ideals A and B of S , $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$. Kehayopulu also showed that an ideal of an ordered semigroup is prime if and only if it is both semiprime and weakly prime. Particularly, if the ordered semigroup is commutative, then the concepts of prime and weakly prime ideals coincide. Then Kehayopulu

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use prime and weakly prime ideal to characterize intra-regular and fully idempotent ordered semigroups, respectively.

In this work, we study prime, weakly prime, semiprime and weakly semiprime properties in a similar way of results of Kehayopulu [7] but generalize them to the case of one-sided (either left or right) ordered ideals and study them on ordered semirings. Moreover, we have that there is a one-sided ordered ideal of an ordered semiring which is both semiprime and weakly prime but not prime. This means that there is a condition which is true in case of two-sided ordered ideals but not true in case of one-sided ordered ideals. Finally, we use prime, weakly prime, semiprime and weakly semiprime one-sided ordered ideals to characterize several kinds of regularities on ordered semirings.

2. PRELIMINARIES

An *ordered semiring* [4] is an algebraic structure $(S, +, \cdot, \leq)$ such that $(S, +, \cdot)$ is a semiring and (S, \leq) is a poset satisfying the property; for any $a, b, c \in S$, if $a \leq b$, then $a + c \leq b + c$, $c + a \leq c + b$, $ac \leq bc$ and $ca \leq cb$. An ordered semiring $(S, +, \cdot, \leq)$ is called *additively commutative* if $a + b = b + a$ for all $a, b \in S$. An element 0 of an ordered semiring S is said to be an *absorbing zero* if $a + 0 = a = 0 + a$ and $a0 = 0 = 0a$ for all $a \in S$. Throughout this work, we simply write S instead of an ordered semiring $(S, +, \cdot, \leq)$ and always assume that it is additively commutative together with an absorbing zero.

For any nonempty subsets A and B of an ordered semiring S , we denote that

$$A + B = \{a + b \in S \mid a \in A, b \in B\},$$

$$AB = \{ab \in S \mid a \in A, b \in B\},$$

$$(A) = \{x \in S \mid x \leq a \text{ for some } a \in A\} \text{ and}$$

$$\Sigma A = \left\{ \sum_{i \in I} a_i \in S \mid a_i \in A \text{ and } I \text{ is a finite index set} \right\}.$$

In a particular case of $a \in S$, we write Σa and (a) instead of $\Sigma\{a\}$ and $(\{a\})$, respectively. If $I = \emptyset$, then we set $\sum_{i \in I} a_i = 0$ for all $a_i \in S$. We review some basic properties of the operator $(\])$ and the finite sums Σ of nonempty subsets of ordered semirings which occur in [8, 9, 10] as follows.

Remark 2.1. Let A and B be nonempty subsets of an ordered semiring S . Then the following statements hold:

- | | |
|--|--|
| (1) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$; | (8) if $A \subseteq B$, then $(A] \subseteq (B]$; |
| (2) if $A \subseteq B$, then $\Sigma A \subseteq \Sigma B$; | (9) $A + (B] \subseteq (A] + (B] \subseteq (A + B]$; |
| (3) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$; | (10) $A(B] \subseteq (A](B] \subseteq (AB]$ and $(A]B \subseteq (A](B] \subseteq (AB]$; |
| (4) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$
$(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$; | (11) $(A \cup B] = (A] \cup (B]$; |
| (5) $\Sigma(A \cup B) \supseteq \Sigma A \cup \Sigma B$; | (12) $(A \cap B] \subseteq (A] \cap (B]$; |
| (6) $\Sigma(A \cap B) \subseteq \Sigma A \cap \Sigma B$; | (13) $\Sigma(A] \subseteq (\Sigma A]$. |
| (7) $A \subseteq (A]$ and $((A]) = (A]$; | |

A nonempty subset A of an ordered semiring S is called a *left (right) ordered ideal* of S if $A + A \subseteq A$, $SA \subseteq A$ ($AS \subseteq A$) and $A = (A]$. If A is either left or right ordered ideal of S , then A is also called a *one-sided ordered ideal* of S . Moreover, if A is both a left and a right ordered ideal of S , then A is called an *ordered ideal* (also called a *two-sided ordered ideal*) of S [4].

Remark 2.2. If A and B are left (resp. right) ordered ideals of an ordered semiring S , then $(\Sigma AB]$ is a left (resp. right) ordered ideals of S as well.

Let $a \in S$. We denote $L(a)$ and $R(a)$ to be the smallest left ordered ideal and right ordered ideal of S containing a , respectively. We recall their constructions which occur in [8, 9] as follows.

Lemma 2.3. *Let S be an ordered semiring. Then $L(a) = (\Sigma a + Sa]$ and $R(a) = (\Sigma a + aS]$ for all $a \in S$.*

Definition 2.4. [8] An ordered semiring S is called *left regular* (resp. *right regular*) if $a \in (Sa^2]$ (resp. $a \in (a^2S]$) for all $a \in S$.

A generalization of a left regular (resp. right regular) ordered semiring is defined as follows.

Definition 2.5. An ordered semiring S is called *left weakly regular* (resp. *right weakly regular*) if $a \in (\Sigma SaSa]$ (resp. $a \in (\Sigma aSaS]$) for all $a \in S$.

Remark 2.6. If S is a left (resp. right) regular ordered semiring, then S is also left (resp. right) weakly regular.

Lemma 2.7. *If an ordered semiring S is left (resp. right) weakly regular, then $L(a) = (Sa]$ (resp. $R(a) = (aS]$) for all $a \in S$.*

Proof. Let $a \in S$. Since $a \in (\Sigma SaSa] \subseteq (\Sigma Sa] = (Sa]$ and $(Sa]$ is a left ordered ideal of S , we get $L(a) \subseteq (Sa]$. If $x \in (Sa]$, then $x \leq y$ for some $y \in Sa$. Since $y = 0 + y \in \Sigma a + Sa$, we get $x \in (\Sigma a + Sa] = L(a)$. So, $(Sa] \subseteq L(a)$. Hence, $L(a) = (Sa]$. \square

In consequences of Remark 2.6 and Lemma 2.7, we obtain the following corollary.

Corollary 2.8. *If an ordered semiring S is left (resp. right) regular, then $L(a) = (Sa]$ (resp. $R(a) = (aS]$) for all $a \in S$.*

Now, we give some useful characterizations of left weakly regular and right weakly regular ordered semirings as the following two lemmas.

Lemma 2.9. *Let S be an ordered semiring. Then the following conditions are equivalent:*

- (1) S is left weakly regular;
- (2) $A \cap B \subseteq (\Sigma AB]$ for any left ordered ideals A and B of S ;
- (3) $L = (\Sigma L^2]$ for every left ordered ideal L of S ;
- (4) $L(a) = (\Sigma(L(a))^2]$ for any $a \in S$.

Proof. (1) \Rightarrow (2) Assume that S is left weakly regular and let A and B be left ordered ideals of S . If $x \in A \cap B$, then by assumption, we get that $x \in (\Sigma SxSx] \subseteq (\Sigma SASB] \subseteq (\Sigma AB]$. Hence, $A \cap B \subseteq (\Sigma AB]$.

(2) \Rightarrow (3) Let L be a left ordered ideal of S . By (2), we immediately obtain $L \subseteq (\Sigma L^2]$. On the other hand, we get that $(\Sigma L^2] \subseteq (\Sigma SL] \subseteq (\Sigma L] = L$. Hence, $L = (\Sigma L^2]$.

(3) \Rightarrow (4) It is obvious.

(4) \Rightarrow (1) Assume that (4) holds and let $a \in S$. Then by assumption, Remark 2.1 and Lemma 2.3, we obtain that

$$\begin{aligned} a \in L(a) &= (\Sigma(L(a))^2] = (\Sigma(\Sigma a + Sa](\Sigma a + Sa]) \\ &\subseteq (\Sigma(\Sigma a^2 + aSa + Sa^2 + SaSa]) \\ (2.1) \quad &\subseteq (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \end{aligned}$$

Using the condition (2.1) and Remark 2.1, we obtain that

$$(2.2) \quad a \in (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa] \subseteq (Sa + \Sigma SaSa]$$

Using the condition (2.2) and Remark 2.1, we obtain that

$$\begin{aligned} a^2 = aa &\in (Sa + \Sigma SaSa](Sa + \Sigma SaSa] \\ (2.3) \quad &\subseteq ((Sa + \Sigma SaSa)(Sa + \Sigma SaSa]) \subseteq (\Sigma SaSa]. \end{aligned}$$

Using the condition (2.2) and Remark 2.1 again, we obtain that

$$(2.4) \quad aSa \subseteq (Sa + \Sigma SaSa]S(Sa + \Sigma SaSa] \subseteq (\Sigma SaSa].$$

Using the condition (2.3) and Remark 2.1, we obtain that

$$(2.5) \quad Sa^2 \subseteq S(\Sigma SaSa] \subseteq (\Sigma SaSa].$$

Using the conditions (2.1), (2.3), (2.4), (2.5) and Remark 2.1, it turns out that

$$\begin{aligned} a &\in (\Sigma a^2 + aSa + Sa + \Sigma SaSa] \\ &\subseteq (\Sigma(\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa] + (\Sigma SaSa]) \subseteq (\Sigma SaSa]. \end{aligned}$$

Therefore, S is left weakly regular. □

We obtain the following lemma as a duality of Lemma 2.10.

Lemma 2.10. *Let S be an ordered semiring. Then the following conditions are equivalent:*

- (1) S is right weakly regular;
- (2) $A \cap B \subseteq (\Sigma AB]$ for any right ordered ideals A and B of S ;
- (3) $R = (\Sigma R^2]$ for every right ordered ideal R of S ;

$$(4) R(a) = (\Sigma R(a)^2] \text{ for any } a \in S.$$

3. MAIN RESULTS

Now, we give the notions of prime and semiprime subsets of an ordered semiring as follows.

Definition 3.1. A nonempty subset T of an ordered semiring S is called *prime* if for any $a, b \in S$, $ab \in T$ implies $a \in T$ or $b \in T$.

The following remark can be directly obtained by Definition 3.1.

Remark 3.2. A nonempty subset T of an ordered semiring S is prime if and only if for any nonempty subsets A and B of S , $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$.

Definition 3.3. A nonempty subset T of an ordered semiring S is called *semiprime* if for any $a \in S$, $a^2 \in T$ implies $a \in T$.

The following remark can be directly obtained by Definition 3.3.

Remark 3.4. A nonempty subset T of an ordered semiring S is semiprime if and only if for any nonempty subset A of S , $A^2 \subseteq T$ implies $A \subseteq T$.

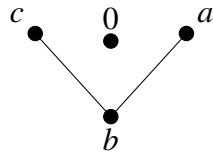
It is obvious that every prime subset of an ordered semiring is semiprime. The converse of this condition is not generally true as shown by the following example.

Example 3.5. Let $S = \{0, a, b, c\}$. Define binary operations $+$ and \cdot on S by the following tables:

$+$	0	a	b	c		\cdot	0	a	b	c
0	0	a	b	c		0	0	0	0	0
a	a	a	b	b	and	a	0	a	b	c
b	b	b	b	b		b	0	b	b	c
c	c	b	b	c		c	0	b	b	c

Define a binary relation \leq on S by

$$\leq := \{(0, 0), (a, a), (b, b), (c, c), (b, a), (b, c)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring with an absorbing zero 0. It is easy to see that $\{0, b\}$ is a left ordered ideal of S which is not a right ordered ideal. We have that $\{0, b\}$ is a semiprime left ordered ideal but not prime because $ca = b \in \{0, b\}$ and $c, a \notin \{0, b\}$.

Now, we define the weakly prime property of one-sided ordered ideals of an ordered semiring as follows.

Definition 3.6. A left (resp. right) ordered ideal T of an ordered semiring S is called *weakly prime* if for any left (resp. right) ordered ideals A and B of S , $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$.

It is clear that every prime one-sided ordered ideal is weakly prime. The converse is not true as shown by the following example

Example 3.7. Let $S = \{0, a, b, c, d, e\}$. Define binary operations $+$ and \cdot on S by the following tables:

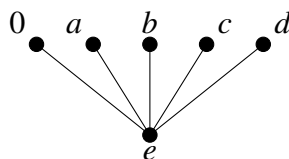
$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	e	e	e	e	e
b	b	e	e	e	e	e
c	c	e	e	e	e	e
d	d	e	e	e	e	e
e	e	e	e	e	e	e

and

\cdot	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	a	b	c	d	e
b	0	b	a	c	d	e
c	0	c	d	c	d	e
d	0	d	c	c	d	e
e	0	e	e	e	e	e

Define a binary relation \leq on S by

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (d,d), (e,e), (e,0), (e,a), (e,b), (e,c), (e,d)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring with an absorbing zero 0. All left ordered ideals of S are as follows: $\{0, e\}$, $\{0, c, e\}$, $\{0, d, e\}$, $\{0, c, d, e\}$ and S . We have that $\{0, c, e\}$ is a weakly prime left ordered ideal of S . However, it is not a prime left ordered ideal since $db = c \in \{0, c, e\}$ but $b, d \notin \{0, c, e\}$.

In an ordered semigroup S , Kehayopulu [7] proved that if S is commutative, then the concepts of weakly prime (two-sided) ideals and prime (two-sided) ideals coincide. It is not difficult to prove that this condition is true in case of one-sided ordered ideals on a commutative ordered semiring as well. However, we are able to use a condition which is more general than the commutative condition as follows.

Remark 3.8. If an ordered semiring S is commutative, then $xSy \subseteq Sxy$ for all $x, y \in S$.

We give the following example to show that the condition $xSy \subseteq Sxy$ for all $x, y \in S$, is a generalization of the commutative condition.

Example 3.9. Let $S = \{0, a, b, c, d\}$. Define binary operations $+$ and \cdot on S by the following tables:

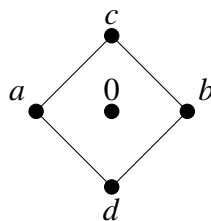
$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	d	d	d	d
b	b	d	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d

and

\cdot	0	a	b	c	d
0	0	0	0	0	0
a	0	c	c	c	d
b	0	b	b	c	d
c	0	c	c	c	d
d	0	d	d	d	d

Define a binary relation \leq on S by

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (d,d), (a,c), (b,c), (d,a), (d,b), (d,c)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring with an absorbing zero 0. Clearly, if $x = 0$ or $y = 0$, then $xSy = \{0\} = Sxy$ for all $x, y \in S$. Obviously, if $x = d$ or $y = d$, then $xSy = \{0, d\} = Sxy$ for all $x, y \in S \setminus \{0\}$. Moreover, we get that $xSy \subseteq \{0, b, c, d\} = Sxy$ for all $x, y \in S \setminus \{0, d\}$. Hence, $xSy \subseteq Sxy$ for all $x, y \in S$. However, S is not commutative since $ab \neq ba$.

Now, we show that prime and weakly prime properties of left ordered ideals of an ordered semiring are coincidence without the commutative condition.

Theorem 3.10. *Let S be an ordered semiring. If $xSy \subseteq Sxy$ for all $x, y \in S$, then prime left ordered ideals and weakly prime left ordered ideals coincide.*

Proof. Assume that $xSy \subseteq Sxy$ for all $x, y \in S$. Let T be a weakly prime left ordered ideal of S . We show that T is also prime. Let $a, b \in S$ such that $ab \in T$. Then by Remark 2.1 and Lemma 2.3, we obtain that

$$\begin{aligned} L(a)L(b) &= (\Sigma a + Sa)(\Sigma b + Sb) \subseteq ((\Sigma a + Sa)(\Sigma b + Sb)) \\ &\subseteq (\Sigma ab + aSb + Sab + \Sigma SaSb) \subseteq (\Sigma ab + Sab + Sab + \Sigma SSab) \\ &= (\Sigma ab + Sab) = L(ab) \subseteq T. \end{aligned}$$

Since T is weakly prime, we have that $a \in L(a) \subseteq T$ or $b \in L(b) \subseteq T$. Therefore, T is prime. \square

As a duality of Remark 3.8 and Theorem 3.10, we obtain the following remark and theorem.

Remark 3.11. If an ordered semiring S is commutative, then $xSy \subseteq xyS$ for all $x, y \in S$.

Theorem 3.12. *Let S be an ordered semiring. If $xSy \subseteq xyS$ for all $x, y \in S$, then prime right ordered ideals and weakly prime right ordered ideals coincide.*

We give the following two examples to show that the concepts of semiprime and weakly prime properties of one-sided ordered ideals are independent.

Example 3.13. We have that the semiprime left ordered ideal $\{0, b\}$ defined as Example 3.5, is not weakly prime because $\{0, b, c\}\{0, a, b\} = \{0, b\}$, $\{0, b, c\} \not\subseteq \{0, b\}$ and $\{0, a, b\} \not\subseteq \{0, b\}$ where $\{0, b, c\}$ and $\{0, a, b\}$ are left ordered ideals of S .

Example 3.14. Let $S = \{0, a, b, c, d, e\}$. Define a binary operation \cdot on S by the following table;

\cdot	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	a	e	c	e	e
b	0	d	b	b	d	e
c	0	a	c	c	a	e
d	0	d	e	b	e	e
e	0	e	e	e	e	e

Then $(S, +, \cdot, \leq)$ is an additively commutative with an absorbing zero 0 where $+$ is the operation and \leq is the relation defined as Example 3.7, respectively. We have that all left ordered ideals of S are as follows: $\{0, e\}$, $\{0, a, d, e\}$, $\{0, b, c, e\}$ and S . It is easy to show that $\{0, b, c, e\}$ is a weakly prime left ordered ideal of S . However, $\{0, b, c, e\}$ is not semiprime because $dd = e \in \{0, b, c, e\}$ and $d \notin \{0, b, c, e\}$.

In consequences of Example 3.13 and 3.14, we are able to say that the concepts of semiprime property and weakly prime property of one-sided ordered ideals are independent.

In ordered semigroups, Kehayopulu [7] proved that if a two-sided ideal is both weakly prime and semiprime, then it is prime as well. However, this condition fails in cases of one-sided ordered ideals. On an ordered semiring, we refer the weakly prime left ordered ideal $\{0, c, e\}$ defined in Example 3.7 again. It is easy to see that $\{0, c, e\}$ is also semiprime but not prime. In case of an ordered semigroup, we give the following example.

Example 3.15. Let $S = \{a, b, c, d\}$. Define a binary operations \cdot on S by the following table:

\cdot	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

Define a binary relation on S by $\leq := \{(x, x) \mid x \in S\}$. Then (S, \cdot, \leq) is an ordered semigroup. It is not difficult to verify that $\{c\}$ is a left ideal of S which is both weakly prime and semiprime but not prime.

Now, we define the weakly semiprime property of one-sided ordered ideals of an ordered semiring as follows.

Definition 3.16. A left (resp. right) ordered ideal T of an ordered semiring S is called *weakly semiprime* if for each left (resp. right) ordered ideal A of S , $A^2 \subseteq T$ implies $A \subseteq T$.

It is clear that every semiprime one-sided ordered ideal is weakly semiprime. In general, the converse is not true as a consequence of Example 3.14.

Now, we show that semiprime and weakly semiprime properties of left ordered ideals of an ordered semiring are coincidence without the commutative condition.

Theorem 3.17. *Let S be an ordered semiring. If $xSy \subseteq Sxy$ for all $x, y \in S$, then semiprime left ordered ideals and weakly semiprime left ordered ideals coincide.*

Proof. Assume that $xSy \subseteq Sxy$ for all $x, y \in S$. Let T be a weakly semiprime left ordered ideal of S . We show that T is also semiprime. Let $a \in S$ such that $a^2 \in T$. Then by Remark 2.1 and Lemma 2.3, we obtain that

$$\begin{aligned} L(a)L(a) &= (\Sigma a + Sa)(\Sigma b + Sb) \subseteq ((\Sigma a + Sa)(\Sigma a + Sa)) \\ &\subseteq (\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa) \subseteq (\Sigma a^2 + Sa^2 + Sa^2 + \Sigma SSaa) \\ &= (\Sigma a^2 + Sa^2) = L(a^2) \subseteq T. \end{aligned}$$

Since T is weakly semiprime, we have that $a \in L(a) \subseteq T$. So, T is semiprime. □

As a duality of Theorem 3.17, we obtain the following theorem.

Theorem 3.18. *Let S be an ordered semiring. If $xSy \subseteq xyS$ for all $x, y \in S$, then semiprime right ordered ideals and weakly semiprime right ordered ideals coincide.*

Furthermore, we are able to obtain that every weakly prime left (resp. right) ordered ideal of an ordered semiring is weakly semiprime. The converse of this condition is not true in general as shown by the weakly semiprime left ordered ideal $\{0, b\}$ of the ordered semiring $(S, +, \cdot, \leq)$ defined in Example 3.13.

Now, we use prime, semiprime, weakly prime and weakly semiprime properties of one-sided ordered ideals to characterize several kinds of regularities of ordered semirings.

Theorem 3.19. *Let S be an ordered semiring. Then S is left (resp. right) weakly regular if and only if every left (resp. right) ordered ideal of S is weakly semiprime.*

Proof. Assume that S is left weakly regular. Let A and L be left ordered ideals of S such that $A^2 \subseteq L$. By assumption, if $a \in A$, then $a \in (\Sigma SaSa] \subseteq (\Sigma SASA]$. It follows that $A \subseteq (\Sigma SASA] \subseteq (\Sigma SAA] = (\Sigma SA^2] \subseteq (\Sigma SL] \subseteq (\Sigma L] = L$. Hence, L is weakly semiprime.

Conversely, assume that every left ordered ideal of S is weakly semiprime. Let L be a left ordered ideal of S . By Remark 2.2, we get that $(\Sigma L^2]$ is a left ordered ideal of S as well. Clearly, $L^2 \subseteq (\Sigma L^2]$. Since $(\Sigma L^2]$ is semiprime, we get $L \subseteq (\Sigma L^2]$. On the opposite inclusion, $(\Sigma L^2] \subseteq (\Sigma SL] \subseteq (\Sigma L] = L$. Hence, $L = (\Sigma L^2]$. By Lemma 2.10(3), we obtain that S is left weakly regular. \square

It is easy to verify that the ordered semiring $(S, +, \cdot, \leq)$ defined in Example 3.13 is left weakly regular. Moreover, we have that the left ordered ideal $\{0, b\}$ of S is weakly semiprime but not weakly prime. This shows that the left weakly regular condition of S is not sufficient to get that every left ordered ideal of S is weakly prime. We add a condition to obtain it as the following theorem.

Theorem 3.20. *Let S be an ordered semiring and the set of all left (resp. right) ordered ideals of S be a chain. Then S is left (resp. right) weakly regular if and only if every left (resp. right) ordered ideal of S is weakly prime.*

Proof. Assume that S is left weakly regular. Let A, B and L be left ordered ideals of S such that $AB \subseteq L$. Since the set of all left ordered ideals of S is a chain, we get that $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$, then by assumption and Lemma 2.10(2), we get $A = A \cap B \subseteq (\Sigma AB] \subseteq (\Sigma L] = L$. Similarly, if $B \subseteq A$, then we are able to obtain that $B = A \cap B \subseteq (\Sigma AB] \subseteq (\Sigma L] = L$. Consequently, L is weakly prime.

Conversely, assume that every left ordered ideal of S is weakly prime. Then we immediately have that every left ordered ideal of S is weakly semiprime as well and thus S is left weakly regular by Theorem 3.19. \square

Theorem 3.21. *Let S be an ordered semiring. Then S is left (resp. right) regular if and only if every left (resp. right) ordered ideal of S is semiprime.*

Proof. Assume that S is left regular. Let L be a left ordered ideal of S and $a \in S$ such that $a^2 \in L$. By assumption, we get that $a \in (Sa^2] \subseteq (SL] \subseteq (L] = L$. Hence, L is semiprime.

Conversely, assume that every left ordered ideal of S is semiprime. Let $x \in S$. We have that $x^4 \in SSx^2 \subseteq Sx^2 \subseteq (Sx^2]$. Since $(Sx^2]$ is a semiprime left ordered ideal of S , we get that $x^2 \in (Sx^2]$ and so $x \in (Sx^2]$. Therefore, S is left regular. \square

Lemma 3.22. *Let S be an ordered semiring. If S is left regular and $xSy \subseteq Sxy$ for all $x, y \in S$, then $L(x) \cap L(y) \subseteq L(xy)$ for all $x, y \in S$.*

Proof. Assume that S is left regular and $xSy \subseteq Sxy$ for all $x, y \in S$. Let $a, b \in S$. By Remark 2.6 and Lemma 2.10(2), we have that $L(a) \cap L(b) \subseteq (\Sigma L(a)L(b)]$. Using Remark 2.1 and Corollary 2.8, we obtain that

$$\begin{aligned} L(a) \cap L(b) &\subseteq (\Sigma L(a)L(b)] \\ &= (\Sigma(Sa](Sb)] \subseteq (\Sigma(SaSb)] \subseteq (\Sigma(SSab)] \\ &\subseteq (\Sigma(Sab)] = (\Sigma L(ab)] = L(ab). \end{aligned}$$

Hence, the proof is done. \square

Theorem 3.23. *Let S be an ordered semiring, the set of all left ordered ideals of S be a chain and $xSy \subseteq Sxy$ for all $x, y \in S$. Then S is left regular if and only if every left ordered ideal of S is prime.*

Proof. Assume that S is left regular. Let L be a left ordered ideal of S and $a, b \in S$ such that $ab \in L$. Since the set of all left ordered ideals of S is a chain, we have that $L(a) \subseteq L(b)$ or $L(b) \subseteq L(a)$. If $L(a) \subseteq L(b)$, then by Lemma 3.22, we get that $a \in L(a) = L(a) \cap L(b) \subseteq L(ab) \subseteq L$. Similarly, if $L(b) \subseteq L(a)$, then we have that $b \in L(b) = L(a) \cap L(b) \subseteq L(ab) \subseteq L$. Hence, L is prime.

Conversely, assume that every left ordered ideal of S is prime. Accordingly, we now have that every left ordered ideal of S is semiprime as well. By Theorem 3.21, we get that S is left regular. \square

As a duality of Lemma 3.22 and Theorem 3.23, we obtain the following lemma and theorem.

Lemma 3.24. *Let S be an ordered semiring. If S is right regular and $xSy \subseteq xyS$ for all $x, y \in S$, then $R(x) \cap R(y) \subseteq R(xy)$ for all $x, y \in S$.*

Theorem 3.25. *Let S be an ordered semiring, the set of all right ordered ideals of S be a chain and $xSy \subseteq xyS$ for all $x, y \in S$. Then S is right regular if and only if every right ordered ideal of S is prime.*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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