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SOME INTEGRAL MEAN INEQUALITIES CONCERNING POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, then Dubinin [J. Math. Sci., 143(2007), 3069-3076.] proved

$$\max_{|z|=1} |P'(z)| \geq \left\{ \frac{n}{2} + \frac{1}{2} \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} \max_{|z|=1} |P(z)|.$$

In this paper, we shall first obtain an integral inequality for the polar derivative of the above inequality. As an application of this result, we prove another inequality which is the L^r analogue of an inequality in polar derivative proved recently by Mir et al. [J. Interdisciplinary Math. 21(2018), 1387-1393].

Keywords: polynomial; polar derivatives; integral mean inequalities.

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1. Introduction

Let $\mathbb{P}_n$ be the class of polynomials $P(z) = \sum_{j=0}^{n} c_j z^j$ of degree n and $P'(z)$ be the derivative of $P(z)$. It was shown by Turán [15] that if $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

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Inequality (1.1) was refined by Aziz and Dawood [2], who under the same hypothesis proved that

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=1}|P(z)|\right\}. \tag{1.2}$$

Equalities hold in (1.1) and (1.2) for polynomial $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta|$.

In the literature, there exist several refinements and generalisations of (1.1) and (1.2), for example see Shah [13], Malik [8], Mir [10], Govil [7], Dewan et al. [5], Dewan and Mir [4], Dubinin [6] etc.

Dubinin [6] used the Classical Schwarz Lemma and obtained an interesting refinement of (1.1) by proving that if $P \in \mathbb{P}_n$ and $P(z)$ has all it zeros in $|z| \leq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \left\{\frac{n}{2} + \frac{1}{2}\frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{1.3}$$

For $P \in \mathbb{P}_n$, the polar derivative [9] of $P(z)$ with respect to a point α, real or complex, is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is polynomial of degree at most $(n-1)$. It generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha P(z)}{\alpha} = P^{'}(z).$$

It is of interest to extend ordinary inequalities into polar derivatives because the later versions are the generalizations of the former.

Shah [13] extended inequality (1.1) to the polar derivative of $P(z)$ and proved the following result.

Theorem 1.1. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n(|\alpha|-1)}{2}\max_{|z|=1}|P(z)|. \tag{1.4}$$

Equality holds in (1.4) *for* $P(z)=\left(\frac{z-1}{2}\right)^n$.

Clearly Theorem 1.1 generalizes inequality (1.1) and to obtain (1.1) we simply divide both sides of (1.4) by $|\alpha|$ and let $|\alpha|\to\infty$.

Recently, Mir et al. [11] extended inequality (1.3) into its polar derivative version by proving:

Theorem 1.2. *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $|z|\leq 1$, then for every real or complex number α with $|\alpha|\geq 1$,*

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{|\alpha|-1}{2}\left\{n+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\max_{|z|=1}|P(z)|. \tag{1.5}$$

Inequality (1.5) *is best possible and the extremal polynomial is $p(z)=(z-1)^n$ with real $\alpha\geq 1$.*

We know from analysis ([12], [14]) that if $P\in\mathbb{P}_n$, then for each $r>0$

$$\lim_{r\to\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} = \max_{|z|=1}|P(z)|. \tag{1.6}$$

2. Main Results

In this paper, we extend inequality (1.3) to its integral analogue for the polar derivative of a polynomial and thereby obtain a generalization of it. Further, as an application of Theorem 2.1, we obtain a more general result which, as special cases, yield interesting generalizations and refinements of (1.2) and (1.3). First, we prove the following, which is the corresponding L^r extension of Theorem 1.2.

Theorem 2.1. *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $|z|\leq 1$, then for every complex number α with $|\alpha|\geq 1$ and $r>0$,*

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \geq \frac{(|\alpha|-1)}{2}\left\{n+\frac{|c_n|-|c_0|}{|c_n|+|c_0|}\right\}\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{2.1}$$

Remark 2.2. Since $P(z)$ has all its zeros in $|z| \leq 1$, therefore $|c_n| \geq |c_0|$. Thus, it follows that Theorem 2.1 strengthens the inequality (1.4). If we divide both sides of inequality (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get L^r version of inequality (1.3) due to Dubinin [6].

Further, we prove the following theorem as an application of Theorem 2.1.

Theorem 2.3. *If* $P \in \mathbb{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq 1$*, then for every complex number* α *with* $|\alpha| \geq 1$ *and* $0 \leq t < 1$*,*

$$\left\{\int_0^{2\pi}\left(\left|D_\alpha P(e^{i\theta})\right| - mnt|\alpha|\right)^r d\theta\right\}^{\frac{1}{r}} \geq \frac{(|\alpha|-1)}{2}\left\{n + \frac{|c_n| - tm - |c_0|}{|c_n| - tm + |c_0|}\right\} \times \left\{\int_0^{2\pi}\left(\left|P(e^{i\theta})\right| - tm\right)^r d\theta\right\}^{\frac{1}{r}}, \tag{2.2}$$

where $m = \min_{|z|=1} |P(z)|$.

Remark 2.4. If we let $t = 0$ in inequality (2.2) of Theorem 2.3, we get inequality (2.1) of Theorem 2.1.

Taking limit as $r \to \infty$ on both sides of (2.2) we have the following result concerning polar derivative recently proved by Mir et al. [11].

Corollary 2.5. *If* $P \in \mathbb{P}_n$ *and* $P(z)$ *has all its zeros in* $|z| \leq 1$*, then for every complex number* α *with* $|\alpha| \geq 1$ *and* $0 \leq t < 1$*,*

$$\max_{|z|=1} |D_\alpha P(z)| \geq \frac{n}{2}\left\{(|\alpha|-1)\max_{|z|=1}|P(z)| + (|\alpha|+1)tm\right\} + \frac{|\alpha|-1}{2}\left(\frac{|c_n| - tm - |c_0|}{|c_n| - tm + |c_0|}\right)\left\{\max_{|z|=1}|P(z)| - tm\right\}. \tag{2.3}$$

where $m = \min_{|z|=1} |P(z)|$.

Equality hold in (2.3) *for* $P(z) = (z-1)^n$ *with real* $\alpha \geq 1$.

Remark 2.6. Corollary 2.5 reduces to Theorem 1.2 when we put $t = 0$.

Remark 2.7. Divide both sides of inequality (2.3) of corollary 2.5 by $|\alpha|$ and making $|\alpha| \to \infty$, we have the following improvement as well as generalization of inequality (1.2) proved by Aziz and Dawood [2].

Corollary 2.8. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for $0 \leq t < 1$,*

$$\begin{aligned} \max_{|z|=1} |P'(z)| &\geq \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + tm\right\} \\ &+ \frac{1}{2}\left(\frac{|c_n| - tm - |c_0|}{|c_n| - tm + |c_0|}\right)\left\{\max_{|z|=1} |P(z)| - tm\right\}. \end{aligned} \tag{2.4}$$

Remark 2.9. Taking limit as $t \to 1$ in inequality (2.4) and using (1.6) we obtain an improved bound of inequality (1.2).

3. Lemmas

For the proof of the theorems, we need the following lemmas.

The first lemma is due to Malik [8].

Lemma 3.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for $|z| = 1$,*

$$k|P'(z)| \leq |Q'(z)|, \tag{3.1}$$

where $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$.

By applying Lemma 3.1 to $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, we immediately get the following result.

Lemma 3.2. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then for $|z| = 1$,*

$$|Q'(z)| \leq k|P'(z)|. \tag{3.2}$$

where $Q(z)$ is defined as in Lemma 3.1.

Lemma 3.3. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for each point z on $|z| = 1$ at which $P(z) \neq 0$,*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \left\{\frac{n}{2} + \frac{1}{2}\left(\frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right)\right\}. \tag{3.3}$$

The above Lemma is due to Dubinin [6].

4. Proof of the Theorems

Proof of Theorem 2.1. If $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, it can be easily verified that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|.$$

Since $P(z)$ has all its zeros in $|z| \leq 1$, therefore, by Lemma 3.2 for $k = 1$, we have

$$\begin{aligned} |P'(z)| &\geq |Q'(z)| \\ &= |nP(z) - zP'(z)| \quad \text{for } |z| = 1. \end{aligned} \tag{4.1}$$

Now for every complex number α with $|\alpha| \geq 1$, we have for $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &= \left|nP(z) + (\alpha - z)P'(z)\right| \\ &\geq |\alpha||P'(z)| - \left|nP(z) - zP'(z)\right|, \end{aligned}$$

which gives with the help of (4.1)

$$|D_\alpha P(z)| \geq (|\alpha| - 1)|P'(z)| \text{ for } |z| = 1. \tag{4.2}$$

For any $r > 0$ and $0 \leq \theta < 2\pi$, from (4.2) we have

$$\left|D_\alpha P(e^{i\theta})\right|^r \geq (|\alpha| - 1)^r \left|P'(e^{i\theta})\right|^r,$$

which equivalently gives

$$\left\{\int_0^{2\pi} \left|D_\alpha P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \geq (|\alpha| - 1)\left\{\int_0^{2\pi} \left|P'(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{4.3}$$

By Lemma 3.3 , we have for each z on $|z| = 1$ at which $P(z) \neq 0$,

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \left\{\frac{n}{2} + \frac{1}{2}\left(\frac{|c_n| - |c_0|}{|c_n| + |c_0|}\right)\right\},$$

which implies by using the fact

$$Re\left(\frac{zP'(z)}{P(z)}\right) \leq \left|\frac{zP'(z)}{P(z)}\right|$$

that

$$|P'(z)| \geq \left\{ \frac{n}{2} + \frac{1}{2} \left| \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right| \right\} |P(z)| \quad \text{for } |z| = 1. \tag{4.4}$$

Further, it is evident that inequality (4.4) follows trivially for those z on $|z| = 1$ at which $P(z) = 0$ as well.

Also from (4.4), we have for $0 \leq \theta < 2\pi$ and $r > 0$

$$\left\{ \int_0^{2\pi} \left| P'(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \geq \left\{ \frac{n}{2} + \frac{1}{2} \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{4.5}$$

Combining (4.3) and (4.5), we get

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \geq (|\alpha| - 1) \left\{ \frac{n}{2} + \frac{1}{2} \frac{|c_n| - |c_0|}{|c_n| + |c_0|} \right\} \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{4.6}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. Let $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$. If $P(z)$ has a zero on $|z| = 1$, then $m = \min_{|z|=1} |P(z)| = 0$ and the result follows from Theorem 2.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < 1$ so that $m > 0$.

Now, as $m \leq |P(z)|$ for $|z| = 1$, therefore, if λ is any complex number such that $|\lambda| < 1$, then

$$|m\lambda z^n| < |P(z)| \text{ for } |z| = 1. \tag{4.7}$$

Since, all the zeros of $P(z)$ lie in $|z| < 1$, it follows by Rouche's Theorem that all zeros of $P(z) - \lambda m z^n$ also lie in $|z| < 1$. Hence, by Theorem 2.1, we have for $|\alpha| \geq 1$ and for any $r > 0$,

$$\begin{aligned} \left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta}) - \lambda m n \alpha e^{i(n-1)\theta} \right|^r d\theta \right\}^{\frac{1}{r}} \;\; &\geq \;\; \frac{|\alpha| - 1}{2} \left\{ n + \frac{|c_n - \lambda m| - |c_0|}{|c_n - \lambda m| + |c_0|} \right\} \\ &\times \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) - \lambda m e^{in\theta} \right|^r d\theta \right\}^{\frac{1}{r}}. \end{aligned} \tag{4.8}$$

Since, for every λ with $|\lambda| < 1$, we have

$$|c_n - \lambda m| \geq |c_n| - m|\lambda|.$$

and because the function

$$\frac{x-|c_0|}{x+|c_0|} \tag{4.9}$$

is a non-decreasing function of x, we have

$$\frac{|c_n-\lambda m|-|c_0|}{|c_n-\lambda m|+|c_0|} \geq \frac{|c_n|-m|\lambda|-|c_0|}{|c_n|-m|\lambda|+|c_0|}.$$

Also by triangle inequality, we have for $|z|=1$,

$$\begin{aligned} |P(z)-\lambda m z^n| &\geq \big||P(z)|-|\lambda|m\big| \\ &= |P(z)|-|\lambda|m. \ [by\ (4.7)]. \end{aligned} \tag{4.10}$$

Applying the argument of (4.9) to the second factor and inequality (4.10) to the third factor of (4.8) respectively, we have

$$\begin{aligned} \left\{\int_0^{2\pi}\left|D_\alpha(P(e^{i\theta})-\lambda mn\alpha e^{i(n-1)\theta})\right|^r d\theta\right\}^{\frac{1}{r}} &\geq \frac{(|\alpha|-1)}{2}\left\{n+\frac{|c_n|-|\lambda|m-|c_0|}{|c_n|-|\lambda|m+|c_0|}\right\} \\ &\quad \times\left\{\int_0^{2\pi}\left(\left|P(e^{i\theta})\right|-|\lambda|m\right)^r d\theta\right\}^{\frac{1}{r}}. \end{aligned} \tag{4.11}$$

It is a simple consequence of Laguerre Theorem [9, p.52] on the polar derivative of polynomial that for every α with $|\alpha|\geq 1$, the polynomial

$$D_\alpha(P(z)-\lambda m z^n)=D_\alpha P(z)-\lambda mn\alpha z^{n-1} \tag{4.12}$$

has all its zeros in $|z|<1$. This implies that,

$$|D_\alpha P(z)|\geq mn|\alpha||z|^{n-1} \quad \text{for} \quad |z|\geq 1. \tag{4.13}$$

Now choosing the argument of λ suitably on the left hand side of (4.11) such that

$$\left|D_\alpha P(z)-\lambda mn\alpha z^{n-1}\right|=|D_\alpha P(z)|-mn|\lambda||\alpha| \quad \text{for} \quad |z|=1,$$

which is possible by (4.13), we get

$$\begin{aligned} \left\{\int_0^{2\pi}\left(\left|D_\alpha P(e^{i\theta})\right|-mn|\lambda||\alpha|\right)^r d\theta\right\}^{\frac{1}{r}} &\geq \frac{(|\alpha|-1)}{2}\left\{n+\frac{|c_n|-|\lambda|m-|c_0|}{|c_n|-|\lambda|m+|c_0|}\right\} \\ &\quad \times\left\{\int_0^{2\pi}\left(\left|P(e^{i\theta})\right|-|\lambda|m\right)^r d\theta\right\}^{\frac{1}{r}}. \end{aligned} \tag{4.14}$$

Put $|\lambda| = t$ in inequality (4.14), we get

$$\left\{\int_0^{2\pi}\left(\left|D_\alpha P(e^{i\theta})\right| - mnt|\alpha|\right)^r d\theta\right\}^{\frac{1}{r}} \geq \frac{(|\alpha|-1)}{2}\left\{n+\frac{|c_n|-tm-|c_0|}{|c_n|-tm+|c_0|}\right\} \times \left\{\int_0^{2\pi}\left(\left|P(e^{i\theta})\right| - tm\right)^r d\theta\right\}^{\frac{1}{r}}, \tag{4.15}$$

where $0 \leq t < 1$ and this completes the proof of Theorem 2.3.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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