Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4229-4247
https://doi.org/10.28919/jmcs/5829
ISSN: 1927-5307

A NEW UNIFICATION OF VARIOUS GENERALIZED METRIC SPACES WITH APPLICATIONS TO FIXED POINT THEORY

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Abstract. In this paper we introduce a new generalized metric space that extends JS metric space and many variants of extended b-metric spaces. We also give extension to some fixed point results for known contractions in this new setting. Some examples are presented to support the obtained results.

Keywords: fixed point; JS metric space; extended b-metric space; Banach contraction; Kannan contraction; Chatterjea contraction.

2010 AMS Subject Classification: 37C25, 47H10, 54H25.

1. Introduction

JS metric space introduced by Jleli and Samet in 2015 [1], is a generalized metric space in which the major modification was changing triangular inequality by this one :

(D_3) there exists $C > 0$ such that if $(x,y) \in X \times X$ and $\{x_n\} \subset X$ converges to x (i.e., $\lim_{n\to+\infty} D(x_n,x) = 0$), then $D(x,y) \leq C \limsup_{n\longrightarrow+\infty} D(x_n,y)$.

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Received April 07, 2021

They showed that this new concept recovers standard metric spaces, b-metric spaces, dislocated metric spaces and modular spaces with Fatou property. The authors obtained some remarkable results concerning Banach's contraction principle, fixed point theorem of Ciric quasi-contraction and a version of Ran-Reurings theorem in these spaces. Since then JS metric concept has inspired many authors, for instance RS metric space initiated by A.F.Roldán, L.de Hierro and N.Shahzad [2], subordinate semimetric space introduced by José Villa-Morales [3] and JS partial metric space introduced by M.Asim and M.Imdad [4].

On the other hand the notion of extended b-metric space has been introduced by Kamran et al. [5] where b-metric space has been extended by replacing the constant in the relaxed triangular inequality by two variables function or by three variables function as seen recently in the work of H. Aydi et al. [6].

Following this line of investigation we introduce a new generalization of metric space that recovers both JS metric space and many variants of extended b-metric space. We begin by recalling definition of JS generalized metric space. For every $x \in X$, let us define the set

$$C(D,X,x) = \left\{ \{x_n\} \subset X : \lim_{n\to+\infty} D(x_n,x) = 0 \right\}.$$

Definition 1.1. *We say that $D : X \times X \to [0,+\infty]$ is a JS metric on X if it satisfies the following conditions:*

(D_1) *for every $(x,y) \in X \times X$, we have $D(x,y) = 0 \Longrightarrow x = y$,*

(D_2) *for every $(x,y) \in X \times X$, we have $D(x,y) = D(y,x)$,*

(D_3) *there exists $C > 0$ such that if $(x,y) \in X \times X, \{x_n\} \in C(D,X,x)$, then $D(x,y) \leq C \limsup_{n\longrightarrow+\infty} D(x_n,y)$.*

Also we say the pair (X,D) is a JS metric space.

The extended b-metric space is defined as follows.

Definition 1.2. *Let X be a nonempty set and $\theta : X \times X \longrightarrow [1,+\infty[$ a real valued mapping. A function $d : X \times X \to [0,\infty)$ is said to be an extended b-metric if and only if for all $x,y,z \in X$ the following conditions are satisfied:*

$(d_{\theta 1})$ *$d(x,y) = 0$ if and only if $x = y$,*

$(d_{\theta 2})$ *$d(x,y) = d(y,x)$,*

$(d_{\theta 3})$ *$d(x,z) \leq \theta(x,z)[d(x,y) + d(y,z)]$.*

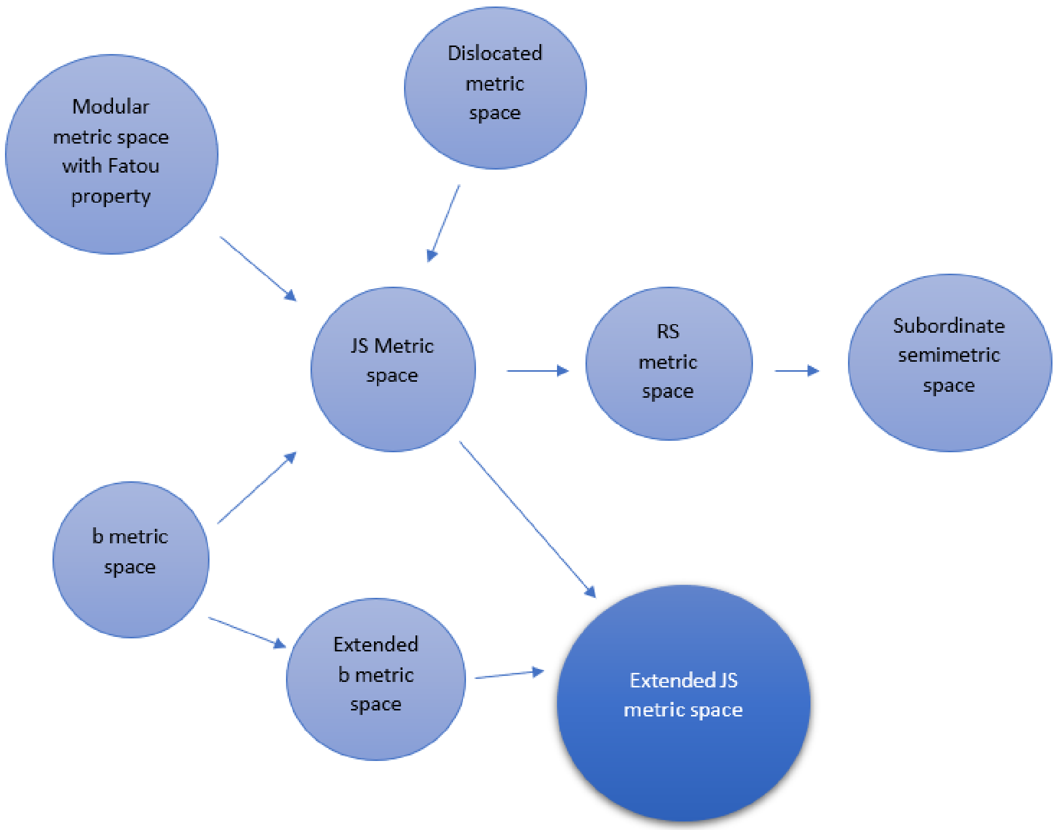

FIGURE 1. Connection between different GMS

The pair (X, d_θ) is called an extended b-metric space.

Remark 1.3.

- *If we take $\theta(x,z) = s$ ($s \geq 1$ a positive real) we obtain the definition of a b-metric space.*
- *If we replace the condition ($d_{\theta 3}$) by $d(x,z) \leq \theta(x,y)\, d(x,y) + \theta(y,z)\, d(y,z)$ we get the definition of controlled metric type space* [7].
- *If we replace ($d_{\theta 3}$) by $d(x,z) \leq \theta(x,y,z)\,[d(x,y) + d(y,z)]$ for all $x,y,z \in X$, where $\theta : X \times X \times X \longrightarrow [1,+\infty[$ is a three variables function we get the definition of extended b-metric space introduced by H. Aydi et al.* [6].

2. PRELIMINARIES

In the following we introduce the notion of extended JS metric space which is a generalization of both JS metric space and many variants of extended b-metric spaces.

Definition 2.1. *Let X be a non empty set and $\theta : X \times X \longrightarrow [1,+\infty[$ a real valued mapping. A function $D_\theta : X \times X \longrightarrow [0,+\infty]$ is called an extended JS metric if for all $x,y \in X$ it satisfies:*
$(D_{\theta 1}) D_\theta(x,y) = 0 \Longrightarrow x = y$,
$(D_{\theta 2}) D_\theta(x,y) = D_\theta(y,x)$,
$(D_{\theta 3}) D_\theta(x,y) \leq \theta(x,y) \limsup_{n\to\infty} D_\theta(x_n,y)$, *with* $\{x_n\} \in C(D,X,x)$.
The pair (X,D_θ) is called an extended JS metric space.

Remark 2.2. *If we take $\theta(x,y) = C$ for all $x,y \in X$($C \geq 1$ a positive real), we obtain the definition of a JS metric space.*

We mention that convergent sequences and Cauchy sequences can be introduced in a similar manner as in metric space.

Definition 2.3. *Let (X,D_θ) be an extended JS metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*

- *We say that $\{x_n\}$ converges to x if $\{x_n\} \in C(D_\theta,X,x)$ (i.e., $\lim_{n\to\infty} D_\theta(x_n,x) = 0$).*
- *We say that $\{x_n\}$ is Cauchy sequence if $\lim\limits_{m,n\to\infty} D_\theta(x_n,x_{n+m}) = 0$.*
- *(X,D_θ) is said to be complete if every Cauchy sequence in X is convergent to some element in X.*

Proposition 2.4. *Let (X,D_θ) be an extended JS metric space. Let $\{x_n\}$ and $x,y \in X$. If $\lim_{n\to\infty} D_\theta(x_n,x) = 0$ and $\lim_{n\to\infty} D_\theta(x_n,y) = 0$ then $x = y$.*

Proof. Let $\{x_n\}$ be a sequence that converges to both x and y in X.
We have $\lim_{n\to\infty} D_\theta(x_n,x) = 0$ and by axiom $(D_{\theta 3})$

$$D_\theta(x,y) \leq \theta(x,y) \limsup_{n\to\infty} D_\theta(x_n,y).$$

Since $\lim_{n\to\infty} D_\theta(x_n,y) = 0$, we get $D_\theta(x,y) = 0$.
Hence by $(D_{\theta 1})$ we have $x = y$ which guarantees the uniqueness of the limit. □

Remark 2.5. *Convergent sequences in extended JS metric space are not necessarily Cauchy sequences, the lack of this property is inherited from JS metric space as shown by T.Senapati et al. (see Example 2.3 in* [11]*).*

Now we present some examples of Extended JS metric space.

Example 1. *Let $X=[a,b]$ with $a<b$, c a real number such that $a<c<b$ and α,β two positive real numbers such that $\alpha<\beta$, let $D_\theta : X\times X \to \mathbb{R}$ be defined by*

$$D_\theta(x,y)=\begin{cases}\alpha|x-y| & \text{, If } x\neq c \text{ and } y\neq c\\ \beta|x-y| & \text{, If } x=c \text{ or } y=c\end{cases}$$

We claim that D_θ is an Extended JS metric over X with an adequate function θ. It is clear that the conditions $D_{\theta 1}$ and $D_{\theta 2}$ are satisfied, therefore we shall only check $D_{\theta 3}$ condition. We have the following inequalities

$$\alpha|x-y|\leq D_\theta(x,y)\leq\beta|x-y|,$$

from which we get the equivalence

$$\lim_{n\to\infty} D_\theta(x_n,x)=0\Leftrightarrow\lim_{n\to\infty}|x_n-x|=0.$$

- *Case 1: $x\neq c$ and $y\neq c$.*
  We have $D_\theta(x,y)=\alpha|x-y|$ and for every sequence $\{x_n\}$ that converges to x, there is an integer N large enough such that $x_n\neq c, \forall n\geq N$.
  Then we have
  $$D_\theta(x_n,y)=\alpha|x_n-y|, \forall n\geq N.$$
  And hence we obtain
  $$D_\theta(x,y)=\limsup_{n\to\infty} D_\theta(x_n,y).$$
  Consequently, in this case we take $\theta(x,y)=1$.
- *Case 2: $x\in X$ and $y=c$.*
  We have $D_\theta(x,y)=\beta|x-y|$ and $D_\theta(x_n,y)=\beta|x_n-y|, \forall n\in\mathbb{N}$.
  Clearly we get
  $$D_\theta(x,y)=\limsup_{n\to\infty} D_\theta(x_n,y).$$
  And we take $\theta(x,y)=1$.

- *Case 3: $x = c$ and $y \neq c$.*

  First we have $D_\theta(x,y) = \beta|x-y|$, let $\{x_n\}$ be a sequence that converges to x, we have the following three subcases :

  - *Subcase 1: If $x_n = c, \forall n \geq N$ for some integer N.*

    We have

    $$D_\theta(x_n,y) = \beta|x_n - y|, \forall n \geq N,$$

    then we get

    $$D_\theta(x,y) = \limsup_{n\to\infty} D_\theta(x_n,y).$$

  - *Subcase 2: If $x_n \neq c, \forall n \geq N$*

    We have

    $$D_\theta(x_n,y) = \alpha|x_n - y|, \forall n \geq N,$$

    then

    $$D_\theta(x,y) = \frac{\beta}{\alpha}\limsup_{n\to\infty} D_\theta(x_n,y).$$

  - *Subcase 3: If the sequence $\{x_n\}$ contains two subsequences $\{x_n^{'}\}$ and $\{x_n^{''}\}$ such that $x_n^{'} = c$ and $x_n^{''} \neq c$ $\forall n \in \mathbb{N}$.*

    In consequence, we obtain

    $$\alpha|x-y| \leq \beta|x-y| \leq \limsup_{n\to\infty} D_\theta(x_n,y).$$

  Hence, in these all three subcases we have

  $$D_\theta(x,y) \leq \frac{\beta}{\alpha}\limsup_{n\to\infty} D_\theta(x_n,y),$$

  therefore we take $\theta(x,y) = \frac{\beta}{\alpha}$.

To recapitulate we have

$$D_\theta(x,y) \leq \theta(x,y)\limsup_{n\to\infty} D_\theta(x_n,y), \forall x,y \in X \text{ and } (x_n) \in C(D_\theta, X, x) \neq \emptyset,$$

with the function θ defined by:

$$\theta(x,y) = \begin{cases} \dfrac{\beta}{\alpha} & \text{, If } x = c \text{ and } y \neq c \\ 1 & \text{, otherwise.} \end{cases}$$

Example 2. *Let (X_i, D_i), with $i \in \{1,2,\ldots,n\}$, be a family of JS metric spaces such that, $\forall i, j \in \{1,2,\ldots,n\} X_i \cap X_j = \emptyset$ and for each (X_i, D_i), let $C_i \geq 1$ be the constant forementioned in definition 1.1, set $X = \bigcup_{i=1}^{n} X_i$ and D_θ the mapping defined by:*

$$D_\theta(x,y) = \begin{cases} D_i(x,y) & \text{, If } x, y \in X_i \\ a_{i,j} & \text{, If } x \in X_i \text{ and } y \in X_j \text{ with } i \neq j \end{cases}$$

a_{ij} are real numbers such that for all $i, j \in \{1,2,\ldots,n\}$ with $i \neq j$, $a_{ij} = a_{ji}$ and $a_{ij} > 0$.
D_θ is an extended JS metric over X with $\theta : X \times X \to [1,+\infty[$ defined by

$$\theta(x,y) = \begin{cases} C_i & \text{, If } x, y \in X_i \\ 1 & \text{, If } x \in X_i \text{ and } y \in X_j \text{ with } i \neq j \end{cases}$$

Clearly $D_{\theta 1}$ and $D_{\theta 2}$ are satisfied, so we have to check the condition $(D_{\theta 3})$.
Suppose that $x \in X_i$, for some $i \in \{1,2,\ldots,n\}$.
First if there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} D_\theta(x_n, x) = 0$.
Then for ε such that $0 < \varepsilon < \min_{i,j} a_{ij}$.
There exists $N \in \mathbb{N}$ such that $D_\theta(x_n, x) < \varepsilon < \min_{i,j} a_{ij}$ for all $n \geq N$.
Which implies $D_\theta(x_n, x) = D_i(x_n, x)$ and $x_n \in X_i$ for all $n \geq N$.
Now let $y \in X$, we have to discuss two cases:

- *Case 1 : If $y \in X_i$.*
  We have $D_\theta(x,y) = D_i(x,y)$ and $D_\theta(x_n,y) = D_i(x_n,y)$ for all $n \geq N$.
  Hence we get
  $$D_\theta(x,y) \leq C_i \limsup_{n\to\infty} D_\theta(x_n,y).$$
- *Case 2 : If $y \in X_j$ with $i \neq j$.*
  In this case, we have $D_\theta(x,y) = D_\theta(x_n,y) = a_{ij}$ for all $n \geq N$.
  Thus $D_\theta(x,y) = \limsup_{n\to\infty} D_\theta(x_n,y)$.

Example 3. *Consider $I =]0,1]$, let $(I_i)_{i\in\mathbb{N}}$ be the partition of I defined by $I_i =]\frac{1}{2^{i+1}}, \frac{1}{2^i}]$, for each $i \in \mathbb{N}$ and c_{ij} are real numbers such that for all $i, j \in \{1,2,\ldots,n\}$ with $i \neq j$, $c_{ij} = c_{ji}$ and*

$c_{ij} > 0$. *The mapping* D_θ *defined by*

$$D_\theta(x,y) = \begin{cases} |x-y| & , \text{If } x,y \in I_i, x \neq \frac{3}{2^{i+2}} \text{ and } y \neq \frac{3}{2^{i+2}} \\ 2^i|x-y| & , \text{If } x,y \in I_i, x = \frac{3}{2^{i+2}} \text{ or } y = \frac{3}{2^{i+2}} \\ c_{i,j} & , \text{If } x \in I_i \text{ and } y \in I_j \text{ with } i \neq j \end{cases}$$

is an extended JS metric over I. Let $\{x_n\}$ *be a sequence that converges to x.*

- *Case 1: If* $x,y \in I_i$.

  We proceed as in example 1, we have similar result, take $a = \frac{1}{2^{i+1}}, b = \frac{1}{2^i}, \alpha = 1, \beta = 2^i$ *and* $c = \frac{3}{2^{i+2}}$.

  Consequently, in this case we obtain

$$\theta(x,y) = \begin{cases} 2^i & , \text{If } x,y \in I_i, x = \frac{3}{2^{i+2}} \text{ and } y \neq \frac{3}{2^{i+2}} \\ 1 & , \text{Otherwise.} \end{cases} \tag{1}$$

- *Case 2: If* $x \in I_i$ *and* $y \in I_j$ *with* $i \neq j$.

  We proceed as in example 2, we have this immediate implication

$$\lim_{n\to\infty} D_\theta(x_n,x) = 0 \Longrightarrow x_n \in I_i, \forall n \geq N.$$

  Hence we have $D_\theta(x,y) = c_{i,j}$ *and* $D_\theta(x_n,y) = c_{ij}$, $\forall n \geq N$.

  In this case we obtain

$$D_\theta(x,y) = \limsup_{n\to\infty} D_\theta(x_n,y). \tag{2}$$

Therefore from (1) and (2), we get

$$\theta(x,y) = \begin{cases} 2^i & , \text{If } x,y \in I_i, x = \frac{3}{2^{i+2}} \text{ and } y \neq \frac{3}{2^{i+2}} \\ 1 & , \text{Otherwise.} \end{cases} \tag{3}$$

Proposition 2.6. *Any extended b-metric on X is an extended JS metric on X.*

Proof. Let d_θ be an extended metric on X, in order to show that d_θ is an extended JS metric, we only need to check that d_θ satisfies $(D_{\theta 3})$ axiom.

Let $x,y \in X$ and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} d_\theta(x_n,x) = 0$.

We have $d_\theta(x,y) \leq \theta(x,y)[d_\theta(x,x_n) + d_\theta(x_n,y)]$.

It follows that $d_\theta(x,y) \le \theta(x,y)\limsup_{n\to\infty} d_\theta(x_n,y)$.

Consequently an extended b-metric on X is an extended JS metric on X. $\square$

Proposition 2.7. *Let d_θ be a controlled metric type on X with $\theta : X\times X \longrightarrow [1,+\infty[$ satisfying the following conditions:*

(1) θ is continuous in the first variable,

(2) $\limsup\limits_{n\longrightarrow+\infty}\theta(x,x_n) < \infty$, for every $x \in X$ and $\{x_n\} \in C(D,X,x)$.

Then d_θ is an extended JS metric on X.

Proof. We only need to check that d_θ satisfies $(D_{\theta 3})$ axiom.

Let $x,y \in X$ and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} d_\theta(x_n,x) = 0$.

We have $d_\theta(x,y) \le \theta(x,x_n)d_\theta(x,x_n) + \theta(x_n,y)d_\theta(x_n,y)$.

Which yields $d_\theta(x,y) \le \theta(x,y)\limsup_{n\to\infty} d_\theta(x_n,y)$.

Consequently, the controlled metric type d_θ is an extended JS metric on X. $\square$

Remark 2.8. *If the function θ is continuous in the second variable, assume that $\limsup\limits_{n\longrightarrow+\infty}\theta(x_n,x) < \infty$, for every $x \in X$, then we have*

$$d_\theta(y,x) \le \theta(y,x_n)d_\theta(y,x_n) + \theta(x_n,x)d_\theta(x_n,x).$$

Hence by symmetry of d_θ, we get $d_\theta(x,y) \le \tilde{\theta}(x,y)\limsup_{n\to\infty} d_\theta(x_n,y)$ such that $\tilde{\theta}(x,y) = \theta(y,x)$, so we conclude that (X,d_θ) is an extended JS metric space with related function $\tilde{\theta}$.

In a similar way we have the following result.

Proposition 2.9. *Let d_θ be a extended b-metric on X in the sense of by H. Aydi et al., for which the mapping $\theta : X\times X\times X \longrightarrow [1,+\infty[$ is continuous in the second variable, then d_θ is an extended JS metric on X.*

Proposition 2.10. *Let (X,D_θ) be an extended JS metric space and $x \in X$, if there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} D_\theta(x_n,x) = 0$, then $D_\theta(x,x) = 0$.*

Proof. Let $x \in X$ and $\{x_n\}$ be a sequence such that $\lim_{n\to\infty} D_\theta(x_n,x) = 0$.
By $(D_{\theta 3})$ we have $D_\theta(x,x) \leq \theta(x,x) \limsup_{n\to\infty} D_\theta(x_n,x)$.
We get $D_\theta(x,x) = 0$. □

Remark 2.11. *This proposition show that the condition $(D_{\theta 3})$ should be verified only for elements $x \in X$ such that $D_\theta(x,x) = 0$.*

3. Main Results

In the following we state our first result, concerning a version of Banach contraction principle in this new setting.

Definition 3.1. *Let (X,D_θ) be an extended JS metric space. The mapping $T : X \longrightarrow X$ is called a k-contraction mapping if there exists $k \in [0,1)$ such that*

$$D_\theta(Tx,Ty) \leq kD_\theta(x,y), \tag{4}$$

for all $x,y \in X$.

Proposition 3.2. *Let (X,D_θ) be an extended JS generalized metric space and $T : X \longrightarrow X$ a k-contraction for some $k \in [0,1)$. Then any fixed point $\omega \in X$ of T satisfies:*

$$D_\theta(\omega,\omega) < \infty \Rightarrow D_\theta(\omega,\omega) = 0.$$

Proof. The proof is obvious. □

Theorem 3.3. *Let (X,D_θ) be a complete extended JS metric space and $T : X \longrightarrow X$ a k-contraction mapping.*
If there exists $x_0 \in X$ such that $\delta_0 = \sup_{n\in\mathbb{N}} D_\theta(x_0,T^n x_0) < \infty$, then the sequence $\{T^n x_0\}$ converges to $\omega \in X$ a fixed point of T. Moreover if $\omega' \in X$ is a fixed point of T such that $D_\theta(\omega,\omega') < \infty$, then $\omega = \omega'$.

Proof. Let $\{x_n\} \subset X$ be the sequence defined by $x_0 \in X$ and $x_n = T^n x_0$ (for every $n \in \mathbb{N}$).
Then for $n,m \in \mathbb{N}$ such that $n \geq m$, we have $D_\theta(x_n,x_m) \leq kD_\theta(x_{n-1},x_{m-1})$.
By induction we get

$$D_\theta(x_n,x_m) \leq k^m D_\theta(x_{n-m},x_0) \leq k^m \delta_0. \tag{5}$$

We have $k \in [0,1)$ then $\lim_{n,m\to\infty} D_\theta(x_n,x_m) = 0$.

Since (X, D_θ) is complete, it follows that the sequence $\{x_n\}$ converges to some $\omega \in X$.

Now we shall prove that ω is a fixed point of T.

From (4) we have

$$D_\theta(x_{n+1}, T\omega) \leq kD_\theta(x_n, \omega).$$

As $n \to +\infty$ we obtain

$$\lim_{n\to\infty} D_\theta(x_{n+1}, T\omega) = 0.$$

By uniqueness of the limit we get $\omega = T\omega$, so we deduce that ω is a fixed point of T.

To prove the uniqueness of fixed point, suppose that $\omega^{'}$ is another fixed point of T such that $D_\theta(\omega, \omega^{'}) < \infty$, then from (4) we get

$$D_\theta(\omega, \omega^{'}) = D_\theta(T\omega, T\omega^{'}) \leq kD_\theta(\omega, \omega^{'}).$$

Since $k \in [0,1)$ we have $D_\theta(\omega, \omega^{'}) = 0$ this implies that $\omega = \omega^{'}$.

□

From Theorem 2.3 we obtain an improved version of Banach contraction principle in the context of extended b-metric space (see[5]), here the condition $\lim_{n,m\longrightarrow\infty} \theta(x_n, x_m) < \frac{1}{k}$, with $x_n = T^n x_0$, $n \in \mathbb{N}$ is removed.

Corollary 3.4. *Let (X, d_θ) be a complete extended b-metric space. Suppose that $T : X \longrightarrow X$ satisfy*

$$d_\theta(Tx, Ty) \leq kd_\theta(x, y), \textit{ for all } x, y \in X,$$

for some $k \in [0,1)$. If there exists $x_0 \in X$ such that $\delta_0 = \sup_{n\in\mathbb{N}} D_\theta(x_0, T^n x_0) < \infty$. Then T has precisely one fixed point u. Moreover, $T^n x_0 \longrightarrow u$.

In order to prove Theorem 3.7 and Theorem 3.14 we need the following Lemma (see [9]).

Lemma 3.5. *If $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 0$ and $\lambda \in [0,1)$, then for any sequence of positive numbers $\{d_n\}$ satisfying $d_{n+1} \leq \lambda d_n + b_n$, for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} d_n = 0$.*

Definition 3.6. *Let (X,D_θ) be an extended JS metric space. The mapping $T : X \longrightarrow X$ is said to be Banach-Kannan-Ciric quasi-contraction (BKC quasi-contraction), if there exists $k \in [0,1)$ such that :*

$$D_\theta(Tx,Ty) \leq k \max\{D_\theta(x,y), D_\theta(Tx,x), D_\theta(Ty,y)\},\ \forall x,y \in X. \tag{6}$$

In what follows, we adopt the following notation, for every $x \in X$, let

$$\delta(D_\theta,T,x) = \sup\left\{D_\theta(T^i x, T^j x) | i,j \in \mathbb{N}\right\}.$$

Theorem 3.7. *Let (X,D_θ) be a complete extended JS metric space and $T : X \longrightarrow X$ a BKC quasi-contraction. If there exists $x_0 \in X$ such that $\delta(D_\theta,T,x_0) < \infty$, then the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover if $\theta(\omega,T\omega) < \frac{1}{k}$ and $D_\theta(\omega,T\omega) < \infty$, then ω is a fixed point of T.*

Proof. Let $\{x_n\} \subset X$ be the sequence defined by $x_0 \in X$ and $x_n = T^n x_0$ (for every $n \in \mathbb{N}$).
We consider the sequence $\left\{d_{n,p}\right\}$ defined by $d_{n,p} = D_\theta(x_n,x_{n+p})$, for all $n,p \in \mathbb{N}$).
If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$, then $\omega = x_{n_0}$ forms a fixed point for T and the proof is completed.
Hence, we assume that $d_{n,1} > 0$ for all $n \in \mathbb{N}$. From (6) we have

$$d_{n+1,1} \leq k \max\left\{d_{n+1,1}, d_{n,1}\right\}.$$

If $\max\left\{d_{n+1,1}, d_{n,1}\right\} = d_{n+1,1}$, then we get

$$d_{n+1,1} \leq k d_{n+1,1}.$$

Since $d_{n,1} > 0$ for all $n \in \mathbb{N}$, we deduce that $k \geq 1$ which is contradiction.
Consequently for all $n \in \mathbb{N}$ we have

$$\max\left\{d_{n+1,1}, d_{n,1}\right\} = d_{n,1} \text{ and } d_{n+1,1} \leq k d_{n,1}.$$

By induction we obtain

$$d_{n,1} \leq k^n d_{0,1}.$$

We have $k \in [0,1)$, hence by taking the limit $n \to +\infty$, we get

$$\lim_{n\to\infty} d_{n,1} = 0. \tag{7}$$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence.

From (6) we have

$$d_{n+1,p} \leq k \max\left\{d_{n,p}, d_{n,1}, d_{n+p,1}\right\}$$

Hence, we obtain

$$d_{n+1,p} \leq k d_{n,p} + k\left[d_{n,1} + d_{n+p,1}\right]. \tag{8}$$

Using (7) we have

$$\lim_{n\to\infty} d_{n+p,1} = \lim_{n\to\infty} d_{n,1} = 0 \text{ and } k \in \left]0,1\right[.$$

Therefore from Lemma (3.5), we have

$$\lim_{n\to\infty} d_{n,p} = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence. As (X, D_θ) is complete, it follows that there exists $\omega \in X$ such that

$$\lim_{n\to\infty} D_\theta(x_n, \omega) = 0. \tag{9}$$

We have from (6)

$$D_\theta(x_{n+1}, T\omega) \leq k \max\left\{D_\theta(x_n,\omega), D_\theta(x_{n+1}, x_n), D_\theta(T\omega, \omega)\right\}. \tag{10}$$

Taking into account (7) and (9) let $n \to +\infty$ in above inequality, we obtain that

$$\limsup_{n\to+\infty} D_\theta(x_{n+1}, T\omega) \leq k D_\theta(T\omega, \omega). \tag{11}$$

By $(D_{\theta 3})$ we have

$$D_\theta(\omega, T\omega) \leq \theta(\omega, T\omega) \limsup_{n\to+\infty} D_\theta(x_{n+1}, T\omega). \tag{12}$$

From (11) and (12) we get

$$D_\theta(\omega, T\omega) \leq \theta(\omega, T\omega) k D_\theta(T\omega, \omega) \tag{13}$$

We have $\theta(\omega, T\omega)k < 1$ and $D_\theta(\omega, T\omega) < \infty$, hence we get $D_\theta(T\omega, \omega) = 0$. It follows that ω is a fixed point of T. □

Proposition 3.8. *Let (X, D_θ) be an extended JS metric space and $T : X \longrightarrow X$ a BKC quasi-contraction.*

(1) If $\omega \in X$ is a fixed point of T such that $D_\theta(\omega,\omega) < \infty$ then $D_\theta(\omega,\omega) = 0$.

(2) If $\omega, \omega' \in X$ are two fixed points of T such that $\max\left\{D_\theta(\omega,\omega'), D_\theta(\omega,\omega), D_\theta(\omega',\omega')\right\} < \infty$ then $\omega = \omega'$.

Proof. (1) Let $\omega \in X$ be a fixed point of T satisfying $D_\theta(\omega,\omega) < \infty$.

We have from (6)

$$D_\theta(\omega,\omega) = D_\theta(T\omega,T\omega) \le k\max\left\{D_\theta(\omega,\omega), D_\theta(T\omega,\omega), D_\theta(T\omega,\omega)\right\}.$$

That is $D_\theta(\omega,\omega) \le kD_\theta(\omega,\omega)$.

Since $D_\theta(\omega,\omega) < \infty$ and $0 < k < 1$ we get $D_\theta(\omega,\omega) = 0$.

(2) Let $\omega, \omega' \in X$ be two fixed point of T such that

$$\max\left\{D_\theta(\omega,\omega'), D_\theta(\omega,\omega), D_\theta(\omega',\omega')\right\} < \infty.$$

Using (6) we have

$$\begin{aligned} D_\theta(\omega,\omega') = D_\theta(T\omega,T\omega') &\le k\max\left\{D_\theta(\omega,\omega'), D_\theta(T\omega,\omega), D_\theta(T\omega',\omega')\right\}, \\ &\le k\max\left\{D_\theta(\omega,\omega'), D_\theta(\omega,\omega), D_\theta(\omega',\omega')\right\}. \end{aligned} \tag{14}$$

Taking into account the assertion (1), we get $D_\theta(\omega,\omega') \le kD_\theta(\omega,\omega')$.

Since $D_\theta(\omega,\omega') < \infty$ and $0 < k < 1$ we get $D_\theta(\omega,\omega') = 0$, which implies that $\omega = \omega'$. □

Corollary 3.9. *Let (X,D) be a complete JS metric space, and let $T : X \to X$ be a BKC contraction for some constant $k \in [0, \inf\{1, \frac{1}{C}\})$. If there exists an element $x_0 \in X$ such that $\delta(D,T,x_0) < \infty$, then the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(\omega,T\omega) < \infty$, then ω is a fixed point of T. Moreover, for each fixed point ω' of T in X such that $D(\omega',\omega') < \infty$, we have $\omega' = \omega$.*

Proof. Immediate consequence by taking $\theta(x,y) = C$ for all $x,y \in X$. □

Now, by considering the inequality

$$\frac{k}{2}\left(D(Tx,x) + D(Ty,y)\right) \le k\max\left\{D(x,y), D(Tx,x), D(Ty,y)\right\}, \forall x,y \in X.$$

We obtain fixed point theorem for Kannan mapping in extended JS metric space.

Corollary 3.10. *Let (X,D_θ) be a complete extended JS metric space and let $T:X\to X$ be a Kannan contraction, that is a mapping satisfying:*

$$D_\theta(Tx,Ty)\le\frac{k}{2}\left(D_\theta(Tx,x)+D_\theta(Ty,y)\right),\forall x,y\in X, \tag{15}$$

for some constant $k\in[0,1)$. If there exists $x_0\in X$ such that $\delta(D_\theta,T,x_0)<\infty$ then the sequence $\{T^nx_0\}$ converges to some $\omega\in X$. Moreover if $\theta(\omega,T\omega)<\frac{1}{k}$ and $D_\theta(T\omega,\omega)<\infty$ then ω is a fixed point of T.

Using a direct proof, this result can be improved as follows.

Corollary 3.11. *Let (X,D_θ) be a complete extended JS metric space and let $T:X\to X$ be a Kannan contraction for some constant $k\in[0,1)$. If there exists $x_0\in X$ such that $\delta(D_\theta,T,x_0)<\infty$ then the sequence $\{T^nx_0\}$ converges to some $\omega\in X$. Moreover if $\theta(\omega,T\omega)<\frac{2}{k}$ and $D_\theta(T\omega,\omega)<\infty$, then ω is a fixed point of T.*

Proof. By similar way as in the proof of Theorem 2.7 the sequence $\left\{d_{n,1}\right\}$ converges to 0. On the other hand, we have

$$\begin{aligned}D_\theta(T^nx_0,T^mx_0)&\le\frac{k}{2}\left(D_\theta(T^nx_0,T^{n-1}x_0)+D_\theta(T^mx_0,T^{m-1}x_0)\right),\\&\le\frac{k}{2}\left(d_{n-1,1}+d_{m-1,1}\right).\end{aligned} \tag{16}$$

Now, let $n\to+\infty$ we get $\lim_{n\to\infty}D_\theta(T^nx_0,T^mx_0)=0$, which proves that $\{T^nx_0\}$ is a Cauchy sequence, hence it converges to some $\omega\in X$.

From (15) we have

$$D_\theta(T^nx_0,T\omega)\le\frac{k}{2}\left(D_\theta(T^nx_0,T^{n-1}x_0)+D_\theta(T\omega,\omega)\right).$$

It follows that

$$\limsup_{n\to\infty}D_\theta(T^nx_0,T\omega)\le\frac{k}{2}D_\theta(T\omega,\omega). \tag{17}$$

By the condition $(D_{\theta 3})$, we have

$$\begin{aligned}D_\theta(\omega,T\omega)&\le\theta(\omega,T\omega)\limsup_{n\to\infty}D_\theta(T^nx_0,T\omega),\\&\le\frac{k}{2}\theta(\omega,T\omega)D_\theta(T\omega,\omega).\end{aligned} \tag{18}$$

Since $\theta(\omega,T\omega) < \frac{2}{k}$ and $D_\theta(T\omega,\omega) < \infty$ we obtain $D_\theta(T\omega,\omega) = 0$, which ends the proof.

□

From the previous corollary we get the following result (Theorem 3.2) established by Y. Elkouch and E. Marhrani in [10].

Corollary 3.12. *Let (X,D) be a complete JS metric space and let $T : X \to X$ be a Kannan contraction for some constant $k \in [0,\inf\{1,\frac{2}{C}\})$. If there exists an element $x_0 \in X$such that $\delta(D,T,x_0) < \infty$, then the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover, if $D(\omega,T\omega) < \infty$, then ω is a fixed point of T. Moreover, for each fixed point $\omega^{'}$ of T in X such that $D(\omega^{'},\omega^{'}) < \infty$, we have $\omega^{'} = \omega$.*

To emphasize the strength of our approach and to highlight the usefulness of this new concept, we present an example based on a JS metric space given by E.Karapinar et al. (see [12]).

Example 4. *Let $X = [0,1] \cup \{2\}$ let $D_\theta : X \times X \to [0,\infty]$ be the function*

$$D_\theta(x,y) = \begin{cases} 10 & \text{, if either } (x,y) = (0,2) \text{ or } (x,y) = (2,0), \\ |x-y| & \text{, otherwise.} \end{cases}$$

According to E.Karapinar et al., (X,D_θ) is a complete JS metric space with $C = 5$, while in our approach (X,D_θ) is seen as a complete extended JS metric space with the function $\theta : X \times X \to [1,\infty)$ defined by:

$$\theta(x,y) = \begin{cases} 5 & \text{, If } x = 0 \text{ and } y = 2, \\ 1 & \text{, otherwise.} \end{cases}$$

We consider the mapping $T : X \to X$ defined by: $Tx = \frac{x}{4}$.
It is easy to see that the mapping T satisfy (15) with $k = \frac{2}{3} \in [0,1)$. Since D_θ is bounded, it follows that all conditions of Corollary 2.12 are satisfied except that $k > \frac{2}{C}$, so the Corollary 2.12 is not applicable and fails to guarantees the existence of fixed point whereas all conditions of Corollary 2.11 are satisfied, in lieu to check $Ck < 2$, we need to check that $\theta(0,T0)k < 2$ which is true and T has 0 as a unique fixed point.

Definition 3.13. *Let (X,D_θ) be an extended JS metric space. The mapping $T : X \longrightarrow X$ is said to be Banach-Chatterjea-Ciric quasi-contraction (BCC quasi-contraction) if there exists $k \in [0,1)$ such that:*

$$D_\theta(Tx,Ty) \leq k\max\{D_\theta(x,y), D_\theta(Tx,y), D_\theta(Ty,x)\},\ \forall x,y \in X. \tag{19}$$

Theorem 3.14. *Let (X,D_θ) be a complete extended JS metric space and $T : X \longrightarrow X$ a BCC quasi-contraction. If there exists $x_0 \in X$ such that $\delta(D_\theta,T,x_0) < \infty$, then the sequence $\{T^n x_0\}$ converges to some $\omega \in X$. Moreover if $D(x_0,T\omega) < \infty$ then ω is a fixed point of T.*

Proof. Let $n \in \mathbb{N}(n \geq 1)$, from (19) we have for all $i,j \in \mathbb{N}$,

$$\begin{aligned} D_\theta(T^{n+i}x_0, T^{n+j}x_0) \leq k\max\{&D_\theta(T^{n+i-1}x_0, T^{n+j-1}x_0), \\ &D_\theta(T^{n+i}x_0, T^{n+j-1}x_0), D_\theta(T^{n+i-1}x_0, T^{n+j}x_0)\}. \end{aligned} \tag{20}$$

Hence, we have

$$\delta(D_\theta,T,T^n x_0) \leq k\delta(D_\theta,T,T^{n-1}x_0).$$

Which implies

$$\delta(D_\theta,T,T^n x_0) \leq k^n\delta(D_\theta,T,x_0).$$

Using the above inequality, for every $n,m \in \mathbb{N}$ we have

$$D_\theta(T^n x_0, T^{n+m}x_0) \leq \delta(D_\theta,T,T^n x_0) \leq k^n\delta(D_\theta,T,x_0).$$

Since $\delta(D_\theta,T,x_0) < \infty$ and $k \in]0,1[$, we get

$$\lim_{n,m\to\infty} D_\theta(T^n x_0, T^{n+m}x_0) = 0,$$

which implies that $\{T^n x_0\}$ is a Cauchy sequence. Since (X,D_θ) is complete, there exists some $\omega \in X$ such that $\{T^n x_0\}$ is convergent to $\omega \in X$.

From (19) we have

$$D_\theta(x_{n+1},T\omega) \leq k\max\{D_\theta(x_n,\omega), D_\theta(x_{n+1},\omega), D_\theta(x_n,T\omega)\}, \tag{21}$$

which yields

$$D_\theta(x_{n+1},T\omega) \leq k\left(D_\theta(x_n,\omega) + D_\theta(x_{n+1},\omega)\right) + kD_\theta(x_n,T\omega). \tag{22}$$

We have $D(x_0, T\omega) < \infty$ and using (22) by induction, we get

$$D_\theta(x_n, T\omega) < \infty, \text{ for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} D_\theta(x_n, \omega) = 0$ and $k \in [0,1)$.

Using Lemma (3.5) we get $\lim_{n\to\infty} D_\theta(x_n, T\omega) = 0$.

Hence, by uniqueness of the limit we have $T\omega = \omega$.

□

Proposition 3.15. *Let (X, D_θ) be an extended JS metric space and $T : X \longrightarrow X$ a BCC quasi-contraction.*

(1) If $\omega \in X$ is a fixed point of T such that $D_\theta(\omega, \omega) < \infty$ then $D_\theta(\omega, \omega) = 0$.
(2) If $\omega, \omega' \in X$ are two fixed points of T such that $D_\theta(\omega, \omega') < \infty$, then $\omega = \omega'$.

Proof. (1) Let $\omega \in X$ be a fixed point of T satisfying $D_\theta(\omega, \omega) < \infty$.

We have from (19)

$$D_\theta(\omega, \omega) = D_\theta(T\omega, T\omega) \le k \max\{D_\theta(\omega, \omega), D_\theta(T\omega, \omega), D_\theta(T\omega, \omega)\}.$$

That is $D_\theta(\omega, \omega) \le kD_\theta(\omega, \omega)$.

Since $D_\theta(\omega, \omega) < \infty$ and $0 < k < 1$, we get $D_\theta(\omega, \omega) = 0$.

(2) Let $\omega, \omega' \in X$ be two fixed point of T such that $D_\theta(\omega, \omega') < \infty$.

Using (19), we have

$$D_\theta(\omega, \omega') = D_\theta(T\omega, T\omega') \le k \max\left\{D_\theta(\omega, \omega'), D_\theta(T\omega, \omega'), D_\theta(T\omega', \omega)\right\}. \tag{23}$$

It follows that $D_\theta(\omega, \omega') \le kD_\theta(\omega, \omega')$.

We have $D_\theta(\omega, \omega') < \infty$ and $0 \le k < 1$, which implies that $D_\theta(\omega, \omega') = 0$, then we get $\omega = \omega'$.

□

4. CONCLUSION

In this paper, a new Extended JS metric is introduced and some fixed point results for known contractions in this new setting are proved, some examples are presented to support the obtained results. We also show that Extended JS metric space generalize and unify various generalized metric spaces and one more reason to introduce such a space, is to improve fixed point results that use conditions related to the constant C, which is defined and fixed for all $x, y \in X$ over a JS metric space, while in this new approach we gain more flexibility via the θ function as illustrated by example 3.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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