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ON ROUGH CONTINUITY

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Abstract. The correlation of rough set theory with topology has been a captivating field of research since the inception of this relatively new theory. However, there has not been much study related to rough topology. In the present paper, the topological definition of continuity of a function is extended to rough topological spaces. The concept of rough continuity is defined and the properties are explored.

Keywords: approximation space; rough continuity; rough open set; rough set; rough topology.

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1. Introduction

At present, the rough set theory, initiated by Zdzislaw Pawlak in 1982 [12], has become a promising realm of research in various theoretical and applicational perspectives [3, 4, 5, 22, 25]. From the beginning itself, this theory has been constantly being correlated with topology in many ways. Several authors studied the topological properties of rough set approximations [2, 7, 8, 26, 10, 14, 15, 23]. A few others defined rough set approximations on topological spaces [1, 9, 17, 18]. The topological framework of rough sets has provided a solid basis for information processing and knowledge discovery [17, 20, 21].

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Even though the topological properties of rough set approximations has been investigated extensively, only a few studies have been conducted on the concept of rough topology. Q. Wu et al. [21] used a metric to define rough topology on a rough set and then generalized it to topological spaces. M. L. Thivagar et al. [20] defined rough topology on a subset of X as the family consisting of the universal set, the null set, the lower and upper approximations and the boundary region of the subset. B. P. Mathew and S. J. John [11] defined a pair of topologies of exact subsets of the lower and upper approximations of a rough set as a rough topology on that rough set. M. Ravindran and A. J. Divya [16] studied compactness, connectedness and the separation axioms on such rough topological spaces. But neither of them considered rough topology as a rough subset of the power set of X. In [19], the present authors studied the approximations of subfamilies of $P(X)$ and proposed a new definition of $\theta_{\approx}$-rough topology on an approximation space using the rough subsets of the extended approximation space. Analogous to the definition of a rough set, which is a pair of subsets of X, a rough topology is defined as a pair of subfamilies of the power set $P(X)$.

In this paper, the rough image and inverse rough image of rough sets under a function are defined and the properties are explored. Further, the concept of $\theta_{\approx}$-rough continuity is defined in line with the topological definition of continuity of a function. Moreover, it is proved that the discrete rough topology on the approximation space makes each function rough continuous. The organization of the paper is as follows: section 2 presents some of the basic definitions and propositions related to rough topological spaces, section 3 introduces $\theta_{\approx}$-rough continuous functions and discusses its properties and section 4 gives the conclusion.

2. Preliminaries

Let X be a non-empty set of objects and θ be an equivalence relation on X.

Definition 2.1. [12] The pair (X,θ) is called an *approximation space*. The *lower approximation* and *upper approximation* of $A \subseteq X$ defined by θ are given by $\underline{\theta}(A) = \{u \in X : [u]_\theta \subseteq A\}$ and $\overline{\theta}(A) = \{u \in X : [u]_\theta \cap A \neq \phi\}$ respectively, where $[u]_\theta$ represents the equivalence class containing u.

Definition 2.2. [12] The pair $(A,B) \in P(X) \times P(X)$ is called a *rough set* on (X,θ) if and only if $\exists H \subseteq X$ such that $\underline{\theta}(H) = A$ and $\overline{\theta}(H) = B$.

A rough set is denoted by $\theta(A) = \langle \underline{\theta}(A), \overline{\theta}(A) \rangle$. The collection of all rough sets on (X,θ) is denoted by $RS(X)$.

Definition 2.3. [12] The relation $\theta_{\approx}$ on the power set $P(X)$ given by $(A,B) \in \theta_{\approx}$ if and only if $\underline{\theta}(A) = \underline{\theta}(B)$ and $\overline{\theta}(A) = \overline{\theta}(B)$, $\forall A, B \in P(X)$ is called the *rough equality relation* on $P(X)$.

Proposition 2.4. [12] *The relation $\theta_{\approx}$ is an equivalence relation on $P(X)$.*

Definition 2.5. [13] The rough sets $\theta(A) = \theta(B)$ if and only if $\underline{\theta}(A) = \underline{\theta}(B)$ and $\overline{\theta}(A) = \overline{\theta}(B)$

Definition 2.6. [13] A rough set $\theta(A)$ is *contained in* $\theta(B)$ if $\underline{\theta}(A) \subseteq \underline{\theta}(B)$ and $\overline{\theta}(A) \subseteq \overline{\theta}(B)$

Definition 2.7. [24] The *rough union* and *intersection* of the rough sets $\theta(A)$ and $\theta(B)$ are respectively defined as

$$\theta(A) \Cup \theta(B) = \langle \underline{\theta}(A) \cup \underline{\theta}(B), \overline{\theta}(A) \cup \overline{\theta}(B) \rangle \tag{1}$$

$$\theta(A) \Cap \theta(B) = \langle \underline{\theta}(A) \cap \underline{\theta}(B), \overline{\theta}(A) \cap \overline{\theta}(B) \rangle \tag{2}$$

Definition 2.8. [24] The *rough complement* of $\langle \underline{\theta}(A), \overline{\theta}(A) \rangle$ is given by

$$\langle \underline{\theta}(A), \overline{\theta}(A) \rangle^{\complement} = \langle (\overline{\theta}(A))^{C}, (\underline{\theta}(A))^{C} \rangle = \theta(A^{C})$$

Definition 2.9. [19] The $\theta_{\approx}$ -lower and upper approximations of the sub family $\mathcal{A} \subseteq P(X)$ are respectively given by

$$\underline{\theta_{\approx}}(\mathcal{A}) = \{H \in P(X) : [H]_{\theta_{\approx}} \subseteq \mathcal{A}\} \tag{3}$$

$$\overline{\theta_{\approx}}(\mathcal{A}) = \{H \in P(X) : [H]_{\theta_{\approx}} \cap \mathcal{A} \neq \emptyset\} \tag{4}$$

Definition 2.10. [19] Let $\mathcal{T}$ be a subfamily of $P(X)$. If $\underline{\theta_{\approx}}(\mathcal{T})$ and $\overline{\theta_{\approx}}(\mathcal{T})$ both form topologies on X, then, $\theta_{\approx}(\mathcal{T}) = \langle \underline{\theta_{\approx}}(\mathcal{T}), \overline{\theta_{\approx}}(\mathcal{T}) \rangle$ is said to be a $\theta_{\approx}$*-rough topology* on X. Also, $(X, \theta_{\approx}(\mathcal{T}))$ is called a $\theta_{\approx}$-rough topological space.

Definition 2.11. [19] The rough set $\theta(A) = \langle \underline{\theta}(A), \overline{\theta}(A) \rangle$, where $A \subseteq X$ is said to be a $\theta_{\approx}$*-rough open set* if $\underline{\theta}(A)$ is an open set in $\underline{\theta_{\approx}}(\mathcal{T})$ and $\overline{\theta}(X)$ is an open set in $\overline{\theta_{\approx}}(\mathcal{T})$.

Theorem 2.12. [19] *The family of all $\theta_{\approx}$-rough open sets in $(X, \theta_{\approx}(\mathcal{T}))$, denoted by $\mathfrak{T}_{\mathcal{T}}$, forms a topology on $P(U) \times P(U)$ with respect to the operations of rough union and rough intersection.*

Definition 2.13. [19] Let τ_D represent the discrete topology on X. Any $\theta_{\approx}$-rough topology on X which is equivalent to $\theta_{\approx}(\tau_D)$ is called a *discrete $\theta_{\approx}$-rough topology* on X.

Proposition 2.14. [19] $\underline{\theta_{\approx}}(\tau_\theta) = \tau_\theta = \overline{\theta_{\approx}}(\tau_\theta)$*, where τ_θ is the topology on X induced by θ. Hence, $\langle \tau_\theta, \tau_\theta \rangle$ is a $\theta_{\approx}$-rough topology on X in which every rough set is $\theta_{\approx}$-rough open. Also, $\langle \tau_\theta, \tau_\theta \rangle$ is a discrete $\theta_{\approx}$ topology.*

3. $\theta_{\approx}$-ROUGH CONTINUITY

Consider two approximation spaces (X, θ) and (X', θ') and a function g from X to X'. Let $\theta(A) = \langle \underline{\theta}(A), \overline{\theta}(A) \rangle$ be a rough set on (X, θ). Then, its image under g ie; $\langle g(\underline{\theta}(A)), g(\overline{\theta}(A)) \rangle$ need not be a rough set on (X', θ'). Similarly, the inverse image of $\theta'(A') = \langle \underline{\theta'}(A'), \overline{\theta'}(A') \rangle$ ie; $\langle g^{-1}(\underline{\theta'}(A')), g^{-1}(\overline{\theta'}(A')) \rangle$ need not be a rough set on (X, θ). So, the concept of rough image and inverse rough image of a rough set under g is introduced in the following definitions.

Definition 3.1. The *rough image* of a rough set $\theta(A)$ on (X, θ) under $g : X \to X'$ is defined by

$$g(\theta(A)) = \Cap \{ \theta'(A') \in RS(X') : g(\underline{\theta}(A)) \subseteq \underline{\theta'}(A'), \, g(\overline{\theta}(A)) \subseteq \overline{\theta'}(A') \} \tag{5}$$

Definition 3.2. The *inverse rough image* of a rough set $\theta'(A')$ on (X', θ') under g is given by

$$g^{-1}(\theta'(A')) = \Cup \{ \theta(A) \in RS(X) : g^{-1}(\underline{\theta'}(A')) \supseteq \underline{\theta}(A), \, g^{-1}(\overline{\theta'}(A')) \supseteq \overline{\theta}(A) \} \tag{6}$$

Example 3.3. Let $X = \{a, b, c, d\}$ and $X/\theta = \{\{a, b\}, \{c\}, \{d\}\}$. Then,

$P(X)/\theta_{\approx} = \{\{\emptyset\}, \{X\}, \{\{a\}, \{b\}\}, \{\{c\}\}, \{\{d\}\}, \{\{a, c\}, \{b, c\}\}, \{\{a, d\}, \{b, d\}\}, \{\{a, b\}\},$
$\{\{c, d\}\}, \{\{a, b, c\}\}, \{\{a, b, d\}\}, \{\{a, c, d\}, \{b, c, d\}\}\}$.

Now, each equivalence class determines a rough set on (X, θ) and so,

$RS(X) = \{\langle \emptyset, \emptyset \rangle, \langle X, X \rangle, \langle \emptyset, \{a, b\} \rangle, \langle \{c\}, \{c\} \rangle, \langle \{d\}, \{d\} \rangle, \langle \{c\}, \{a, b, c\} \rangle, \langle \{d\}, \{a, b, d\} \rangle,$
$\langle \{a, b\}, \{a, b\} \rangle, \langle \{c, d\}, \{c, d\} \rangle, \langle \{a, b, c\}, \{a, b, c\} \rangle, \langle \{a, b, d\}, \{a, b, d\} \rangle, \langle \{c, d\}, X \rangle\}$.

Take $X' = \{p, q, r, s, t\}$ and $X'/\theta' = \{\{p, q\}, \{r, s\}, \{t\}\}$.Then,

$P(X')/\theta'_{\approx} = \{\{\emptyset\}, \{X'\}, \{\{p\}, \{q\}\}, \{\{r\}, \{s\}\}, \{\{t\}\}, \{\{p, q\}\}, \{\{r, s\}\}, \{\{p, t\}, \{q, t\}\},$
$\{\{r, t\}, \{s, t\}\}, \{\{p, r\}, \{p, s\}, \{q, r\}, \{q, s\}\}, \{\{p, q, t\}\}, \{\{r, s, t\}\}, \{\{p, q, r\}, \{p, q, s\}\},$

$\{\{p,r,t\},\{p,s,t\},\{q,r,t\},\{q,s,t\}\},\{\{p,r,s\},\{q,r,s\}\},\{\{p,q,r,t\},\{p,q,s,t\}\},$
$\{\{p,r,s,t\},\{q,r,s,t\}\},\{\{p,q,r,s\}\}\}.$

$RS(X') = \{\langle\emptyset,\emptyset\rangle,\langle X',X'\rangle,\langle\emptyset,\{p,q\}\rangle,\langle\emptyset,\{r,s\}\rangle,\langle\{t\},\{t\}\rangle,\langle\{p,q\},\{p,q\}\rangle,\langle\{r,s\},\{r,s\}\rangle,$
$\langle\{t\},\{p,q,t\}\rangle,\langle\{t\},\{r,s,t\}\rangle,\langle\emptyset,\{p,q,r,s\}\rangle,\langle\{p,q,t\},\{p,q,t\}\rangle,\langle\{r,s,t\},\{r,s,t\},\rangle,$
$\langle\{p,q\},\{p,q,r,s\}\rangle,\langle\{t\},X'\rangle,\langle\{r,s\},\{p,q,r,s\}\rangle,\langle\{p,q,t\},X'\rangle,\langle\{r,s,t\},X'\rangle,$
$\langle\{p,q,r,s\},\{p,q,r,s\}\rangle\}.$

Consider the function $g : X \to X'$ defined by $g(a) = p, g(b) = q, g(c) = r, g(d) = s$ and the rough set $\theta(A) = \langle\{c\},\{a,b,c\}\rangle$. Then, $g(\{c\}) = \{r\}$ and $g(\{a,b,c\}) = \{p,q,r\}$.
But, the pair $\langle\{r\},\{p,q,r\}\rangle$ is not a rough set on (X',θ').
From definition 3.1, $g(\theta(A)) = \langle\{r,s\},p,q,r,s\}\rangle$.
Take $\theta'(A') = \langle\{p,q\},\{p,q,r,s,\}\rangle$. Then, $g^{-1}(\{p,q\}) = \{a,b\}$ and $g^{-1}(\{p,q,r,s\}) = X$.
In this case also, the pair $\langle\{a,b\},X\rangle$ is not a rough set on (X,θ).
From definition 3.2, $g^{-1}(\theta'(A')) = \langle\{a,b\},\{a,b\}\rangle$.

Obviously, $g(\langle\underline{\theta}(A),\overline{\theta}(A)\rangle) = \langle g(\underline{\theta}(A)), g(\overline{\theta}(A))\rangle$, if $\langle g(\underline{\theta}(A)), g(\overline{\theta}(A))\rangle$ is a rough set. Also, $g^{-1}(\langle\underline{\theta'}(A'),\overline{\theta'}(A')\rangle) = \langle g^{-1}(\underline{\theta'}(A')), g^{-1}(\overline{\theta'}(A'))\rangle$ if $\langle g^{-1}(\underline{\theta'}(A')), g^{-1}(\overline{\theta'}(A'))\rangle$ forms a rough set on (X,θ).

Theorem 3.4. *Let g be be a function from (X,θ) to (X',θ'). Then, for all $\langle\theta(A),\theta(B)\rangle \in RS(X)$ and $\langle\theta'(A'),\theta'(B')\rangle \in RS(X')$,*

(i) $g(\langle\emptyset,\emptyset\rangle) = \langle\emptyset,\emptyset\rangle$
(ii) $g^{-1}(\langle\emptyset,\emptyset\rangle) = \langle\emptyset,\emptyset\rangle$
(iii) $g^{-1}(\langle X',X'\rangle) = \langle X,X\rangle$
(iv) $\theta(A) \subseteq \theta(B) \Rightarrow g(\theta(A)) \subseteq g(\theta(B))$
(v) $\theta'(A') \subseteq \theta'(B') \Rightarrow g^{-1}(\theta'(A')) \subseteq g^{-1}(\theta'(B'))$
(vi) $g(\theta(A)) \subseteq \theta'(A') \Leftrightarrow \theta(A) \subseteq g^{-1}(\theta'(A'))$

Proof. Let $\langle\theta(A),\theta(B)\rangle \in RS(X)$ and $\langle\theta'(A'),\theta'(B')\rangle \in RS(X')$.

(i) Because $g(\emptyset) = \emptyset$ and $\langle\emptyset,\emptyset\rangle$ is a rough set on (X',θ'), we get $g(\langle\emptyset,\emptyset\rangle) = \langle\emptyset,\emptyset\rangle$.
(ii) $g^{-1}(\langle\emptyset,\emptyset\rangle) = \langle\emptyset,\emptyset\rangle$, as $g^{-1}(\emptyset) = \emptyset$ and $\langle\emptyset,\emptyset\rangle$ is a rough set on (X,θ).

(iii) $g^{-1}(X') = X$ and $\langle X, X\rangle$ is a rough set on (X,θ).

Thus, $g^{-1}(\langle X', X'\rangle) = \langle X, X\rangle$.

(iv) $\theta(A) \subseteq \theta(B) \Rightarrow \underline{\theta}(A) \subseteq \underline{\theta}(B), \overline{\theta}(A) \subseteq \overline{\theta}(B) \Rightarrow g[\underline{\theta}(A)] \subseteq g[\underline{\theta}(B)],\ g[\overline{\theta}(A)] \subseteq g[\overline{\theta}(B)]$.

So, $g(\underline{\theta}(B)) \subseteq \underline{\theta'}(A')$ and $g(\overline{\theta}(B)) \subseteq \overline{\theta'}(A')\} \Rightarrow g(\underline{\theta}(A)) \subseteq \underline{\theta'}(A')$ *and* $g(\overline{\theta}(A)) \subseteq \overline{\theta'}(A')\}$.

Therefore, $\{\theta'(A') : g(\underline{\theta}(B)) \subseteq \underline{\theta'}(A'),\ g(\overline{\theta}(B)) \subseteq \overline{\theta'}(A')\}$ forms a subfamily of the family $\{\theta'(A') : g(\underline{\theta}(A)) \subseteq \underline{\theta'}(A'),\ g(\overline{\theta}(A)) \subseteq \overline{\theta'}(A')\}$.

From definition 3.1, it follows that $g(\theta(A)) \subseteq g(\theta(B))$.

(v) $\theta'(A') \subseteq \theta'(B') \Rightarrow \underline{\theta'}(A') \subseteq \underline{\theta'}(B'),\ \overline{\theta'}(A') \subseteq \overline{\theta'}(B')$

$$\Rightarrow g^{-1}[\underline{\theta'}(A')] \subseteq g^{-1}[\underline{\theta'}(B')],\ g^{-1}[\overline{\theta'}(A')] \subseteq g^{-1}[\overline{\theta'}(B')].$$

So, $\underline{\theta}(A) \subseteq g^{-1}(\underline{\theta'}(A'))$ and $\overline{\theta}(A) \subseteq g^{-1}(\overline{\theta'}(A'))\}$

$$\Rightarrow \underline{\theta}(A) \subseteq g^{-1}(\underline{\theta'}(B'))\ \textit{and}\ \overline{\theta}(A) \subseteq g^{-1}(\overline{\theta'}(B'))\}.$$

Hence, $\{\theta(A) : g^{-1}(\underline{\theta'}(A')) \supseteq \underline{\theta}(A),\ g^{-1}(\overline{\theta'}(A')) \supseteq \overline{\theta}(A)\}$ forms a subfamily of the family $\{\theta(A) : g^{-1}(\underline{\theta'}(B')) \supseteq \underline{\theta}(A),\ g^{-1}(\overline{\theta'}(B')) \supseteq \overline{\theta}(A)\}$.

Therefore, by definition 3.2, $g^{-1}(\theta'(A')) \subseteq g^{-1}(\theta'(B'))$.

(vi) From definition 3.1, $g(\theta(A)) \subseteq \theta'(A') \Leftrightarrow g[\underline{\theta}(A)] \subseteq \underline{\theta'}(A')$ and $g[\overline{\theta}(A)] \subseteq \overline{\theta'}(A')$

$$\Leftrightarrow \underline{\theta}(A) \subseteq g^{-1}\{\underline{\theta'}(A')\} \text{ and } \overline{\theta}(A) \subseteq g^{-1}\{\overline{\theta'}(A')\}$$

$$\Leftrightarrow \theta(A) \subseteq g^{-1}\langle \underline{\theta'}(A'), \overline{\theta'}(A')\rangle, \text{ from definition 3.2.}$$

□

Theorem 3.5. *Let g be be a function from* (X,θ) *to* (X',θ'). *Then,*

(i) $g(\theta(A) \Cup \theta(B)) = g(\theta(A)) \Cup g(\theta(B))$

(ii) $g(\theta(A) \Cap \theta(B)) \subseteq g(\theta(A)) \Cap g(\theta(B))$

(iii) $g^{-1}(\theta'(A') \Cup \theta'(B')) \supseteq g^{-1}(\theta'(A')) \Cup g^{-1}(\theta'(B'))$

(iv) $g^{-1}(\theta'(A') \Cap \theta'(B')) = g^{-1}(\theta'(A')) \Cap g^{-1}(\theta'(B'))$

(v) $g^{-1}(\theta'(A'^C)) \subseteq [g^{-1}(\theta'(A'))]^C$.

Proof. (i) Let $\theta'(Z') = g(\theta(A) \Cup \theta(B)) = g(\langle \underline{\theta}(A) \cup \underline{\theta}(B), \overline{\theta}(A) \cup \overline{\theta}(B)\rangle)$.

Then, $g(\underline{\theta}(A) \cup \underline{\theta}(B)) \subseteq \underline{\theta'}(Z')$ and $g(\overline{\theta}(A) \cup \overline{\theta}(B)) \subseteq \overline{\theta'}(Z')$.

ie; $g(\underline{\theta}(A)) \cup g(\underline{\theta}(B)) \subseteq \underline{\theta'}(Z')$ and $g(\overline{\theta}(A)) \cup g(\overline{\theta}(B)) \subseteq \overline{\theta'}(Z')$

Hence, $g(\underline{\theta}(A)) \subseteq \underline{\theta'}(Z')$, $g(\underline{\theta}(B)) \subseteq \underline{\theta'}(Z')$, $g(\overline{\theta}(A)) \subseteq \overline{\theta'}(Z')$ and $g(\overline{\theta}(B)) \subseteq \overline{\theta'}(Z')$.

So, $g(\theta(A)) \subseteq \theta'(Z')$ and $g(\theta(B)) \subseteq \theta'(Z')$. Therefore, $g(\theta(A)) \Cup g(\theta(B)) \subseteq \theta'(Z')$.

Now, let $g(\theta(A)) = \theta'(A')$ and $g(\theta(B)) = \theta'(B')$.

Then, $g[\underline{\theta}(A)] \subseteq \underline{\theta'}(A')$, $g[\overline{\theta}(A)] \subseteq \overline{\theta'}(A')$, $g[\underline{\theta}(B)] \subseteq \underline{\theta'}(B')$ and $g[\underline{\theta}(B)] \subseteq \underline{\theta'}(B')$.

So, $g(\underline{\theta}(A)) \cup g(\underline{\theta}(B)) \subseteq \underline{\theta'}(A') \cup \underline{\theta'}(B')$ and $g(\overline{\theta}(A)) \cup g(\overline{\theta}(B)) \subseteq \overline{\theta'}(A') \cup \overline{\theta}(B')$.

Hence, $g[\underline{\theta}(A) \cup \underline{\theta}(B)] \subseteq \underline{\theta'}(A') \cup \underline{\theta'}(B')$ and $g[\overline{\theta}(A)] \cup \overline{\theta}(B)] \subseteq \overline{\theta'}(A') \cup \overline{\theta'}(B')$.

Also, $\langle \underline{\theta'}(Z'), \overline{\theta'}(Z') \rangle$ is the smallest rough set having this property.

So, $\langle \underline{\theta'}(Z'), \overline{\theta'}(Z') \rangle \subseteq \langle \underline{\theta'}(A') \cup \underline{\theta'}(B'), \overline{\theta'}(A') \cup \overline{\theta'}(B') \rangle$.

ie; $\langle \underline{\theta'}(Z'), \overline{\theta'}(Z') \rangle \subseteq \langle \underline{\theta'}(A'), \overline{\theta}(A') \rangle \Cup \langle \underline{\theta'}(B'), \overline{\theta}(B') \rangle$.

Thus, $\theta'(Z') \subseteq g(\theta(A)) \Cup g(\theta(B))$. Therefore, $g(\theta(A) \Cup \theta(B)) = g(\theta(A)) \Cup g(\theta(B))$

(ii) Let $\theta'(W') = g(\theta(A) \Cap \theta(B))$, $g(\theta(A)) = \theta'(A')$ and $g(\theta(B)) = \theta'(B')$.

Then, $g[\underline{\theta}(A)] \subseteq \underline{\theta'}(A')$, $g[\overline{\theta}(A)] \subseteq \overline{\theta'}(A')$, $g[\underline{\theta}(B)] \subseteq \underline{\theta'}(B')$ and $g[\underline{\theta}(B)] \subseteq \underline{\theta'}(B')$.

So, $g[\underline{\theta}(A)] \cap g[\underline{\theta}(B)] \subseteq \underline{\theta'}(A') \cap \underline{\theta'}(B')$ and $g[\overline{\theta}(A)] \cap g[\overline{\theta}(B)] \subseteq \overline{\theta'}(A') \cap \overline{\theta'}(B')$.

Hence, $g[\underline{\theta}(A) \cap \underline{\theta}(B)] \subseteq \underline{\theta'}(A') \cap \underline{\theta'}(B')$ and $g[\overline{\theta}(A)] \cap \overline{\theta}(B)] \subseteq \overline{\theta'}(A') \cap \overline{\theta'}(B')$.

Also, $\langle \underline{\theta'}(W'), \overline{\theta'}(W') \rangle$ is the smallest rough set having this property.

Thus, $\langle \underline{\theta'}(W'), \overline{\theta'}(W') \rangle \subseteq \langle \underline{\theta}(A') \cap \underline{\theta}(B'), \overline{\theta}(A') \cap \overline{\theta}(B') \rangle = \theta'(A') \Cap \theta'(B')$.

ie; $\langle \underline{\theta'}(W'), \overline{\theta'}(W') \rangle \subseteq g(\theta(A)) \Cap g(\theta(B))$

Therefore, $g(\theta(A) \Cap \theta(B)) \subseteq g(\theta(A)) \Cap g(\theta(B))$

(iii) Let $\theta(Z) = g^{-1}[\theta'(A') \Cup \theta'(B')]$, $\theta(A) = g^{-1}(\theta'(A'))$ and $\theta(B) = g^{-1}(\theta'(B'))$. Then, $g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(A)$, $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(A)$, $g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(B)$ and $g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(B)$.

So, $g^{-1}[\underline{\theta'}(A')] \cup g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(A) \cup \underline{\theta}(B)$ and $g^{-1}[\overline{\theta'}(A')] \cup g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(A) \cup \overline{\theta}(B)$.

Hence, $g^{-1}[\underline{\theta'}(A') \cup \underline{\theta'}(B')] \supseteq \underline{\theta}(A) \cup \underline{\theta}(B)$ and $g^{-1}[\overline{\theta'}(A')] \cup \overline{\theta'}(B')] \supseteq \overline{\theta}(A) \cup \overline{\theta}(B)$.

Also, $\langle \underline{\theta}(Z), \overline{\theta}(Z) \rangle$ is the largest rough set having this property.

Hence, $\langle \underline{\theta}(Z), \overline{\theta}(Z) \rangle \supseteq \langle \underline{\theta}(A) \cup \underline{\theta}(B), \overline{\theta}(A) \cup \overline{\theta}(B) \rangle = \theta(A) \Cup \theta(B)$.

ie; $\langle \underline{\theta}(Z), \overline{\theta}(Z) \rangle \supseteq g^{-1}(\theta'(A')) \Cup g^{-1}(\theta'(B'))$.

Therefore, $g^{-1}(\theta'(A') \Cup \theta'(B')) \supseteq g^{-1}(\theta'(A')) \Cup g^{-1}(\theta'(B'))$

(iv) Let $\theta(W) = g^{-1}[\theta'(A') \Cap \theta'(B')] = g^{-1}[\langle \underline{\theta'}(A') \cap \underline{\theta'}(B'), \overline{\theta'}(A') \cap \overline{\theta'}(B') \rangle]$.

Then, $g^{-1}[\underline{\theta'}(A') \cap \underline{\theta'}(B')] \supseteq \underline{\theta}(W)$ and $g^{-1}[\overline{\theta'}(A') \cap \overline{\theta'}(B')] \supseteq \overline{\theta}(W)$.

Hence, $g^{-1}[\underline{\theta'}(A')] \cap g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(W)$ and $g^{-1}[\overline{\theta'}(A')] \cap g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(W)$.

$\therefore g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(W)$, $g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(W)$, $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(W)$ and $g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(W)$.

ie; $g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(W)$, $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(W)$ and $g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(W)$, $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(W)$.

Thus, $g^{-1}[\theta'(A')] \supseteq \theta(W)$ and $g^{-1}[\theta'(B')] \supseteq \theta(W)$. $\therefore g^{-1}[\theta'(A')] \Cap g^{-1}[\theta'(B')] \supseteq \theta(W)$.

Now let $g^{-1}[\langle\underline{\theta'}(A'),\overline{\theta'}(A')\rangle] = \langle\underline{\theta}(A),\overline{\theta}(A)\rangle$ and $g^{-1}[\langle\underline{\theta'}(B'),\overline{\theta'}(B')\rangle] = \langle\underline{\theta}(B),\overline{\theta}(B)\rangle$

$\therefore g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(A)$, $g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(B)$, $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(A)$ and $g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(B)$.

$\therefore g^{-1}[\underline{\theta'}(A')] \cap g^{-1}[\underline{\theta'}(B')] \supseteq \underline{\theta}(A) \cap \underline{\theta}(B)$ and $g^{-1}[\overline{\theta'}(A')] \cap g^{-1}[\overline{\theta'}(B')] \supseteq \overline{\theta}(A) \cap \overline{\theta}(B)$.

Since $\theta(W)$ is the largest rough set having this property, we get

$\langle\underline{\theta}(W),\overline{\theta}(W)\rangle \supseteq \langle\underline{\theta}(A)\cap\underline{\theta}(B),\overline{\theta}(A)\cap\overline{\theta}(B)\rangle = \theta(A) \Cap \theta(B)$.

Hence, $\langle\underline{\theta}(W),\overline{\theta}(W)\rangle \supseteq g^{-1}[\langle\underline{\theta'}(A'),\overline{\theta'}(A')\rangle] \Cap g^{-1}[\langle\underline{\theta'}(B'),\overline{\theta'}(B')\rangle]$.

Therefore, $g^{-1}[\theta'(A') \Cap \theta'(B')] \supseteq g^{-1}[\theta'(A')] \Cap g^{-1}[\theta'(B')]$.

Thus, $g^{-1}(\theta'(A') \Cap \theta'(B')) = g^{-1}(\theta'(A')) \Cap g^{-1}(\theta'(B'))$

(v) Let $g^{-1}(\theta'(A')) = \theta(A)$.

Then, from definition 3.2, $g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(A)$ and $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(A)$.

Taking complements on both sides, $(g^{-1}[\underline{\theta'}(A')])^C \subseteq (\underline{\theta}(A))^C$ and $(g^{-1}[\overline{\theta'}(A')])^C \subseteq (\overline{\theta}(A))^C$.

Hence, $g^{-1}([\underline{\theta'}(A')]^C) \subseteq \overline{\theta}(A^C)$ and $g^{-1}([\overline{\theta'}(A')]^C) \subseteq \underline{\theta}(A^C)$.

Thus, $g^{-1}[\overline{\theta'}(A'^C)] \subseteq \overline{\theta}(A^C)$ and $g^{-1}[\underline{\theta'}(A'^C)]) \subseteq \underline{\theta}(A^C)$.

Now, $g^{-1}(\theta'(A'^C)) = g^{-1}\langle\underline{\theta'}(A'^C),\overline{\theta'}(A'^C)\rangle) = \langle\underline{\theta}(V),\overline{\theta}(V)\rangle$ (say).

Then, $\underline{\theta}(V) \subseteq g^{-1}[\underline{\theta'}(A'^C)]$ and $\overline{\theta}(V) \subseteq g^{-1}[\overline{\theta'}(A'^C)]$.

Hence, $\underline{\theta}(V) \subseteq \underline{\theta}(A^C)$ and $\overline{\theta}(V) \subseteq \overline{\theta}(A^C)$. Thus, $\langle\underline{\theta}(V),\overline{\theta}(V)\rangle \subseteq \langle\underline{\theta}(A^C),\overline{\theta}(A^C)\rangle$.

So, $\langle\underline{\theta}(V),\overline{\theta}(V)\rangle \subseteq (\theta(A))^C$. Thus, $g^{-1}(\theta'(A')^C) \subseteq [g^{-1}(\theta'(A'))]^C$. □

Theorem 3.6. *Let g be be a function from (X,θ) to (X',θ'). Then, for all $\theta(A) \in RS(X)$ and $\theta'(A') \in RS(X')$,*

(i) $\theta(A) \subseteq g^{-1}\{g(R(A))\}$.

(ii) $g\{g^{-1}(\theta'(A'))\} \subseteq \theta'(A')$

Proof. Let $\theta(A) \in RS(X)$ and $\theta'(A') \in RS(X')$.

(i) Let $g(\theta(A)) = \theta'(A')$.

By definition 3.1, $g[\underline{\theta}(A)] \subseteq \underline{\theta'}(A')$ and $g[\overline{\theta}(A)] \subseteq \overline{\theta'}(A')$.

Hence, $\underline{\theta}(A) \subseteq g^{-1}[\underline{\theta'}(A')]$ and $\overline{\theta}(A) \subseteq g^{-1}[\overline{\theta'}(A')]$.

$g^{-1}\{g(\theta(A))\}$ is the largest rough set having this property. So, it contains $\theta(A)$.

Hence, $\theta(A) \subseteq g^{-1}\{g(\theta(A))\}$.

(ii) Let $g^{-1}(\theta'(A')) = \theta(A)$.

From definition 3.2, $g^{-1}[\underline{\theta'}(A')] \supseteq \underline{\theta}(A)$ and $g^{-1}[\overline{\theta'}(A')] \supseteq \overline{\theta}(A)$.

So, $\underline{\theta'}(A')) \supseteq g[\underline{\theta}(A)]$ and $\overline{\theta'}(A') \supseteq g[\overline{\theta}(A)]$.

$g(\theta(A))$ is the smallest rough set satisfying this property. So, it is contained in $\theta'(A')$.

Hence, $g\{g^{-1}(\theta'(A'))\} \subseteq \theta'(A')$, $\forall \theta'(A') \in RS(X')$. □

Definition 3.7. Consider two rough topological spaces $(X, \theta_{\approx}(\mathcal{T}))$ and $(X', \theta'_{\approx}(\mathcal{T}'))$ and a function $g : X \to X'$. Then, g is called a $\theta_{\approx}$*-rough continuous* function if the inverse rough image of every $\theta'_{\approx}$-rough open subset on $(X', \theta'_{\approx}(\mathcal{T}'))$ is a $\theta_{\approx}$-rough open set on $(X, \theta_{\approx}(\mathcal{T}))$.

Theorem 3.8. *Every function defined from* (X, θ) *to* (X, θ') *is a* $\theta_{\approx}$*-rough continuous function with reference to the induced* $\theta_{\approx}$*-rough topology* $\langle \tau_\theta, \tau_\theta \rangle$ *on* (X, θ).

Proof. If $\langle \tau_\theta, \tau_\theta \rangle$ is the induced $\theta_{\approx}$-rough topology on (X, θ), then every rough set on (X, θ) is $\theta_{\approx}$-rough open with respect to $\langle \tau_\theta, \tau_\theta \rangle$.
If $\theta'(A')$ is a $\theta_{\approx}$-rough open set, then, $g^{-1}(\theta'(A')) = \langle \underline{\theta}(A), \overline{\theta}(A) \rangle$ is a $\theta_{\approx}$-rough open set.
Therefore, inverse image of every $\theta'_{\approx}$-rough open set is $\theta_{\approx}$-rough open.
Hence, g is $\theta_{\approx}$-rough continuous. □

The next corollary follows form the fact that every discrete $\theta_{\approx}$-rough topology is equivalent to the $\theta_{\approx}$-rough topology corresponding to τ_θ.

Corollary 3.9. *Every function defined from* (X, θ) *to* (X, θ') *is* $\theta_{\approx}$*-rough continuous with reference to the discrete* $\theta_{\approx}$*-rough topology on* X.

4. Conclusion

The strong association of rough set theory with topology theory has been a well-discussed topic of research. In this paper, the concept of rough topology introduced in [19] which is a rough subset of the power set of the the set under consideration is explored further. The rough image and inverse rough image of a rough set under a function have been defined and investigated in detail. The continuity of a function on a topological space is extended to the rough topological space and the properties of the rough continuous functions are presented.

Also, it is found that all functions defined on discrete $\theta_{\approx}$-rough topological spaces are $\theta_{\approx}$-rough continuous.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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