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RESULTS ON MEROMORPHIC AND ENTIRE FUNCTIONS SHARING CM AND IM WITH THEIR DIFFERENCE OPERATORS

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Abstract. In this paper, we study the value distribution of finite order meromorphic, entire functions and their difference operators sharing CM and IM. Our results in this paper improve and generalizes the corresponding results from Dong-Mei Wei and Zhi-Gang Huang.

Keywords: meromorphic functions; entire functions; difference operator; uniqueness.

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1. Introduction

Whole of this article, every function like meromorphic function, integral function are defined on complex plane $\mathbb{C}$ be a open complex plane and two functions f and g are non constant meromorphic functions in $\mathbb{C}$. The fundamentals of value distribution theory of meromorphic function can be read in [10], [17]. For a meromorphic function f, the order and the lower order of f is given by

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r} \text{ and } \mu(f) = \underline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r}$$

While

$$\lambda(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ N(r,\frac{1}{f})}{\log r} \text{ and } \delta(\alpha,f) = \underline{\lim_{r\to\infty}} \frac{m(r,\frac{1}{f-\alpha})}{T(r,f)}{=}1 - \overline{\lim_{r\to\infty}} \frac{N(r,\frac{1}{f-\alpha})}{T(r,f)}$$

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stand for the exponent of convergence of zero sequence of f and the deficiency of f at the point α, respectively. For a non constant meromorphic function v, we denote by $T(r,v)$ the Nevanlinna characteristic of v and by $S(r,v)$ any quantity satisfying $S(r,v) = o(T(r,v))$, as r runs to infinity outside of a set $\mathbb{E} \subset (0,+\infty)$ of finite linear measure. We say that v is a small function of f if $T(r,v)$ = S(r,v).

Let b be a small function and k be a positive integer, then denote $N_{(k}(r,b;f)$ the counting function for zeros of $f(z)-b$ with multiplicity atleast k, and $\overline{N}_{(k}(r,b;f)$ if multiplicity is not counted and $N_{k)}(r,b;f)$ is the counting function for zeros of $f(z)-b$ with multiplicity at most k and $\overline{N}_{k)}(r,b;f)$ if multiplicity is not counted.

2. Preliminaries

Definition 1. [12] *Let k be a non-negative integer or infinity. For $\alpha \in \mathbb{C} \cup \{\infty\}$, we denote by $\mathbb{E}_k(\alpha; f)$ the set of all α points of $f(z)$ where an α point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $\mathbb{E}_k(\alpha; f) = \mathbb{E}_k(\alpha; g)$, then we say that f, g share the value α with weight k.*

We write f and g share (α, k) to mean that f, g share the value α with weight k.

Definition 2. [12] *If s is a positive integer, then we denote by $N(r, \frac{1}{f-\alpha} \mid = s)$ the counting function of those α points of f whose multiplicity is s, where each α point is counted according to its multiplicity. For a positive integer m, denote by $N(r, \alpha; f \mid \geq m)$ the counting function of those α points of f whose multiplicities are not less than m where each α point is counted according to its multiplicity.*

Definition 3. [13] *Denote by $N_2(r, \frac{1}{f-\alpha})$ the sum of $\overline{N}(r, \frac{1}{f-\alpha}) + \overline{N}(r, \frac{1}{f-\alpha} \mid \geq 2)$.*

The classical four point and five point theorems of Nevanlinna [15] show f is a Mobius transformation of g if two meromorphic functions f and g share four distinct values CM, and $f = g$ if f and g share five distinct values IM. The assumption 4 CM of the four point theorem and 5 IM of the five-point theorem have been improved to 2 CM+2 IM and 3 CM+1 IM , while 1 CM+3 IM remains an open problem.

Some researchers also considered whether the conditions of shared values can be replaced by other conditions. Ozawa [16] obtained the following.

Theorem A. Let f and g be non constant entire functions of finite order such that f and g share 0 and 1 CM. If $\delta(0,f) > \frac{1}{2}$, then $f.\ g = 1$ or $f = g$.

Removing the order restriction Ueda [19] and Yi [22] obtained some improvements of Theorem A. Especially, Yi [23] obtained the following.

Theorem B. [23] Suppose that f and g are non constant meromorphic functions. If f, g share 0, 1, ∞ CM and $\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) < (d+o(1))T(r,f)$ for $r \in 1$ and $r \in \infty$, where d is a positive number satisfying $0 < d < \frac{1}{2}$, while $I \subset (0,+\infty)$ is a subset of infinite linear measure, then $f.\ g = 1$ or $f = g$.

For the sake of relaxing the nature of sharing of values and improving Theorem B, Lahiri in [12] obtained the following result in terms of the weighted value sharing.

Theorem C. [12] Suppose that f and g are non constant meromorphic functions. Let f and g share (0,1), $(\infty,0)$, $(1,\infty)$. If

$$N\left(r,\frac{1}{f-\alpha} \mid = 1\right) + 4\overline{N}(r,f) < (d+o(1))T(r,f),$$

then either $f.\ g = 1$ or $f = g$.

Rubel and Yang [18] in 1977 initiated the study of entire functions sharing values with their derivatives instead of studying the problem of sharing value of two meromorphic functions f and g.

Theorem D. Let f be a non constant entire function. If f shares two distinct finite values CM with f', then $f \equiv f'$.

More results on the uniqueness of f' with its nth derivative $f^{(n)}$ were obtained by several authors; see [[1], [8], [20]]. In view of the progress on the difference analogues of classical Nevanlinna theory of meromorphic functions [[4], [9]], it is quite natural to investigate the uniqueness problems of meromorphic functions and their difference operators; see [3], [6], [11], [25]. So a natural question arises, that is, how about the uniqueness of the derivatives and the difference operators of $f(z)$?

In 2018, Qi et al. [17] obtained some results in the case that $f'(z)$ shares values with $\triangle f$ or $f(z+c)$.

Theorem E. Let f be a meromorphic function of finite order. Suppose that f' and $\triangle f$ share a_1, a_2, a_3, a_4 *IM*, where a_1, a_2, a_3, a_4 are four distinct finite values. Then $f' = \triangle f$.

Theorem F. Let f be a transcendental entire function of finite order, and α be a nonzero finite value. If $f'(z)$ and $f(z+c)$ share $0, \alpha$ *CM*, then $f'(z) = f(z+c)$ for all $z \in \mathbb{C}$.

In 2020 Dong-Mei Wei and Zhi-Gang Huang proved the 1^{st} result which investigates the uniqueness of meromorphic functions in terms of weighted value sharing and 2^{nd} and 3^{rd} results on difference operators of $f(z)$ shares some values with its derivatives.

Theorem G. [26] Let f and g be meromorphic functions with finite order, and let $c \in \mathbb{C} \setminus \{0\}$. Suppose that f^n and g^n share $(R(z),\ l)$, where $R(z)$ is a rational function and l, n are integer. If one of the following cases holds:

(1) $l = 0, n \geq 15$;

(2) $l = 1, n \geq 10$;

(3) $l \geq 2, n \geq 9$,

then $f = tg$ or $f.\ g = t\alpha$, where $t^n = 1$, $\alpha^n = R^2$.

Theorem H. [26] Let f be a non constant entire function of finite order with periodic $\eta \neq 0$ such that $\mu(f) > 1$, where η is a finite nonzero value, and let $a_1$and a_2 be two distinct finite values, and k be a positive integer. If $\triangle f$ and $f^{(k)}$ share a_1 CM, $\triangle f$ and $f^{(k)}$ share a_2 IM, then $\triangle f = f^{(k)}$.

Theorem I. [26] Let f be a non constant meromorphic function of finite order, and let c be a finite nonzero value, let k be a positive integer satisfying $k \geq 2$, and let a_1, a_2, a_3 be three finite values such that $a_1 \neq 0$, $a_2 \neq 0$ and $N(r, \frac{1}{f-a_3}) = S(r, f)$. If $f(z+c)$ and $f^{(k)}(z)$ share a_1 CM and a_2 IM, then $f(z+c) = f^{(k)}(z)$ for all $z \in \mathbb{C}$.

3. Main Results

Theorem 1. Let f and g be meromorphic functions with finite order, and let $c \in \mathbb{C} \setminus \{0\}$. Suppose that $f^n P(f)$ and $g^n P(g)$ share $(R(z), l)$, where $R(z)$ is a rational function and l, m, n are integers. If one of the following cases holds:

(1) $l = 0$, $n \geq 13m + 15$;

(2) $l = 1$, $n \geq 8m + 10$;

(3) $l \geq 2$, $n \geq 7m + 9$,

then $f = tg$ or $f.\ g = t\alpha$, where $t^{n+m} = 1$, $\alpha^{n+m} = R^2$.

Theorem 2. Let f be a non constant entire function of finite order with periodic $\eta \neq 0$ such that $\mu(f) > 1$, where η is a finite nonzero value, and let $a_1$and a_2 be two distinct finite values, and k be a positive integer. If $\triangle(f^n P(f))$ and $f^{(k)}$ share a_1 CM, $\triangle(f^n P(f))$ and $f^{(k)}$ share a_2 IM, then

$$\triangle(f^n P(f)) = f^{(k)}.$$

Theorem 3. Let f be a non constant meromorphic function of finite order, and let c be a finite nonzero value, let k be a positive integer satisfying $k \geq 2$, and let a_1, a_2, a_3 be three finite values such that $a_1 \neq 0$, $a_2 \neq 0$ and $N(r, \frac{1}{f-a_3}) = S(r, f)$. If $f^n(z+c)P(f)$ and $f^{(k)}(z)$ share a_1 CM and a_2 IM, then

$$f^n(z+c)P(f) = f^{(k)}(z)$$

for all $z \in \mathbb{C}$.

4. LEMMAS

Let F and G be two non constant meromorphic functions defined in $\mathbb{C}$. The function H is defined by:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F-1}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G-1}\Big).$$

The following Lemmas are used to prove the main results of this paper.

Lemma 1. [2] *Let F and G be two non constant meromorphic functions sharing (1,0) and $H\neq 0$. Then*

$$\begin{aligned}T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+2\overline{N}\left(r,\frac{1}{F}\right)\\+\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)+\overline{N}(r,G)+S(r,F)+S(r,G)\end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2. [2] *Let F and G be two non constant meromorphic functions, sharing (1,1) and $H\neq 0$. Then*

$$\begin{aligned}T(r,F)\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+\frac{1}{2}\overline{N}(r,F)\\+S(r,F)+S(r,G)\end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 3. [12] *Let f and g be two non constant meromorphic functions sharing (1,2). Then one of the following cases holds:*

i) $T(r)\leq N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})+N_2(r,f)+N_2(r,g)+S(r)$,

ii) $f=g$,

iii) $fg=1$,

where $T(r)=max\{T(r,f),T(r,g)\}$ and $S(r)=o\{T(r)\}$, as $r\notin\mathbb{E}$, where $\mathbb{E}\subset(0,+\infty)$ is a subset of finite linear measure.

Lemma 4. [7] *Let f and g be two meromorphic functions, and let k be a positive integer. If $E_k(1;f)=E_k(1;g)$, then one of the following cases must occur:*

$$\begin{aligned}(i)\ T(r,f)+T(r,g) \leq\ & N_2(r,\frac{1}{f})+N_2(r,\frac{1}{g})+N_2(r,f)+N_2(r,g)\\ & +\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)-N_{11}\left(r,\frac{1}{f-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right)\\ & +\overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right)+S(r,f)+S(r,g);\end{aligned}$$

$$(ii)\ f=\frac{(b+1)g+(a-b-1)}{bg+(a-b)},\ \textit{where } a(\neq 0),\ b \textit{ are two constants}.$$

Lemma 5. [21] *Let f be a meromorphic function. If*

$$g=\frac{af+b}{cf+d},$$

where a, b, c, $d \in S(f)$ and $ad-bc \neq 0$, then

$$T(r,g)=T(r,f)+S(r,f).$$

Lemma 6. [4] *Let f be a non constant meromorphic function, let $\varepsilon > 0$ and let $c \in \mathbb{C}$. If f is of finite order, then there exists a set $E=E(f,\varepsilon) \subset (0,+\infty)$ satisfying*

$$\overline{\lim_{r\to\infty}}\frac{\int_{E\cap[1,r)}\frac{dt}{t}}{\log r} \leq \varepsilon,$$

i.e, of logarithmic density at most ε, such that

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=O\left(\frac{\log r}{r}T(r,f(z))\right)$$

for all out of the set E. If $\rho_2=\rho_2<1$ and $\varepsilon>0$, then

$$m\left(r,\frac{f(z+c)}{f(z)}\right)=o\left(\frac{T(r,f(z))}{r^{1-\rho_2-\varepsilon}}\right)$$

for all $r \in (0,+\infty)$ outside of a set of finite logarithmic measure.

Lemma 7. *(*[1]*, Lemma 3) Let k be a positive integer, and let f be a non constant meromorphic function such that $f^{(k+1)} \not\equiv 0$. If $\overline{N}\left(r,\frac{1}{f}\right)=S(r,f)$, then*

$$kN_{1)}(r,f) \leq \overline{N}_{(2}(r,f)+N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f).$$

Lemma 8. [24] *Let f be a meromorphic function such that $f^{(k)}$ is not constant. Then*

$$T(r,f) \leq N\left(r,\frac{1}{f})\right)+N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}(r,f)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f).$$

Lemma 9. [24] *Let f be a transcendental meromorphic function. Then, for each positive real number ε and for each positive integer n,*

$$(n-1)\overline{N}(r,f) \leq (1+\varepsilon)N\left(r,\frac{1}{f^{(n)}}\right)+(1+\varepsilon)(N(r,f)-\overline{N}(r,f))+S(r,f).$$

Lemma 10. [4] *Let f be a non constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$T(r,f(z+c)) = T(r,f)+S(r,f),$$

$$N(r,f(z+c)) = N(r,f)+S(r,f),$$
$$N\Big(r,\frac{1}{f(z+c)}\Big) = N\Big(r,\frac{1}{f(z)}\Big)+S(r,f),$$
$$\overline{N}(r,f(z+c)) = \overline{N}(r,f)+S(r,f),$$
$$\overline{N}\Big(r,\frac{1}{f(z+c)}\Big) = \overline{N}\Big(r,\frac{1}{f(z)}\Big)+S(r,f).$$

5. PROOF OF MAIN RESULTS

Proof of Theorem 1

Set $F=\frac{f^nP(f)}{R}$, $G=\frac{g^nP(g)}{R}$, clearly, F and G share $(1,l)$. Write $T(r)=max\{T(r,f),T(r,g)\}$ and $S(r)=o(T(r))$ as $r\notin E$ and $r\to\infty$, where $E\subset(0,+\infty)$ is a subset of finite linear measure.

Case 1. $l=0$ and $n\geq 13m+15$

Assume that $H\neq 0$. By Lemma 1, we have

$$T(r,F) \leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+2\overline{N}\left(r,\frac{1}{F}\right)$$
$$+\overline{N}\left(r,\frac{1}{G}\right)+2\overline{N}(r,F)+\overline{N}(r,G)+S(r,F)+S(r,G).$$

Clearly,

$$\begin{aligned}(n+m)T(r,f) &\leq 4\overline{N}\left(r,\frac{1}{F}\right)+3\overline{N}\left(r,\frac{1}{G}\right)+4\overline{N}(r,F)+3\overline{N}(r,G)+S(r,F)+S(r,G)\\ &\leq 4\overline{N}\left(r,\frac{1}{f^nP(f)}\right)+3\overline{N}\left(r,\frac{1}{g^nP(g)}\right)+4\overline{N}(r,f^nP(f))+3\overline{N}(r,g^nP(g))\\ &+S(r,f)+S(r,g)\end{aligned}$$

$$(n+m)T(r,f) \leq 8(1+m)T(r,f)+6(1+m)T(r,g)+S(r,f)+S(r,g). \tag{1}$$

Similarly, we have

$$(n+m)T(r,g) \leq 8(1+m)T(r,g)+6(1+m)T(r,f)+S(r,f)+S(r,g). \tag{2}$$

Now (1) and (2) yield

$$(n-13m-14)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g).$$

Hence

$$(n-13m-14)T(r) \leq S(r),$$

which contradicts $n \geq 13m+15$.

Therefore, $H \equiv 0$ and we have

$$\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)=0. \tag{3}$$

It follows from (3) that

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{4}$$

where $A \neq 0$, B are constants.

Subcase 1.1. If $B=0$, then (4) leads to $F=\frac{G-1+A}{A}$ and $G=AF-(A-1)$. Suppose that $A=1$. Clearly, we have $F=G$, and thus $f^nP(f)=tg^nP(g)$, where $t^{n+m}=1$. If $A \neq 1$, then we have $\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)$ and $\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{G+(A-1)}\right)$.

By Nevanlinna's second fundamental theorem,

$$\begin{aligned} T(r,F) &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{F-\frac{A-1}{A}}\right)+\overline{N}(r,F)+S(r,F) \\ &\leq \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,F)+S(r,F). \end{aligned}$$

Thus,

$$\begin{aligned} (n+m)T(r,f) &\leq \overline{N}\left(r,\frac{1}{fP(f)}\right)+\overline{N}(r,gP(g))+\overline{N}(r,fP(f))+S(r,f) \\ &\leq 2(1+m)T(r,f)+(1+m)T(r,g)+S(r,f). \end{aligned} \tag{5}$$

Similarly, we have

$$(n+m)T(r,g) \leq 2(1+m)T(r,g)+(1+m)T(r,f)+S(r,g). \tag{6}$$

Combining (5) and (6), we obtain

$$(m+n-3-3m)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)$$

$$(n-2m-3)T(r) \leq S(r)$$

Clearly, it is a contradiction by $n \geq 13m+15$,

Subcase 1.2 $B \neq 0$.

If $A \neq B$, then we have $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$. Applying Nevanlinna's second fundamental theorem to F with considering 0 point, 1 point, and ∞ point, we can also get a contradiction by similar discussion as in Subcase 1.1.

Case 2. $l = 1$ and $n \geq 8m+10$.

Assume that $H \neq 0$. By Lemma 2, we have

$$\begin{aligned} T(r,F) &\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right)+\frac{1}{2}\overline{N}(r,F) \\ &+S(r,F)+S(r,G) \\ &\leq \frac{5}{2}\overline{N}\left(r,\frac{1}{f^nP(f)}\right)+\frac{5}{2}\overline{N}(r,f^nP(f))+2\overline{N}\left(r,\frac{1}{g^nP(g)}\right)+2\overline{N}(r,g^nP(g)) \\ &+S(r,f)+S(r,g). \end{aligned}$$

Thus,

$$(n+m)T(r,f) \leq 5(1+m)T(r,f)+4(1+m)T(r,g)+S(r,f)+S(r,g). \tag{7}$$

Similarly, we have

$$(n+m)T(r,g) \leq 5(1+m)T(r,g)+4(1+m)T(r,f)+S(r,f)+S(r,g). \tag{8}$$

Combining (7) and (8) yields

$$(m+n-9-9m)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

which contradicts $n \geq 8m+10$.

Therefore, $H \equiv 0$. We can deduce the same conclusion by similar discussion as in case 1.

Case 3. $l \geq 2$ and $n \geq 7m+9$.

Subcase 3.1. $l=2$.

From Lemma 3, if (i) holds, then we deduce that

$$(9)\ \ max\{T(r,F),T(r,G)\} \le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G).$$

That is,

$$\begin{aligned}(m+n)T(r) &= (m+n)\ max\{T(r,F),T(r,G)\}\\ &\le 2\overline{N}\left(r,\frac{1}{f^nP(f)}\right)+2\overline{N}\left(r,\frac{1}{g^nP(g)}\right)+2\overline{N}(r,f^nP(f))\\ &+2\overline{N}(r,g^nP(g))+S(r,f)+S(r,g)\\ &\le 8(1+m)T(r)+S(r).\end{aligned}$$

Therefore, $(m+n-8-8m)T(r)\le S(r)$, which contradicts $n\ge 7m+9$. Thus we have $F=G$ or $FG=1$. If $F=G$, then $f^nP(f)=g^nP(g)$, which yields $f=tg$, where $t^{n+m}=1$. If $FG=1$, then $(f^nP(f)g^nP(g))=R^2$, which yields $(fg)^{n+m}=R^2$, $fg=t\alpha$, where $t^{n+m}=1$, $\alpha^{n+m}=R^2$.

Subcase 3.2. $l\ge 3$.

By Lemma 4, either (i) or (ii) holds. If (i) holds, then we obtain

$$\begin{aligned}T(r,F)+T(r,G) &\le N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+N_2(r,F)+N_2(r,G)\\ &+\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_{11}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(k+1}\left(r,\frac{1}{F-1}\right)\\ &+\overline{N}_{(k+1}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)\\ &\le N_2(r,F)+N_2(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F-1}\right)\\ &+\frac{1}{2}\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,F)+S(r,G)\\ &\le N_2(r,F)+N_2(r,G)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)\\ &+T(r,F)+T(r,G)+S(r,F)+S(r,G).\end{aligned}$$

Therefore, we get

$$\begin{aligned}\frac{1}{2}T(r,F)+\frac{1}{2}T(r,G) &\leq 2\overline{N}(r,fP(f))+2\overline{N}(r,gP(g))+2\overline{N}\left(r,\frac{1}{fP(f)}\right)\\ &+2\overline{N}\left(r,\frac{1}{gP(g)}\right)+S(r,f)+S(r,g).\end{aligned} \tag{10}$$

Consequently,

$$\frac{1}{2}(m+n)\{T(r,f)+T(r,g)\} \leq 4(1+m)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which leads to

$$\left(\frac{1}{2}(m+n)-4(1+m)\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

This is a contradiction since $n \geq 7m+9$

Hence, (ii) holds, which means

$$F=\frac{(b+1)G+(a-b-1)}{bG+(a-b)},$$

where $a \neq 0$, b are constants.

Suppose that $b=0$. Then we have $F=G$ when $a-1=0$, that is, $f=tg$, where $t^{n+m}=1$. If $a-1\neq 0$, then we obtain $F=\frac{G+a-1}{a}$ and $G=a\left(F+\frac{1-a}{a}\right)$, and so $\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{G+a-1}\right)$, $\overline{N}\left(r,\frac{1}{G}\right)=\overline{N}\left(r,\frac{1}{F+\frac{1-a}{a}}\right)$. By Nevanlinna's second fundamental theorem we get

$$T(r,G) \leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G+a-1}\right)+\overline{N}(r,G)+S(r,G).$$

This yields

$$(m+n)T(r,g) \leq 2(1+m)T(r,g)+(1+m)T(r,f)+S(r,g).$$

Similarly, we have

$$(m+n)T(r,f) \leq 2(1+m)T(r,f)+(1+m)T(r,g)+S(r,f).$$

Thus we obtain $(n-2m-3)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)$, which is a contradiction with $n \geq 7m+9$. Suppose that $b=-1$. If $a+1=0$, then $F.\ G \equiv 1$. Hence $f.g=t\alpha$, where $t^{n+m}=1$, $\alpha^{n+m}=R^2$. If $a+1\neq 0$, similarly to above, then we can obtain a contradiction. Suppose $b \neq 0$ and $b \neq -1$. By similar reasoning to the case $b=0$, we can also obtain a contradiction.

This completes the proof of Theorem 1.

Proof of Theorem 2

Without loss of any generality, we can assume $a_1 = 0$, $a_2 = 1$. Since $\triangle(f^n(z)P(f))$, $f^{(k)}$ share 0 CM, we have

$$\frac{\triangle(f^n(z)P(f))}{f^{(k)}} = e^R, \tag{11}$$

Where R is a polynomial of degree $n+m$.

Since the period of f is $c \in \mathbb{C} \setminus \{0\}$, we have $e^{R(z)} = e^{R(z+c)}$.

Consequently, $e^{R(z+c)-R(z)} = 1$, which leads to $R'(z+c) = R'(z)$. Then $R'(z)$ has a period c and $R'(z)$ must be a constant.

Now write

$$R(z) = f^{n+m}(z)a_m + f^{n+m-1}(z)a_{m-1} + \ldots + a_0.$$

Where a, n, m are constants.

Since $\triangle(f^n(z)P(f))$, $f^{(k)}$ share 1 IM, we get

$$\frac{\triangle(f^n(z)P(f)) - 1}{f^{(k)} - 1} = \alpha(z), \tag{12}$$

where α is a meromorphic functin.

By (11), (12) and $R(z) = f^{n+m}(z)a_m + f^{n+m-1}(z)a_{m-1} + \ldots + a_0$, we deduce

$$\alpha(z) = \frac{f^{(k)}e^R - 1}{f^{(k)} - 1}. \tag{13}$$

By Lemma 5, we obtain

$$T(r,\alpha) = T(r, f^{(k)}) + S(r,f) = (k+1)T(r,f) + S(r,f). \tag{14}$$

Now we estimate the number of zeros, poles of α. From the assumption that $\mu(f) > 1$, we know that $T(r,e^R) = S(r,f)$.

Since $\triangle(f^n(z)P(f))$, $f^{(k)}$ share 1 IM, it follows from (13) that the zero of $\triangle(f^n(z)P(f)) - 1$ and $f^{(k)} - 1$ must be the zero of $e^R - 1$. Noting that $f^{(k)} - 1$ have the same poles with $f^{(k)}e^R - 1$, then by (12), we have

$$\overline{N}(r,\alpha) = \overline{N}\left(r, \frac{1}{f^{(k)} - 1}\right) \leq \overline{N}\left(r, \frac{1}{e^R - 1}\right) = S(r,f)$$

and

$$\overline{N}\left(r,\frac{1}{\alpha}\right)=\overline{N}\left(r,\frac{1}{\triangle(f^nP(f)-1}\right)\leq\overline{N}\left(r,\frac{1}{e^R-1}\right)=S(r,f).$$

Therefore, from the Nevanlinna second fundamental theorem, we obtain

$$T(r,\alpha)\leq\overline{N}(r,\alpha)+\overline{N}\left(r,\frac{1}{\alpha}\right)+\overline{N}\left(r,\frac{1}{\alpha-e^R}\right)+S(r,\alpha). \tag{15}$$

Combining (11) and (12), we may write $\alpha-e^R=\frac{\triangle(f^n(z)P(f))-1}{f^{(k)}-1}-\frac{\triangle(f^n(z)P(f))}{f^{(k)}}=\frac{e^R-1}{f^{(k)}-1}$.

Then, by (15), we conclude that

$$T(r,\alpha)\leq 3\overline{N}\left(r,\frac{1}{e^R-1}\right)+S(r,f)=S(r,f). \tag{16}$$

It contradicts (14).

Therefore, $\triangle(f^nP(f))=f^{(k)}$. This completes the proof of Theorem 2.

Proof of Theorem 3

Some ideas of our proof come from [1], [26]. Without loss of generality, we assume that $f^n(z+c)P(f)$ and $f^{(k)}(z)$ share 1 CM and ∞ IM, and $N\left(r,\frac{1}{f}\right)=S(r,f)$. For the general case, we take the transformation $T(z)=\frac{z-a_3}{z-a_1}\frac{a_2-a_1}{a_2-a_3}$, and so $T(a_1)=\infty$, $T(a_2)=1$, $T(a_3)=0$. Suppose that $f^n(z+c)P(f)\not\equiv f^{(k)}(z)$. Set

$$G(z)=\frac{1}{f^n(z+c)P(f)}\left(\frac{f^{(k+1)(z)}}{f^{(k)}(z)-1}-\frac{(f^n(z+c)P(f))'}{f^n(z+c)P(f)-1}\right) \tag{17}$$

(18)

$$G(z)=\frac{f^{(k)}(z)}{f^n(z+c)P(f)}\left(\frac{f^{(k+1)(z)}}{f^{(k)}(z)-1}-\frac{f^{(k+1)}(z)}{f^{(k)}(z)}\right)-\left(\frac{(f^n(z+c)P(f)'}{f^n(z+c)P(f)-1}-\frac{(f^n(z+c)P(f))'}{f^n(z+c)P(f)}\right)$$

It follows from the lemma of the logarithmic derivative, Lemma 6 and (18) that $m(r,G)=S(r,f)$.

By (17), we see that the possible poles of G can occur at the zeros of $f^n(z+c)P(f)$, the 1-points of $f^n(z+c)P(f)$ and $f^{(k)}(z)$, and the poles of $f^n(z+c)P(f)$ and $f^{(k)}(z)$. If z_0 is a 1-point of $f^n(z+c)P(f)$, then by a short calculation with Laurent series and (17) we see that $G(z)$ is analytic at z_0. Since $f^n(z+c)P(f)$ and $f^{(k)}(z)$ share 1 CM, we know the 1 points of $f^n(z+c)P(f)$ and $f^{(k)}(z)$ are not the poles of $G(z)$. If $f^n(z+c)P(f)$ has a pole z_0 with multiplicity $(n+m)p$ (≥ 1), we need to consider two cases: (i) z_0 is also a pole of $f^{(k)}(z)$, then by (17) $G(z) = O((z-z_0)^{(n+m)p-1})$; (ii) z_0 is not a pole of $f^{(k)}(z)$, and hence z_0 is not a pole of $f^{(k+1)}(z)$. Then we also have $G(z) = O((z-z_0)^{(n+m)p-1})$. Similarly, the poles of $f^{(k)}(z)$ are not also the poles of $G(z)$. Therefore, the poles of F can only occur at the zeros of $f^n(z+c)P(f)$. By Lemma 7 and the hypothesis of Theorem 3, it follows that $N\left(r, \frac{1}{f^n(z+c)P(f)}\right) = N\left(r, \frac{1}{f^n(z+c)}\right) + N\left(r, \frac{1}{P(f)}\right) + S(r,f) = S(r,f)$, and so we have $N(r,G) = S(r,f)$. Thus,

$$T(r,G) = S(r,f). \tag{19}$$

If $G \equiv 0$, then, by (17), we find that $f^{(k)}(z) - 1 = t(f^n(z+c)P(f) - 1)$, with $t \neq 0$ constant. Thus, $(1-t)m(r, \frac{1}{f}) \leq m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) + m\left(r, \frac{f^n(z+c)P(f)}{f^n P(f)}\right) = S(r,f)$. Since $N\left(r, \frac{1}{f}\right) = S(r,f)$, we have $T\left(r, \frac{1}{f}\right) = S(r,f)$. It is a contradiction. Then $G \not\equiv 0$. And so we deduce from (17) and (19) that

$$\begin{aligned} m(r, f^n(z+c)P(f)) &\leq m\left(r, \frac{1}{G}\right) + m\left(r, \frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} - \frac{(f^n(z+c)P(f))'}{f^n(z+c)P(f)-1}\right) \\ &\leq T(r,G) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{20}$$

If z_0 is a pole of $f^n(z+c)P(f)$ of multiplicity $(n+m)p \geq 2$, then by (17) we know that z_0 is possible a zero of G with multiplicity $(n+m)p-1$. Consequently, it follows from (19) that

$$N_{(2}(r, f^n(z+c)P(f)) \leq 2N\left(r, \frac{1}{G}\right) \leq 2T(r,G) + O(1) = S(r,f). \tag{21}$$

Let z_0 be a simple pole of $f^n(z+c)P(f)$. Set

$$H(z) = \frac{f^{(k+1)}(z)(f^n(z+c)P(f)-1)}{(f^n(z+c)P(f))'(f^{(k)}(z)-1)}. \tag{22}$$

By a short calculation with Laurent series, it follows that $H(z_0) = k+1$. If $H(z) \equiv k+1$, then we have $f^{(k)}(z) - 1 = t(f^n(z+c)P(f)-1)^{k+1}$ with $t \neq 0$ constant. This is a contradiction, since $f^{(k)}(z)$ and $f^n(z+c)P(f)$ share 1 CM. Thus $H \not\equiv k+1$, and so,

$$N_{1)}(r, f^n(z+c)P(f)) \leq N\left(r, \frac{1}{H-(k+1)}\right) \leq T(r,H) + O(1). \tag{23}$$

We now estimate the poles of H. Clearly, the poles of H can only occur at the 1 points of $f^{(k)}(z)$, the zeros of $(f^n(z+c)P(f))'$, and the poles of $f^n(z+c)P(f)$ and $f^{(k+1)}(z)$. Since

$f^{(k)}(z)$, $(f^n(z+c)P(f))$ share 1 CM and ∞ IM, H is holomorphic at the 1 points of $f^{(k)}(z)$ and the poles of $f^n(z+c)P(f)$ and $f^{(k+1)}(z)$. Thus

$$N(r,H) \le N_0\left(r, \frac{1}{(f^n(z+c)P(f))'}\right) + S(r,f), \tag{24}$$

where $N_0\left(r, \frac{1}{(f^n(z+c)P(f))'}\right)$ denotes the zeros of $(f^n(z+c)P(f))'$ which are not zeros of $f^n(z+c)P(f)-1$. Again by (22), we see that

$$m(r,H) = S(r,f), \tag{25}$$

From this, (23) and (24) we find that

$$N_{1)}(r, f^n(z+c)P(f)) \le N_0\left(r, \frac{1}{(f^n(z+c)P(f))'}\right) + S(r,f). \tag{26}$$

Combining this, Nevanlinna's second fundamental theorem ([10], Theorem 3.2]) for $f^n(z+c)P(f)$, (21) and the hypothesis $N\left(r, \frac{1}{f(z)}\right) = S(r,f)$ we have

(27)

$$\begin{aligned}
T(r, f^n(z+c)P(f)) \le\; & N\left(r, \frac{1}{f^n(z+c)P(f)}\right) + \overline{N}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + \overline{N}(r, f^n(z+c)P(f)) \\
& - N_0\left(r, \frac{1}{(f^n(z+c)P(f))'}\right) + S(r, f^n(z+c)P(f)) \\
\le\; & \overline{N}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + S(r,f) \\
\le\; & N_{1)}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + S(r,f).
\end{aligned}$$

From this and Nevanlinna's first fundamental theorem, it is easy to deduce that

(28)

$$m\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + N_{(2}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) \le \overline{N}_{(2}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + S(r,f),$$

which implies

$$m\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) + N_{(2}\left(r, \frac{1}{f^n(z+c)P(f)-1}\right) = S(r,f). \tag{29}$$

From (29), we see that

$$T(r,f^n(z+c)P(f)) = m\left(r,\frac{1}{f^n(z+c)P(f)-1}\right)+N_{1)}\left(r,\frac{1}{f^n(z+c)P(f)-1}\right)$$
$$+N_{(2}\left(r,\frac{1}{f^n(z+c)P(f)-1}\right)+S(r,f) \tag{30}$$
$$= N_{1)}\left(r,\frac{1}{f^n(z+c)P(f)-1}\right)+S(r,f).$$

Since $f^n(z+c)P(f)$ and $f^{(k)}(z)$ share 1 CM, it follows from Lemma 7 and (21) that

$$(k-1)N_{1)}(r,f^n(z+c)P(f)) \leq \overline{N}\left(r,\frac{1}{f^{(k+1)}(z)}\right)+S(r,f). \tag{31}$$

By Lemma 8, (29) (30) and (21), we have

(32)

$$N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right) \leq N\left(r,\frac{1}{f^n(z+c)P(f)}\right)+N_{1)}(r,f^n(z+c)P(f))+N\left(r,\frac{1}{f^n(z+c)P(f)-1}\right)$$
$$-T(r,f^n(z+c)P(f))+S(r,f^n(z+c)P(f))$$
$$\leq N_{1)}(r,f^n(z+c)P(f))+S(r,f).$$

It follows from Lemma 10 that

$$N\left(r,\frac{1}{f^{(k+1)}(z)}\right) = N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right)+S(r,f)$$

From this, (31) and (32), we see that $(k-2)N_{1)}(r,f^n(z+c)P(f)) = S(r,f)$. If $k \geq 3$, then $N_{1)}(r,f^n(z+c)P(f)) = S(r,f)$. Combining this, (20) and (21), we have $T(r,f^n(z+c)P(f)) = S(r,f)$, which is a contradiction.

Let $k = 2$.

Case 1. If $f(z)$ is transcendental, then by Lemma 9, for a positive constant $\varepsilon < 1$ we have

$$2N_{1)}(r,f^n(z+c)P(f)) \leq (1+\varepsilon)N\left(r,\frac{1}{f^{(k+1)}(z+c)}\right)+(1+\varepsilon)[N(r,f^n(z+c)P(f))$$
$$-\overline{N}(r,f^n(z+c)P(f))]+S(r,f) \tag{33}$$

From this and (21) we have

$$(1-\varepsilon)N_{1)}(r,f^n(z+c)P(f)) = S(r,f).$$

Combining this, (20) and (21), we have $T(r, f^n(z+c)P(f)) = S(r, f)$. This is contradiction.

Case 2. If $f(z)$ is rational, then by $N\left(r, \frac{1}{f}\right) = S(r, f)$ we know that f has no zeros, and hence we can write $f^n(z+c)P(f) = \frac{1}{P(z)}$, where $P(z)$ is a non constant polynomial. Set

$$\Phi = \frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} - \frac{(f^n(z+c)P(f))'}{f^n(z+c)P(f))-1} - 2\frac{(f^n(z+c)P(f))'}{f^n(z+c)P(f)}.$$

Clearly $T(r,\Phi) = S(r, f)$. Combining this and (17), we have

$$2(f^n(z+c)P(f))' = Gf^2 - \Phi f. \tag{34}$$

Substituting $f^n(z+c)P(f) = \frac{1}{P(z)}$ into (34) we obtain $-2P' = G - \Phi P$. This shows that $T(r,P') = T(r,P) + S(r, f)$ and so $T(r,P) = S(r, f)$. This is a contradiction.

Therefore, $f^n(z+c)P(f) = f^{(k)}(z)$. This completes the proof of Theorem 3.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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