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OPTIMAL CONTROL OF A SEIR MODEL WITH CONFINEMENT OF COVID'19

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Abstract. In this work, we proposed a mathematical model with an isolation rate and a control strategy, such as confinement, to control the spread of the COVID'19 epidemic in society. We formulated an optimal control problem and solved it. Also, a characterization of the optimal control is given using Pontryagin's Maximum Principle.

Keywords: Covid'19; optimal control; SEIR epidemic model.

2010 AMS Subject Classification: 37N25.

1. Introduction

Corona-virus disease (COVID'19) is an epidemic of viral-looking pneumonia that is different from Sars-Cov viruses. It was officially announced by Chinese health authorities and the World Health Organization (WHO). Unfortunately, this virus was able to spread very quickly leading to an increase in the number of deaths every day. Therefore, it is imperative to control the spread of this disease using non-pharmaceutical strategies such as the confinement.

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Since the announcement of the first cases of COVID'19 in Morocco [9], the country has moved up a gear in terms of measures to keep the Corona-virus under control, which consist of closing schools, restaurants and all gathering places, and announcing a complete quarantine throughout the country. Figure (1) shows the cumulative number of confirmed cases of Corona-virus in Morocco between 2 March and 28 December 2020, showing that the quarantine and isolation measures announced in mid-March contributed to reducing the contact rate. Therefore, the incidence curve did not accelerate due to the slow and structured increase. Whereas after announcing the lifting of the quarantine on 25 June, Morocco recorded a significant increase in the number of confirmed cases.

Mathematical modelling methods are used to understand the dynamics of infectious diseases and how to control their spread using epidemiological models that are considered the most effective for formulating disease and epidemic control strategies. From this standpoint, many mathematical models have been applied to Corona-virus. Some authors have proposed models to estimate the risks of transmission and effects of the virus [1], while others have suggested epidemic models to predict the potential local and global spread of the epidemic Corona-virus [2]. Although most researchers have focused on calculating the baseline reproductive R_0 number and predicting the number of infected and deaths [3, 4, 5, 6], some authors have been interested in formulating models to investigate certain control strategies, such as social distance and health education [7, 8].

In this paper, we applied the optimal control theory to propose the optimal strategy in order to mitigate the spread of the COVID'19 epidemic by minimizing the susceptible individuals and the infected individuals, where confinement was used as a control.

The model inspired by the COVID'19 pandemic associated with confinement is presented in the following section. In section 3, we provide necessary condition for the existence of an optimal control and its characterization using Pontryagin's Maximum Principle.

2. COVID'19 MODEL WITH CONFINEMENT

To study the transmission dynamics of the novel Corona-virus epidemic, we use in our work an extension of a SEIR model. This choice is motivated by the fact that this model takes into account the latency period of the disease. That is to say that during this period, the individual is infected without any transmission of the disease to other individuals. In this model, we consider a time-dependent control $u(t)$ representing the confinement, that is bounded between 0 and U_{max}, with $U_{max} < 1$.

The population is made up of five classes of individuals (compartments), susceptible S, exposed E, infected without protection I and we introduce the compartment R_1 for infectious individuals, but confined. In other words, individuals whose movement is restricted and completely removed from individuals likely to be infected, and their infection is confirmed by tests in laboratories approved by the state. Whereas, the compartment R_2 contains contagious and isolated cases, but they are not counted, that is to say that she has not undergone any test, either because some of them are suffering from symptoms of low severity, either because some live in areas far from laboratories and where live in retirement homes and some are afraid to test and prefer to isolate themselves and adopt a health protocol to deal with the virus. It should be noted that the two compartments R_1 and R_2 also include both cases which are no longer contagious, either because they have cured or because they have died.

The parameter α is defined as the effective contact rate, the parameter β is the incubation rate at which people infected in the latent phase become infectious, η is the isolation rate of infectious people and therefore removed from the chain of transmission, and δ is the fraction of infectious individuals who are counted among the confirmed cases in the time of isolation, it is assumed that this fraction is constant, although it may change over time.

The total population is given by

$N = S(t) + E(t) + I(t) + R_1(t) + R_2(t)$.

Thus, we can write the model mathematically as:

$$
\begin{aligned}
\frac{dS(t)}{dt} &= -\alpha\,(1-u(t))\,\frac{S(t)I(t)}{N},\\
\frac{dE(t)}{dt} &= \alpha\,(1-u(t))\,\frac{S(t)I(t)}{N} - \beta E(t),\\
\frac{dI(t)}{dt} &= \beta E(t) - \eta I(t),\\
\frac{R_1(t)}{dt} &= \delta\eta I(t),\\
\frac{R_2(t)}{dt} &= (1-\delta)\eta I(t),
\end{aligned}
\tag{1}
$$

(2) with $S(0) > 0, E(0) \geq 0, I(0) \geq 0, R_1(0) \geq 0, R_2(0) \geq 0,$ as the initial conditions.

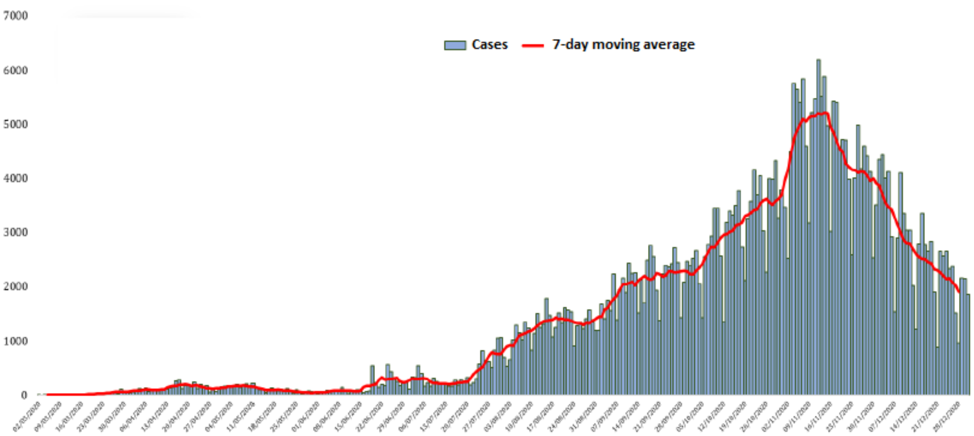

FIGURE 1. Daily new confirmed cases due to COVID'19 (rolling 7-day average, right-aligned).

3. OPTIMAL CONTROL PROBLEM

The aim of our work is to analyse the problem of optimal control to suggest the most effective mitigation strategy for the COVID'19 epidemic by minimizing the level of susceptible and infected individuals and therefore maximizing the number of recovered individuals.

In our model, we have included the control u which represents the confinement, so our goal is to determine an optimal control u^* to stop the spread of the epidemic by minimizing, over a finite time horizon $[0, t_f]$, the objective functional.

We define the objective functional J as follows:

$$J(u) = \int_0^{t_f} \left[C_1 S(t) + C_2 I(t) + \frac{1}{2} \rho u^2(t) \right] dt, \tag{3}$$

where constants C_1, C_2 and ρ are positive. The weight parameter ρ is associated with the control u. C_1 describes the cost related to the susceptible class while C_2 indicates the cost incurred due to the symptomatic compartment. In addition, the objective functional J corresponds to the total cost due to the Corona-virus epidemic and its control strategy. Further, the integrand of the objective functional is given by $\mathscr{L}(S,E,I,R_1,R_2,u) = C_1S(t)+C_2I(t)+\frac{1}{2}\rho u^2(t)$. More precisely, the optimal control problem can be defined as follows:

$$J(u^*) = minJ(u), u \in \Omega, \tag{4}$$

where

$$\Omega = \left\{u : 0 \le u(t) \le u_{max} < 1, t \in [0,t_f], u(t) \text{is lebesgue measurable}\right\} \tag{5}$$

is the set of admissible controls.

3.1. Existence of optimal control.

In order to solve the optimal control problem, it is first necessary to show the existence of the solution of system (1). Consider the state variables $S(t)$, $E(t)$, $I(t)$, $R_1(t)$, $R_2(t)$ and the control $u(t)$ with non negative initial conditions as given (2), then the system can be written as

$$\mathscr{X}_t = \mathscr{A}\mathscr{X} + \mathscr{B}(\mathscr{X}), \tag{6}$$

where

$$\mathscr{X} = \begin{bmatrix} S(t) \\ E(t) \\ I(t) \\ R_1(t) \\ R_2(t) \end{bmatrix}, \quad \mathscr{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 & 0 \\ 0 & \beta & -\eta & 0 & 0 \\ 0 & 0 & \delta\eta & 0 & 0 \\ 0 & 0 & (1-\delta)\eta & 0 & 0 \end{bmatrix},$$

$$\mathscr{B}(\mathscr{X}) = \begin{bmatrix} \dfrac{-\alpha(1-u(t))S(t)I(t)}{N} \\ \dfrac{\alpha(1-u(t))S(t)I(t)}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$\mathscr{X}_t$ is derivative of $\mathscr{X}$ with respect to time t. It is clearly that the system (6) is a non linear system with a bounded coefficient.

We pose

$$\mathscr{H}(\mathscr{X}) = \mathscr{A}\mathscr{X} + \mathscr{B}(\mathscr{X}). \tag{7}$$

Then,

$$\begin{aligned}
\mathscr{B}(\mathscr{X}_1) - \mathscr{B}(\mathscr{X}_2) &= \begin{bmatrix} \dfrac{-\alpha(1-u(t))S_1(t)I_1(t)}{N} \\ \dfrac{\alpha(1-u(t))S_1(t)I_1(t)}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \dfrac{-\alpha(1-u(t))S_2(t)I_2(t)}{N} \\ \dfrac{\alpha(1-u(t))S_2(t)I_2(t)}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{\alpha(1-u(t))}{N}\left[-S_1(t)I_1(t) + S_2(t)I_2(t)\right] \\ \dfrac{\alpha(1-u(t))}{N}\left[S_1(t)I_1(t) - S_2(t)I_2(t)\right] \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

The second term on the right hand side of the equation (7) satisfies

$$\begin{aligned}
|\mathscr{B}(\mathscr{X}_1)-\mathscr{B}(\mathscr{X}_2)| &= \frac{2\alpha(1-u(t))}{N}|S_1(t)I_1(t)-S_2(t)I_2(t)| \\
&\leq \frac{2\alpha(1-u(t))}{N}[|S_1(t)||I_1(t)-I_2(t)|+|I_2(t)||S_1(t)-S_2(t)|] \\
&\leq M_1|S_1(t)-S_2(t)|+M_2|I_1(t)-I_2(t)|,
\end{aligned}$$

where $M_1>0$ and $M_2>0$ are independent of the variables $S(t)$ and $I(t)\leq N$ respectively. Therefore,

$$|\mathscr{B}(\mathscr{X}_1)-\mathscr{B}(\mathscr{X}_2)|\leq M[|S_1(t)-S_2(t)|+|I_1(t)-I_2(t)|], \text{ with } M=max(M_1,M_2).$$

Also, we get

$$|\mathscr{H}(\mathscr{X}_1)-\mathscr{H}(\mathscr{X}_2)|\leq K|\mathscr{X}_1-\mathscr{X}_2| \text{ where } K=||\mathscr{A}||+M.$$

Thus, it follows that the function $\mathscr{H}$ satisfies the Lipschitz condition, uniformly with respect to non negative state variables. Therefore, there exists a solution of the system (1).

Now, we present a result that will show the existence of an optimal control that minimizes the objective functional J in a finite interval, subjected to the system (1).

Theorem 1. *There exists an optimal control u^* in Ω and a corresponding solution $\mathscr{X}^*=\{S^*,E^*,I^*,R_1^*,R_2^*\}$ to the initial value problem (1)-(2) such that*

$$J(u^*)=minJ(u),\ u\in\Omega.$$

Proof. To prove the existence of an optimal control, we use the conditions of theorem (III.4.1) and its corresponding corollary in Fleming and Rischel [10].

Note that the set of solution to the system (1)-(2) with control variables in Ω is nonempty. Let u_1, $u_2\in\Omega$ such that $0\leq u_1\leq u_{max}$ and $0\leq u_2\leq u_{max}$. Then, for any $\varepsilon\in[0,1]$, one has

$$0\leq\varepsilon u_1+(1-\varepsilon)u_2\leq u_{max},$$

which implies that Ω is convex and close. Also the state system is linear in the control variable u with coefficients depending on time and state variables. In addition, it is easy to verify that all $\varepsilon\in[0,1]$ and all u_1, $u_2\in\Omega$,

$$\mathscr{L}(\mathscr{X},\varepsilon u_1+(1-\varepsilon)u_2)\leq\varepsilon\mathscr{L}(\mathscr{X},u_1)+(1-\varepsilon)\mathscr{L}(\mathscr{X},u_2),$$

where $\mathscr{X} = \{S, E, I, R_1, R_2\}$. Thus, the integrand of the objective functional is convex on Ω. Furthermore, $\mathscr{L}(S, E, I, R_1, R_2, u) \geq \frac{1}{2}\rho u^2$. Let $\nu = \frac{1}{2}\rho$ and g is a continuous function defined by $g(u) = \nu |u|^2$. Then,

$$\mathscr{L}(S, E, I, R_1, R_2, u) \geq g(u)$$

such that $|u|^{-1} g(u) \to +\infty$ as $|u| \to +\infty$, $u \in \Omega$. Thus, all conditions are achieved. Therefore, we deduce the existence of an optimal control u^* which minimizes the objective functional $J(u)$. □

3.2. Characterization of optimal control.

Theorem 1 assures us the existence of the solution of the problem (4) before attempting to calculate the optimal control. Thereafter, we characterize this control by applying Pontryagin's Maximum Principle to the Hamiltonian.

Let $\mathscr{X} = \{S, E, I, R_1, R_2\}$, $u \in \Omega$ and $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ the adjoint variable. The Hamiltonian function is defined as

(8)
$$\begin{aligned} \mathrm{H}(\mathscr{X}, u, \Lambda, t) = \quad & C_1 S(t) + C_2 I(t) + \frac{1}{2}\rho u^2(t) + \lambda_1(t)\frac{dS(t)}{dt} + \lambda_2(t)\frac{dE(t)}{dt} + \lambda_3(t)\frac{dI(t)}{dt} \\ & + \lambda_4(t)\frac{dR_1(t)}{dt} + \lambda_5(t)\frac{dR_2(t)}{dt}. \end{aligned}$$

Necessary condition that $\{\mathscr{X}^*(t), u^*(t)\}$ be can optimal solution for the optimal control problem is the existence of a non trivial vector function $\Lambda(t) = \{\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t), \lambda_5(t)\}$ such that

$$(9) \qquad \begin{aligned} \frac{d\mathscr{X}}{dt} &= \frac{\partial \mathrm{H}(\mathscr{X}^*(t), u^*(t), \Lambda(t), t)}{d\Lambda} \\ 0 &= \frac{\partial \mathrm{H}(\mathscr{X}^*(t), u^*(t), \Lambda(t), t)}{du} \\ \Lambda'(t) &= -\frac{\partial \mathrm{H}(\mathscr{X}^*(t), u^*(t), \Lambda(t), t)}{d\mathscr{X}}. \end{aligned}$$

Theorem 2. *Given an optimal control $u^*(t)$ and corresponding solutions S^*, E^*, I^*, R_1^* and R_2^* that minimize $J(u)$ over Ω. Then, there exist adjoint variables λ_i, $i = 1, 2, \cdots, 5$ satisfying*

(10)

$$\begin{aligned}
\frac{d\lambda_1(t)}{dt} &= -C_1 + \frac{\alpha\,(1-u(t))\,I(t)}{N}\left[\lambda_1(t)-\lambda_2(t)\right], \\
\frac{d\lambda_2(t)}{dt} &= \lambda_2(t)\beta - \lambda_3(t)\beta, \\
\frac{d\lambda_3(t)}{dt} &= -C_2 + \frac{\alpha\,(1-u(t)S(t))}{N}\left[\lambda_1(t)-\lambda_2(t)\right] + \eta\left[\lambda_3(t)-\lambda_5(t)\right] - \delta\eta\left[\lambda_4(t)-\lambda_5(t)\right], \\
\frac{d\lambda_4(t)}{dt} &= 0, \\
\frac{d\lambda_5(t)}{dt} &= 0,
\end{aligned}$$

where final time conditions $\lambda_i(t_f) = 0$, $i = 1, 2, \cdots, 5$. Moreover, the following characterization holds:

$$u^*(t) = max\left(0, min\left(u_{max}, \frac{\alpha[-\lambda_1(t)+\lambda_2(t)]S^*(t)I^*(t)}{\rho N}\right)\right). \tag{11}$$

Proof. The form of the adjoint system (10) endowed with terminal conditions results from Pontryagin's Maximum Principle by differentiating the Hamiltonian function (8), at the respective solutions of the state system (1). Also, to get the characterizations of the optimal control given by (11) we use the optimality conditions. After solving the equation $\frac{\partial \mathrm{H}(\mathscr{X}^*(t), u^*(t), \Lambda(t), t)}{du} = 0$, we obtain

$$\rho u^*(t) + \frac{\alpha\left[\lambda_1(t)-\lambda_2(t)\right]S^*(t)I^*(t)}{N} = 0 \text{ at } u = u^*(t).$$

It follows that

$$u^*(t) = \frac{\alpha[-\lambda_1(t)+\lambda_2(t)]S^*(t)I^*(t)}{\rho N}.$$

Taking to account the boundedness condition given in (5), if we set

$$\Sigma_t = \frac{\alpha[-\lambda_1(t)+\lambda_2(t)]S^*(t)I^*(t)}{\rho N}.$$

Then,

$$\begin{cases} u^*(t) = 0 & if \quad \Sigma_t \leq 0, \\ \\ u^*(t) = \Sigma_t & if \quad 0 < \Sigma_t < u_{max}, \\ \\ u^*(t) = u_{max} & if \quad \Sigma_t \geq u_{max}. \end{cases}$$

Consequently, the optimal control $u^*(t)$ is given by

$$u^*(t) = max\left(0, min\left(u_{max}, \frac{\alpha[-\lambda_1(t) + \lambda_2(t)]S^*(t)I^*(t)}{\rho N}\right)\right).$$

□

4. Conclusion

Among the most important measures taken to limit the propagation of Corona-virus is quarantined and isolation of infected individuals, and it is the control option we supported in this work. We have demonstrated the existence of optimal control over our model that minimizes the number of susceptible individuals and infected individuals. The results discussed in this article will be valuable for our future work because we aim to use other optimal control strategies in the form of vaccination, health awareness and intensive prevention.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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