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CERTAIN INTEGRAL TRANSFORMS OF THE PRODUCT OF TWO FUNCTIONS

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Abstract. In this paper, we prove propositions of a class of integral transform, this class includes Laplace transform and Fourier transform. In particular, under certain conditions, we prove an explicit formula for the integral transform of the product of two functions.

Keywords: Integral transform; Laplace transform; Fourier transform.

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1. INTRODUCTION

Differential operators and integral operators, and in general operator theory, play important roles in Mathematics. The details for operator theory details are discussed in [10] and [15]. Recently, properties of difference operators and their applications are discussed in [1]- [7].

Define the integral transform Tf as

$$\widehat{f}(t) = (Tf)(t) = \int_{\Omega} K(u,t) f(u) du, \tag{1.1}$$

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where $\Omega \subseteq \mathbb{R}$ and K is the kernel of Tf that satisfies the following conditions

$$K(a,b) = K(b,a) \tag{1.2}$$

for all $(a,b) \in \Omega \times \Omega$.

$$K(a,b)K(c,b) = K(a+c,b) \tag{1.3}$$

for all $(a,b),(c,b) \in \Omega \times \Omega$.

The kernel of Laplace transform $K(t,u) = e^{-tu}$ with $\Omega = (0,\infty)$ and the kernel of Fourier transform $K(t,u) = (2\pi)^{-1/2}e^{-itu}$ with $\Omega = (-\infty,\infty)$ satisfy the conditions (1.2) and (1.3).

One of the disappointments of the Laplace transform and Fourier transform is that the transform of the product of two functions does not equal the product of their transforms. In fact, the Laplace (Fourier) transform of the convolution of two functions is the product of their Laplace (Fourier) transforms.

In [14], it is proved that the Laplace transform of the product of $f(t)$ and $\frac{1}{t}$ is

$$\mathscr{L}_s\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\xi)d\xi.$$

The Laplace transform of a product of two functions was given in [5] as

Theorem 1.1. *Assume that* $\mathscr{L}(f(t)) = F(s)$ *and* $\mathscr{L}(g(t)) = G(s)$. *If* $\int_0^\infty \int_0^\infty e^{-(s+\xi)t} g(\xi) f(t) dt d\xi$ *converges absolutely for* $s > \alpha$, *then* $\mathscr{L}_s(f(t)G(t))$ *is given as*

$$\mathscr{L}_s\Big(f(t)G(t)\Big) = \mathscr{L}_s\Big(f(t)\mathscr{L}(g(t))\Big) = \int_0^\infty g(\xi)F(\xi+s)d\xi = \int_s^\infty g(\xi-s)F(\xi)d\xi \text{ for } s > \alpha. \tag{1.4}$$

Example 1.2.

$$\begin{aligned}
\mathscr{L}_s\Big(\frac{1-\cos(t)}{t^2}\Big) &= \int_s^\infty (u-s)(\frac{1}{u} - \frac{u}{u^2+1})du \\
&= \int_s^\infty \frac{du}{u^2+1} - s\int_s^\infty (\frac{1}{u} - \frac{u}{u^2+1})du \\
&= \arctan(\frac{1}{s}) + s\ln(\frac{s}{\sqrt{s^2+1}}).
\end{aligned}$$

Example 1.3. For $s > 1$

$$\mathscr{L}_s\left(\frac{e^t - t - 1}{t^2}\right) = \int_s^\infty (\xi - s)\Big(\frac{1}{\xi - 1} - \frac{1}{\xi^2} - \frac{1}{\xi}\Big)d\xi$$
$$= (s-1)\ln(\frac{s-1}{s}) - 1.$$

In the following section, we extend the result in Theorem 1.1 to a more generalized kernel.

2. Main Results

Theorem 2.1. *Let K be the kernel of the operator T which satisfies* (1.2) *and* (1.3). *Moreover, assume that $\int_\Omega \int_\Omega K(t,u)K(\xi,t)g(t)f(\xi)d\xi dt$ converges absolutely for some u, then*

$$\widehat{(g\widehat{f})}(u) = \int_\Omega \widehat{g}(\xi + u) f(\xi) d\xi. \tag{2.1}$$

Proof.

$$\begin{aligned}((\widehat{g\widehat{f}}))(u) &= \int_\Omega K(t,u)(g\widehat{f})(t)dt \\ &= \int_\Omega K(t,u)g(t)\widehat{f}(t)dt \\ &= \int_\Omega K(t,u)g(t)\int_\Omega K(\xi,t)f(\xi)d\xi dt \\ &= \int_\Omega \int_\Omega K(t,u)K(\xi,t)g(t)f(\xi)d\xi dt.\end{aligned}$$

Therefore,

$$\widehat{(g\widehat{f})}(u) = \int_\Omega \int_\Omega K(t,u)K(\xi,t)g(t)f(\xi)d\xi dt. \tag{2.2}$$

Now, interchanging the integrals of (2.2)(By Fubini's Theorem (see [13])) and using conditions (1.2) and (1.3), then

$$\begin{aligned}\widehat{(g\widehat{f})}(u) &= \int_\Omega \int_\Omega K(t,u)K(\xi,t)g(t)f(\xi)dtd\xi \\ &= \int_\Omega \int_\Omega K(u,t)K(\xi,t)g(t)f(\xi)dtd\xi \\ &= \int_\Omega f(\xi)\int_\Omega K(\xi+u,t)g(t)dtd\xi \\ &= \int_\Omega \widehat{g}(\xi+u)f(\xi)d\xi.\end{aligned}$$

□

Example 2.2. Consider $\Omega = (0,\infty)$ and $K(t,u) = e^{-\alpha tu}$, where $\alpha \in \mathbb{C}$. Clearly K satisfies conditions (1.2) and (1.3).

If $f(t) = e^{bt}$, then $\widehat{f}(u) = \frac{1}{\alpha u - b}$ for $\Re(\alpha u - b)>0$. In particular, If $f(t) = 1$, then $\widehat{f}(u) = \frac{1}{\alpha u}$ for $\Re(\alpha u)>0$.

Applying Theorem 2.1 on $h(t) = \frac{1-e^{-t}}{t}$ and for $\Re(\alpha u)>\max\{0,b\}$, we get that

$$\begin{aligned}\widehat{h}(u) &= \int_0^\infty \widehat{g}(\xi+u)f(\xi)d\xi \\ &= \int_0^\infty (\frac{1}{\xi+u} - \frac{\alpha}{\alpha(\xi+u)+1})d\xi \\ &= \log(\frac{1}{\alpha u}+1).\end{aligned}$$

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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