Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4428-4435
https://doi.org/10.28919/jmcs/5850
ISSN: 1927-5307

SOME FIXED POINT THEOREMS IN QUASICONVEX METRIC SPACES

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Abstract. In this paper, we study some fixed point theorems for self-mappings satisfying certain contraction principles on a quasiconvex complete metric space. In addition, we investigate some common fixed point theorems for a Banach operator pair under certain generalized contractions on a quasiconvex complete metric space. Our results generalize and improve several recent results in literature.

Keywords: quasiconvex metric spaces; quasiconvex functions; quasiconvexity; fixed point.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

Metric space is a crucial concept in functional and nonlinear analysis. Its topological structure has attracted the attention of many researchers both in pure and applied mathematics (see [1–16]). In an attempt to generalized metric spaces, [16] introduced the concept of convex metric spaces as seen below:

Definition 1.1. [16] Let (X,d) be a metric space, A mapping $W : X \times X \times [0,1] \to X$ is said to have convex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y) \tag{1}$$

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Received April 12, 2021

Definition 1.2. [16] A metric space (X,d) having convex structure W is called a convex metric space.

Adewale *et al.* in [5] extend the concept to quasiconvex metric spaces and define them as follows:

Definition 1.3. [5] Let (X,d) be a metric space, A mapping $\gamma : X \times X \times [0,1] \to X$ is said to have quasiconvex structure on X if for each $(x,y,\lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, \gamma(x,y,\lambda)) \leq max\{d(u,x), d(u,y)\} \tag{2}$$

Definition 1.4. [5] A metric space (X,d) having quasiconvex structure γ is called a quasiconvex metric space.

Remark 1.5 If $max\{d(u,x),d(u,y)\} = \lambda d(u,x) + (1-\lambda)d(u,y)$ in Definition 2.1 where $\lambda \in [0,1]$, we obtain convex structure in metric spaces as defined by Takahashi [16].

Example 1.6. [5] Considering a linear space, V which is at the same time a metric space with metric, d. For all $x,y \in V$ and $\lambda \in [0,1]$ if:

(i) $d(x,y) = d(x-y,0)$, and

(ii) $d(\lambda x + (1-\lambda)y, 0) = \max\{d(x,0), d(y,0)\}$

Then V is a quasiconvex metric space.

Example 1.7. [5] Considering a linear space, V which is at the same time a metric space with metric, d defined by

$$d(x,y) = \begin{cases} 0, & if \quad x = y = 0; \\ 1, & if \quad x,y \in N; \\ 0.5, & Otherwise. \end{cases}$$

For all $x,y,z \in V$ and $\lambda \in [0,1]$ if:

(i) $d(x, y+z) = d(x-y, z)$, and

(ii) $d(\lambda x + (1-\lambda)y, z) \leq \max\{d(x,z), d(y,z)\}$

Then V is a quasiconvex metric space but not convex metric space because if $x = 0, y = 2, z = 3$ and $\lambda = 0.5$, we obtain $d(1,3) = 1 > 0.5d(0,3) + 0.5d(2,3) = 0.75$.

Definition 1.8. **[5]** A subset C of a quasiconvex metric space X is said to be quasiconvex if $\gamma(x,y,\lambda) \in C$ for all $x.y \in C$ and $\lambda \in [0,1]$.

Definition 1.9. **[5]** Let (X,d,γ) be a complete quasiconvex metric space and E a nonempty closed convex subset of X. A mapping $T : E \to E$ is said to be (k,L)-Lipschitzian if there exists $k \in [1,\infty)$, $L \in [0,1)$ such that

$$d(Tx,Ty) \leq Ld(x,Tx) + kd(x,y), \forall x,y \in E. \tag{3}$$

Definition 1.10. **[5]** Let (X,d,γ) be a quasiconvex metric space. An open ball $S(z,r)$ in (X,d,γ) is defined by

$$S(z,r) = \{(x,y) \in X^2 : d(z,\gamma(x,y,\lambda)) < r\}.$$

Definition 1.11. **[5]** Let (X,d,γ) be a quasiconvex metric space. A closed ball $\bar{S}(z,r)$ in (X,d,γ) is defined by

$$\bar{S}(z,r) = \{(x,y) \in X^2 : d(z,\gamma(x,y,\lambda)) \leq r\}.$$

The following propositions show that an open ball and a closed ball in quasiconvex metric space are respectively open and closed subset of the space.

Proposition 1.12. **[5]** Let X be a quasiconvex metric space. Open ball $S(x,r)$ and closed ball $\bar{S}(x,r)$ in X are quasiconvex subsets of X.

Definition 1.13. **[11]** The ordered pair (T,S) of two self-maps of a metric space (X,d) is called a Banach operator pair if $F(S)$ is T-invariant, namely $T(F(S)) \subseteq F(S)$.

Theorem 1.14. **[5]** Let (X,d,γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a (k,L)-Lipschitzian mapping. Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in F$ there exists $q \in F$ with

$$d(Tq,q) \leq \psi(d(Tx,x)) \tag{4}$$

Then T has a fixed point in F.

Theorem 1.15. **[5]** Let (X,d,γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a (k,L)-Lipschitzian mapping. Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a comparison function such that for arbitrary $x \in F$ there exists $q \in F$ with

(i) $d(Tq,q) \leq \psi(d(Tx,x))$;

(ii) $d(Tq,Tx) \leq cd(Tx,x)$, $c > 0$.

Then T has a fixed point in F.

Theorem 1.16. [5] Let (X,d,γ) be a complete quasiconvex metric space, F, a nonempty closed quasiconvex subset of X and $T : F \to F$, a k-Lipschitzian involution with $k \in [0,1)$. Then T has a fixed point in F.

2. Main Results

To prove the next theorem, we need the following lemma.

Lemma 2.1. Let (X,d,γ) be a quasiconvex metric space, then the following statements hold:

a $d(x,y) = \frac{1}{2}[d(x,\gamma(x,y,\lambda)) + d(y,\gamma(x,y,\lambda))]$ $\forall (x,y,\lambda) \in X \times X \times I$.

b $d(x,\gamma(x,y,\lambda)) = d(y,\gamma(x,y,\lambda)) = d(x,y)$ $\forall x,y \in X$.

Proof:

a For any $(x,y,\lambda) \in X \times X \times I$, we have

$$
\begin{aligned}
d(x,y) &= \frac{1}{2}[d(x,\gamma(x,y,\lambda)) + d(y,\gamma(x,y,\lambda))] && (5)\\
&= \frac{1}{2}[\max\{d(x,x),d(x,y)\} + \max\{d(y,x),d(y,y)\}] && (6)\\
&= \frac{1}{2}[d(x,y) + d(y,x)] && (7)\\
&= \frac{1}{2} \times 2d(x,y) && (8)\\
&= d(x,y) && (9)
\end{aligned}
$$

b $d(x,\gamma(x,y,\lambda)) = \max\{d(x,x),d(x,y)\} = d(x,y)$.

Similarly, $d(y,\gamma(x,y,\lambda)) = \max\{d(y,x),d(y,y)\} = d(y,x)$.

Hence, $d(x,\gamma(x,y,\lambda)) = d(y,\gamma(x,y,\lambda)) = d(x,y) \forall x,y \in X$.

We prove existence and uniqueness of fixed point for $(\gamma - \phi)$-contraction mapping under different assumptions in this section.

Theorem 2.2. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ) and $T : E \to E$. If there exist $a,b,c,k \in R$ such that $a+b+c = 1$, $k \geq 2$, $c+1 < k$ and

$$ad(x,Tx) + bd(y,Ty) + cd(Tx,Ty) \leq kd(x,y),$$

for all $x, y \in E$, then T has at least one fixed point.

Proof: Suppose $x_0 \in E$ is arbitrary. We define a sequence $\{x_n\}_{\infty}^{n=0}$ by:

$$x_n = \gamma(x_{n-1}, Tx_{n-1}, \lambda), n \in \mathbb{N}. \tag{10}$$

Since E is quasiconvex, $x_n \in E$ for all $n \in \mathbb{N}$. Using Lemma 2.1b and (2.6), we obtain:

$$d(x_{n-1}, x_n) = d(x_n, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}). \tag{11}$$

For all $n \in \mathbb{N}$,

$$ad(x_n, Tx_n) + bd(x_{n-1}, Tx_{n-1}) + cd(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}). \tag{12}$$

which implies

$$ad(x_n, x_{n+1}) + bd(x_{n-1}, x_n) + cd(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}). \tag{13}$$

Then

$$d(x_n, x_{n+1}) \leq \left[\frac{b+c}{k-a-c}\right] d(x_{n-1}, x_n) \tag{14}$$

Let $q = \frac{b+c}{k-a-c}$, then

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) = q^2 d(x_{n-2}, x_{n-1}) = \ldots = q^n d(x_0, x_1). \tag{15}$$

So,

$$a+b+c = 1, k \geq 2, c+1 < k \Rightarrow \frac{b+c}{k-a-c} \in [0,1).$$

Hence, $\{x_n\}_{n=1}^{\infty}$ is a contraction sequence in E. Therefore, it is a Cauchy sequence. Since E is a closed subset of a complete space, there exists $u \in E$ such that $\lim_{n\to\infty} x_n = u$ and

$$ad(x_n, Tx_n) + bd(u, Tu) + cd(Tx_n, Tu) \leq kd(x_n, u). \tag{16}$$

As $n \to \infty$, we obtain

$$d(u, Tu) \leq 0. \tag{17}$$

Which implies $Tu = u$. Therefore, u is a fixed point of T.

Corollary 2.3. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ) and $T:E\to E$. If there exist $a,b,k\in R$ such that $a+b=1$, $k\geq 2$, $1<k$ and

$$ad(x,Tx)+bd(y,Ty)\leq kd(x,y),$$

for all $x,y\in E$, then T has at least one fixed point.

Proof: Set $c=0$ in Theorem 2.2.

Corollary 2.4. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ) and $T:E\to E$. If there exist $a,k\in R$ such that $k\geq 2$ and

$$ad(Tx,Ty)\leq kd(x,y),$$

for all $x,y\in E$, then T has at least one fixed point.

Proof: Set $a=b=0$ in Theorem 2.2.

Corollary 2.4. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ) and $T:E\to E$. If there exist $k\in R$ such that $k\geq 2$ and

$$d(Tx,Ty)\leq kd(x,y),$$

for all $x,y\in E$, then T has at least one fixed point.

Proof: Set $a=1$ in Corollary 2.4.

Definition 2.5. Let (X,d,γ) be a quasiconvex metric space and E be a quasiconvex subset of X. A self-mapping T on E has a property I if $T(W(x,y,\lambda))=W(T(x),T(y),\lambda)$ for each $x,y\in E$ and $\lambda\in[0,1]$.

Theorem 2.6. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ), $T,S:E\to E$ maps for which there exists the real numbers $a,b,c,k\in R$ satisfying $a+b+c=1$, $k\geq 2$, $c+1<k$ and

$$ad(Sx,Tx)+bd(Sy,Ty)+cd(Tx,Ty)\leq kd(Sx,Sy),$$

for all $x,y\in E$ with a Banach operator pair, (T,S), property I on S and a nonempty closed subset, $F(S)$, of E, then $F(T,S)$ is nonempty.

Proof: From

$$ad(Sx,Tx)+bd(Sy,Ty)+cd(Tx,Ty)\leq kd(Sx,Sy),$$

we obtain

$$ad(x,Tx)+bd(y,Ty)+cd(Tx,Ty)\leq kd(x,y)\forall x,y\in F(S).$$

$F(S)$ is quasiconvex because S has the property I. It follows from Theorem 2.2 that $F(T,S)$ is nonempty.

Theorem 2.7. Let E be a nonempty closed quasiconvex subset of a quasiconvex complete metric space (X,d,γ), $T,S:E\to E$ maps for which there exists the real numbers $a,b,c,k\in R$ satisfying $a+b+c=1$, $k\geq 2$, $c+1<k$ and

$$ad(Sx,STx)+bd(Sy,STy)+cd(STx,STy)\leq kd(Sx,Sy),$$

for all $x,y\in E$ with a Banach operator pair, (T,S), property I on S and a nonempty closed subset, $F(S)$, of E, then $F(T,S)$ is nonempty.

Proof: Since (S,T) is Banach operator pair,

$$ad(Sx,STx)+bd(Sy,STy)+cd(STx,STy)\leq kd(Sx,Sy),$$

implies

$$ad(x,Tx)+bd(y,Ty)+cd(Tx,Ty)\leq kd(x,y)\forall x,y\in F(S).$$

It follows from Theorem 2.2 that $F(T,S)$ is nonempty (Since S has the property I and $F(S)$ is closed).

Remark 2.8.

(i) Theorem 2,2, Theorem 2.6 and Theorem 2.7 extend and improve Theorem 3.2, Theorem 3.4 and Theorem 3.5 in [11] respectively.

(ii) The results in [11] are corollaries to our result. Theorem 2,2, Theorem 2.6 and Theorem 2.7 are extensions and generalizations of some related work in literature.

Conflict of Interests

The authors declare that there is no conflict of interests.

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