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DIMENSION SUBGROUP CONJECTURE FOR SOME p-GROUPS

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Abstract. The objective of this paper is to discuss some conditions on a finite p-group G, p prime, under which $D_5(G) = \gamma_5(G)$.

Keywords: dimension subgroups; metabelian groups; finite p-groups.

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1. Introduction

Let G be a finite group and $\mathbb{Z}G$ be the integral group ring of G. Let $\Delta(G)$ be the augmentation ideal of $\mathbb{Z}G$. For $n \geq 1$, let $D_n(G) = G \cap \{1 + \Delta^n(G)\}$ be the nth dimension subgroup of G. It is well known that $D_n(G) \supseteq \gamma_n(G)$, where $\gamma_n(G)$ is the nth term of the lower central series of G. The problem, of identifying groups G and $n \geq 1$ for which $D_n(G) = \gamma_n(G)$, is of interest for so many years (see [4, 5, 6, 7, 8]). This famous problem is known as Dimension Subgroup Conjecture. It has been proved that this conjecture holds for all groups and for $n \leq 3$ (see [4], [7]). Rips [8] gave an example of 2-group G for which $D_4(G) \neq \gamma_4(G)$. Tahara [9] gave the structure of $D_5(G)$ and proved that in general $D_5^6(G) \subseteq \gamma_5(G)$. Further, it has been proved in [6] that if G is a metabelian group, then $D_5^2(G) \subseteq \gamma_5(G)$. Thus, it follows that for metabelian p-group G, p odd, $D_5(G) = \gamma_5(G)$. N. Gupta [3] constructed a 2-group G, generated by four

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elements such that $D_5(G) \neq \gamma_5(G)$. Gupta et. al in [1, 2] have obtained certain conditions on p-group G (metabelian as well as non-metabelian), p prime, under which $D_5(G) = \gamma_5(G)$. In continuation to these results, we will find some more conditions on group G under which $D_5(G) = \gamma_5(G)$ (Theorems 3.1, 3.2 and 3.3).

2. PRELIMINARIES

Let G be a finite group of class 4 and $G = \gamma_1(G) \supseteq \gamma_2(G) \supseteq \gamma_3(G) \supseteq \gamma_4(G) \supseteq \gamma_5(G) = \{e\}$, be the lower central series of group G. We write $\gamma_1(G)/\gamma_2(G) \cong C_1 \oplus C_2 \oplus \cdots \oplus C_s$, where C_i is cyclic group generated by $\bar{x}_{1i},\ 1 \leq i \leq s$. Similarly, write $\gamma_2(G)/\gamma_3(G)$ and $\gamma_3(G)/\gamma_4(G)$ as a sum of t and u cyclic groups generated by $\bar{x}_{2k},\ 1 \leq k \leq t$ and $\bar{x}_{3l},\ 1 \leq l \leq u$ respectively. Write

$$x_{1i}^{d(i)} = x_{21}^{b_{i1}} x_{22}^{b_{i2}} \dots x_{2t}^{b_{it}} x_{31}^{c_{i1}} x_{32}^{c_{i2}} \dots x_{3u}^{c_{iu}} y_{4i},\ \ y_{4i} \in \gamma_4(G),\ 1 \leq i \leq s; \tag{1}$$

$$x_{2k}^{e(k)} = x_{31}^{d_{k1}} x_{32}^{d_{k2}} \dots x_{3u}^{d_{ku}} y'_{4k},\ \ y'_{4k} \in \gamma_4(G),\ \ 1 \leq k \leq t; \tag{2}$$

$$x_{3l}^{f(l)} = x_{41}^{f_{l1}} x_{42}^{f_{l2}} \dots x_{4u}^{f_{lv}},\ \ 1 \leq l \leq u; \tag{3}$$

$$[x_{1i}^{d(i)}, x_{1j}] = x_{31}^{\alpha_1^{(ij)}} x_{32}^{\alpha_2^{(ij)}} \dots x_{3u}^{\alpha_u^{(ij)}} x_{41}^{\beta_1^{(ij)}} x_{42}^{\beta_2^{(ij)}} \dots x_{4v}^{\beta_v^{(ij)}},\ 1 \leq i < j \leq s. \tag{4}$$

Lemma 2.1. ([9, Lemma 2.3]). *Let G be a group of class 4. Then for a non-negative integer d, we have*

$$\begin{aligned}[x,y]^d &= [x,y^d][x,y,y]^{-\binom{d}{2}}[x,y,y,y]^{-\binom{d}{3}} \\ &= [x^d,y][x,y,x]^{-\binom{d}{2}}[x,y,x,x]^{-\binom{d}{3}}\end{aligned}$$

$$\begin{aligned}[x,y,z]^d &= [x^d,y,z][y,x,x,z]^{\binom{d}{2}} \\ &= [x,y^d,z][y,x,y,z]^{\binom{d}{2}} \\ &= [x,y,z^d][y,x,z,z]^{\binom{d}{2}}\end{aligned}$$

We now recall the structure of fifth dimension subgroup given by Tahara in [9] as follows:

Theorem 2.2. ([9, Theorem 6.1]). *Let G be a group of class 4. Then $D_5(G)$ is equal to the subgroup generated by the elements*

$$\prod_{1\le i<j\le s}[x_{1i}{}^{d(i)},x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}}\prod_{1\le i\le s}\prod_{\substack{1\le k,l\le t,\\ k<l}}[x_{2l},x_{2k}]^{v_{ik}b_{il}}$$

$$\prod_{1\le i\le j\le k\le s}[x_{1i}{}^{d(i)},x_{1j},x_{ik}]^{w_{ijk}},$$

where u_{ij}, v_{ik}, v'_{ik}, w_{ijk}, w'_{ijk} and w''_{ijk} are the integers satisfying following conditions:

(5) $$w_{iii}=0,\ 1\le i\le s;$$

(6) $$u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{2}+w_{iij}d(i)+w''_{iij}d(j)=0,\ 1\le i<j\le s;$$

(7) $$-u_{ij}\binom{d(j)}{2}+w_{ijj}d(i)+w'_{ijj}d(j)=0,\ 1\le i<j\le s;$$

(8) $$w_{ijk}d(i)+w'_{ijk}d(j)+w''_{ijk}d(k)=0,\ 1\le i<j<k\le s;$$

(9) $$\sum_{i<h}u_{ih}b_{hk}-\sum_{h<i}u_{hi}\frac{d(i)}{d(h)}b_{hk}+v_{ik}d(i)+v'_{ik}e(k)=0,\ 1\le i\le s,\ 1\le k\le t;$$

(10) $$u_{ij}\frac{d(j)}{d(i)}\binom{d(i)}{3}+w_{iij}\binom{d(i)}{2}\equiv 0\,(\mathrm{mod}\,d(i)),\ 1\le i<j\le s;$$

(11) $$w_{ijj}\binom{d(i)}{2}+w''_{iij}\binom{d(j)}{2}\equiv 0\,(\mathrm{mod}\,d(i)),\ 1\le i<j\le s;$$

(12) $$u_{ij}\binom{d(j)}{3}+w'_{ijj}\binom{d(j)}{2}\equiv 0\,(\mathrm{mod}\,d(i)),\ 1\le i<j\le s;$$

(13) $$w_{ijk}\binom{d(i)}{2},\ w'_{ijk}\binom{d(j)}{2},\ w''_{ijk}\binom{d(k)}{2}\equiv 0\,(\mathrm{mod}\,d(i)),\ 1\le i<j<k\le s;$$

(14) $$v_{ik}\binom{d(i)}{2}-\sum_{h\le i}w_{hii}b_{hk}-\sum_{i<h}w''_{iih}b_{hk}\equiv 0\,(\mathrm{mod}\,(d(i),\,e(k))),$$
$$1\le i\le s,\ 1\le k\le t;$$

(15) $$\sum_{h\le i}w_{hij}b_{hk}+\sum_{i<h\le j}w'_{ihj}b_{hk}+\sum_{j<h}w''_{ijh}b_{hk}\equiv 0\,(\mathrm{mod}\,(d(i),\,e(k))),$$
$$1\le i<j\le s,\ 1\le k\le t;$$

$$(16)\qquad -\sum_{h<i} u_{hi}\frac{d(i)}{d(h)}\alpha_l{}^{(hi)} + \sum_{i<h} u_{ih}c_{hl} - \sum_{h<i} u_{hi}\frac{d(i)}{d(h)}c_{hl} - \sum_{k} v'_{ik}d_{kl}$$

$$-\sum_{g\le i\le h} w_{gih}\alpha_l{}^{(gh)} - \sum_{g\le h\le i} w_{ghi}\alpha_l{}^{(gh)} - \sum_{i<g\le h} w'_{igh}\alpha_l{}^{(gh)} \equiv 0\,(\mathrm{mod}\,(d(i),\, f(l))),$$

$$1\le i\le s,\ 1\le l\le u;$$

$$(17)\qquad \sum_i v_{ik}b_{ik} \equiv 0\,(\mathrm{mod}\, e(k)),\ 1\le k\le t;$$

$$(18)\qquad \sum_i v_{ik}b_{il} + \sum_i v_{il}b_{ik} \equiv 0\,(\mathrm{mod}\, e(k)),\ 1\le k< l\le t.$$

3. Main Results

Tahara [9] proved that the exponent of $D_5(G)/\gamma_5(G)$ is divisible by 6. Moreover, it has been proved in [6] that $D_5^2(G) \subseteq \gamma_5(G)[\gamma_2(G),\gamma_2(G)]$. Thus it follows that for metabelian p-group G, p odd prime, $D_5(G) = \gamma_5(G)$. Now, the problem is to find the conditions on metabelian 2-group G under which $D_5(G) = \gamma_5(G)$. In continuation to the results proved by Gupta et. al in [2], the following theorem has been proved:

Theorem 3.1. *Let G be a finite metabelian 2-group and $G/\gamma_2(G) \cong C_1 \oplus C_2 \oplus \cdots \oplus C_n$, where $C_i =< \bar{x}_{1i} >$, is a cyclic group of order $d(i)$, $1\le i\le n$. Let $d(1) = d(2) = \cdots = d(n-3) = 2, d(n-2) = 2^2, d(n-2) < d(n-1) < d(n)$ and $[x_{1n-1}, x_{1n}] = \{e\}$. Then $D_5(G) = \gamma_5(G)$.*

Note: In Thoerem 3.1 notations $w_{i,j,k}$, $w'_{i,j,k}$ and $w''_{i,j,k}$ have been used instead of w_{ijk}, w'_{ijk} and w''_{ijk}, $1\le i\le j\le k$ respectively, as stated in Theorem 2.2.

Proof: It is enough to prove the result for a group G of class 4. It follows from Theorem 2.2 that any element g of $D_5(G)$ is of the form

$$g = \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \cdot \prod_{1\le i\le j\le k\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{i,j,k}}$$

$$= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{1\le i<j<k\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{i,j,k}} \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{w_{i,j,j}}$$

$$(19)\qquad = A.B.C \text{ (say)}$$

where u_{ij} and $w_{i,j,k}$ are integers satisfying conditions given in Theorem 2.2.

$$\begin{aligned}
A &= \prod_{1\le i<j\le n} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} \\
&\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} \\
&\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= A_1.A_2.A_3.A_4 \ \text{(say)}
\end{aligned} \tag{20}$$

Now, for $1 \le i < j \le n-3$, condition (7) becomes $u_{ij} = w_{i,j,j}d(i) + w'_{i,j,j}d(j)$, which implies that,

$$\begin{aligned}
A_1 &= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{u_{ij}} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{w_{i,j,j}d(i)+w'_{i,j,j}d(j)} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}]^{d(j)(w_{i,j,j}+w'_{i,j,j})} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}^{d(j)}]^{w_{i,j,j}+w'_{i,j,j}} \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-(w_{i,j,j}+w'_{i,j,j})\binom{d(j)}{2}} \quad \text{[Using Lemma 2.1]} \\
&= \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{i,j,j}} \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w'_{i,j,j}}, \\
&\qquad \text{as } d(j)=2, \text{ for } 1\le j\le n-3 \text{ and } [x_{1i}^{d(i)}, x_{1j}^{d(j)}] \in [\gamma_2(G), \gamma_2(G)] = \{e\}.
\end{aligned}$$

Now, for $1 \le i < j \le n-3$, condition (12) becomes, $w'_{i,j,j} \equiv 0 \pmod{d(j)}$, thus

$$A_1 = \prod_{1\le i<j\le n-3} [x_{1i}^{d(i)}, x_{1j}, x_{1j}]^{-w_{i,j,j}} \tag{21}$$

For $j = n-2$, condition (6) becomes, $u_{in-2}\frac{d(n-2)}{d(i)} = -w_{i,i,n-2}d(i) - w''_{i,i,n-2}d(n-2)$

Thus

$$A_2 = \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{u_{in-2}\frac{d(n-2)}{d(i)}} = \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{-w_{i,i,n-2}d(i)-w''_{i,i,n-2}d(n-2)}$$

For $j = n-2$, condition (10) becomes; $w_{i,i,n-2} \equiv 0 \,(\mathrm{mod}\, d(i))$, i.e., $w_{i,i,n-2} = 2s$ for some integer s, as $d(i) = 2$ for $i < n-2$.

Thus

$$\begin{aligned}
A_2 &= \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{-2^2 s - w''_{i,i,n-2}d(n-2)} = \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}]^{-d(n-2)(s+w''_{i,i,n-2})} \\
&= \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}^{d(n-2)}]^{-(s+w''_{i,i,n-2})} \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}, x_{1n-2}]^{\binom{d(n-2)}{2}(s+w''_{i,i,n-2})} \quad \text{[Using Lemma 2.1]} \\
&= \prod_{i<n-2} [x_{1i}^{d(i)}, x_{1n-2}, x_{1n-2}]^{\binom{d(n-2)}{2}(s+w''_{i,i,n-2})},
\end{aligned}$$

as $[x_{1i}^{d(i)}, x_{1n-2}^{d(n-2)}] \in [\gamma_2(G), \gamma_2(G)] = \{e\}$.

$$\begin{aligned}
&= \prod_{i<n-2} [x_{1i}, x_{1n-2}, x_{1n-2}]^{d(i)\binom{d(n-2)}{2}(s+w''_{i,i,n-2})} \\
&\prod_{i<n-2} [x_{1n-2}, x_{1i}, x_{1i}, x_{1n-2}]^{-\binom{d(i)}{2}\binom{d(n-2)}{2}(s+w''_{i,i,n-2})} \quad \text{[Using Lemma 2.1]} \\
&= \prod_{i<n-2} [x_{1i}, x_{1n-2}, x_{1n-2}^{d(n-2)}]^{(d(n-2)-1)(s+w''_{i,i,n-2})} \\
&\prod_{i<n-2} [x_{1n-2}, x_{1i}, x_{1n-2}, x_{1n-2}]^{\binom{d(n-2)}{2}(d(n-2)-1)(s+w''_{i,i,n-2})} \\
&\prod_{i<n-2} [x_{1n-2}, x_{1i}, x_{1i}, x_{1n-2}]^{-\binom{d(n-2)}{2}(s+w''_{i,i,n-2})}
\end{aligned}$$

$$= \{e\}, \qquad \text{as } \binom{d(n-2)}{2} \equiv 0 \,(\mathrm{mod}\, d(i)) \tag{22}$$

$$\begin{aligned}
A_3 &= \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \\
&= \prod_{i<n-1} \prod_{1\le k\le t} [x_{2k}^{b_{ik}}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \prod_{i<n-1} \prod_{1\le l\le u} [x_{3l}^{c_{il}}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}} \quad \text{[from (1)]} \\
&= \prod_{i<n-1} \prod_{1\le k\le t} [x_{2k}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}b_{ik}} \prod_{i<n-1} \prod_{1\le l\le u} [x_{3l}, x_{1n-1}]^{u_{in-1}\frac{d(n-1)}{d(i)}c_{il}} \quad \text{[Using Lemma 2.1]}
\end{aligned}$$

$$= \prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}b_{ik}}\prod_{1\le l\le u}[x_{3l},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}}$$

For $i=n-1$, condition (9) becomes, $\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}b_{ik} = u_{n-1n}b_{nk}+v_{n-1k}d(n-1)+v'_{n-1k}e(k)$

$$= \prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{u_{n-1n}b_{nk}+v_{n-1k}d(n-1)+v'_{n-1k}e(k)}$$

$$\prod_{1\le l\le u}[x_{3l},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{u_{n-1n}b_{nk}}\prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{v_{n-1k}d(n-1)}$$

$$\prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{v'_{n-1k}e(k)}\prod_{1\le l\le u}[x_{3l},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{u_{n-1n}b_{nk}}\prod_{1\le k\le t}[x_{2k},x_{1n-1}^{d(n-1)}]^{v_{n-1k}}$$

$$\prod_{1\le k\le t}[x_{2k},x_{1n-1},x_{1n-1}]^{-v_{n-1k}\binom{d(n-1)}{2}}\prod_{1\le k\le t}[x_{2k}^{e(k)},x_{1n-1}]^{v'_{n-1k}}$$

$$\prod_{1\le l\le u}[x_{3l},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}}$$

$$= \prod_{1\le k\le t}[x_{2k},x_{1n-1}]^{u_{n-1n}b_{nk}}\prod_{1\le k\le t}[x_{2k},x_{1n-1},x_{1n-1}]^{-v_{n-1k}\binom{d(n-1)}{2}}$$

$$\prod_{1\le l\le u}[x_{3l},x_{1n-1}]^{\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}+\sum_k v'_{n-1k}d_{kl}}, \qquad \text{[from (2)]} \tag{23}$$

For $i=n-1$, condition (14) and (16) becomes,

$$v_{n-1k}\binom{d(n-1)}{2} = \sum_{i<n-1}w_{i,n-1,n-1}b_{ik}+w''_{n-1,n-1,n}b_{nk}+d(n-1)a+e(k)b,$$

and

$$\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}c_{il}+\sum_k v'_{n-1k}d_{kl} = -\sum_{i<n-1}u_{in-1}\frac{d(n-1)}{d(i)}\alpha_l^{(in-1)}+u_{n-1n}c_{nl}$$

$$-w_{n-1,n-1,n}\alpha_l^{(n-1n)}-\sum_{i<n-1}w_{i,n-1,n}\alpha_l^{(in)}-2\sum_{i<n-1}w_{i,n-1,n-1}\alpha_l^{(in-1)}$$

$$-\sum_{i<j<n-1}w_{i,j,n-1}\alpha_l^{(ij)}-w'_{n-1,n,n}\alpha_l^{(nn)}+d(n-1)c+f(l)d$$

for some integers a, b, c and d. Using these equations in (23), we get the following:

$$A_3 = \prod_{1\le k\le t}[x_{2k}, x_{1n-1}]^{u_{n-1n}b_{nk}} \prod_{1\le k\le t}[x_{2k}, x_{1n-1}, x_{1n-1}]^{-\sum_{i<n-1} w_{i,n-1,n-1}b_{ik}}$$

$$\prod_{1\le k\le t}[x_{2k}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}b_{nk}+d(n-1)a+e(k)b}$$

$$\prod_{1\le l\le u}[x_{3l}, x_{1n-1}]^{-\sum_{i<n-1} u_{in-1}\frac{d(n-1)}{d(i)}\alpha_l^{(in-1)}+u_{n-1n}c_{nl}-w_{n-1,n-1,n}\alpha_l^{(n-1n)}}$$

$$\prod_{1\le l\le u}[x_{3l}, x_{1n-1}]^{-\sum_{i<n-1} w_{i,n-1,n}\alpha_l^{(in)}-2\sum_{i<n-1} w_{i,n-1,n-1}\alpha_l^{(in-1)}}$$

$$\prod_{1\le l\le u}[x_{3l}, x_{1n-1}]^{-\sum_{i<j<n-1} w_{i,j,n-1}\alpha_l^{(ij)}-w'_{n-1,n,n}\alpha_l^{(nn)}+d(n-1)c+f(l)d}$$

$$= \prod_{1\le k\le t}[x_{2k}^{b_{nk}}, x_{1n-1}]^{u_{n-1n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-w_{i,n-1,n-1}}$$

$$[x_{1n}^{d(n)}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-u_{in-1}\frac{d(n-1)}{d(i)}}$$

$$\prod_{1\le l\le u}[x_{3l}^{c_{nl}}, x_{1n-1}]^{u_{n-1n}}[x_{1n-1}^{d(n-1)}, x_{1n}, x_{1n-1}]^{-w_{n-1,n-1,n}}$$

$$\prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-2w_{i,n-1,n-1}}$$

$$\prod_{i<j<n-1}[x_{1i}^{d(i)}, x_{1j}, x_{1n-1}]^{-w_{i,j,n-1}}[x_{1n}^{d(n)}, x_{1n}, x_{1n-1}]^{-w'_{n-1,n,n}}$$

$$= [x_{1n}^{d(n)}, x_{1n-1}]^{u_{n-1n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-w_{i,n-1,n-1}}$$

$$[x_{1n}^{d(n)}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-u_{in-1}\frac{d(n-1)}{d(i)}}$$

$$\prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}} \prod_{i<n-1}[x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-2w_{i,n-1,n-1}}$$

$$\prod_{i<j<n-1}[x_{1i}^{d(i)}, x_{1j}, x_{1n-1}]^{-w_{i,j,n-1}}[x_{1n-1}^{d(n-1)}, x_{1n}, x_{1n-1}]^{-w_{n-1,n-1,n}} \tag{24}$$

Now we will separately solve the terms of (24), firstly consider,

$$[x_{1n}^{d(n)}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}} = [x_{1n}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}d(n)}$$

$$[x_{1n-1}, x_{1n}, x_{1n}, x_{1n-1}]^{w''_{n-1,n-1,n}\binom{d(n)}{2}} \quad \text{[Using Lemma 2.1]}$$

Since $d(n-1) < d(n) \Rightarrow 2|\frac{d(n)}{d(n-1)}$. Let $d(n) = 2^m d(n-1)$, $m \geq 1$. Hence

$$\begin{aligned}
[x_{1n}^{d(n)}, x_{1n-1}, x_{1n-1}]^{-w''_{n-1,n-1,n}} &= [x_{1n}, x_{1n-1}, x_{1n-1}^{d(n-1)}]^{-2^m w''_{n-1,n-1,n}} \\
&\quad [x_{1n-1}, x_{1n}, x_{1n-1}, x_{1n-1}]^{-2^m w''_{n-1,n-1,n}\binom{d(n-1)}{2}} \\
&\quad [x_{1n-1}, x_{1n}, x_{1n}, x_{1n-1}]^{w''_{n-1,n-1,n}\binom{d(n)}{2}} \\
&= \{e\}, \qquad (25)
\end{aligned}$$

because $2^m\binom{d(n-1)}{2} \equiv 0 \,(\mathrm{mod}\, d(n-1))$ and $\binom{d(n)}{2} \equiv 0 \,(\mathrm{mod}\, d(n-1))$.

$$\begin{aligned}
\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-u_{in-1}\frac{d(n-1)}{d(i)}} &= \prod_{i<n-1, i\neq n-2} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-u_{in-1}\frac{d(n-1)}{d(i)}} \\
&\quad [x_{1n-2}^{d(n-2)}, x_{1n-1}, x_{1n-1}]^{-u_{n-2\,n-1}\frac{d(n-1)}{d(n-2)}}
\end{aligned}$$

In a similar way, as in (25), we get the following

$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-u_{in-1}\frac{d(n-1)}{d(i)}} = \{e\} \qquad (26)$$

$$\begin{aligned}
\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}} &= \prod_{i<n-1} [x_{1i}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}d(i)} \\
&\quad [x_{1n}, x_{1i}, x_{1i}, x_{1n-1}]^{w_{i,n-1,n}\binom{d(i)}{2}} \quad \text{[Using Lemma 2.1]}
\end{aligned}$$

For $j = n-1$, $k = n$ condition (8) becomes, $w_{i,n-1,n}d(i) = -w'_{i,n-1,n}d(n-1) - w''_{i,n-1,n}d(n)$ and condition (13) becomes $w_{i,n-1,n}\binom{d(i)}{2} \equiv 0 \,(\mathrm{mod}\, d(i))$. This implies

$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}} = \prod_{i<n-1} [x_{1i}, x_{1n}, x_{1n-1}]^{w'_{i,n-1,n}d(n-1)+w''_{i,n-1,n}d(n)}$$

Again in a similar way, as in (25), we get

$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n}, x_{1n-1}]^{-w_{i,n-1,n}} = \{e\} \qquad (27)$$

For $i = n-1$, condition (14) becomes, $\sum_{i<n-1} w_{i,n-1,n-1} b_{ik} = v_{n-1k}\binom{d(n-1)}{2} + w''_{n-1,n-1,n} b_{nk}$ $+\mathbb{Z}$-linear combination $d(n-1)$ and $e(k)$. Thus

$$\begin{aligned}
\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-2w_{i,n-1,n-1}} &= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}, x_{1n-1}]^{-\sum_{i<n-1} 2w_{i,n-1,n-1} b_{ik}} \quad \text{[from (1)]} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n-1}, x_{1n-1}]^{-2v_{n-1k}\binom{d(n-1)}{2}} \\
&\quad \prod_{1\le k\le t} [x_{2k}, x_{1n-1}, x_{1n-1}]^{-2w''_{n-1,n-1,n} b_{nk}} \\
&= \prod_{1\le k\le t} [x_{1n}^{d(n)}, x_{1n-1}, x_{1n-1}]^{-2w''_{n-1,n-1,n}} \quad \text{[from (1)]} \qquad (28)
\end{aligned}$$

Similarly, it can be seen that

$$\prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-2w_{i,n-1,n-1}} = \{e\} \tag{29}$$

Using condition (14) for $i = n-1,\ j = n$, we get the following

$$[x_{1n-1}^{d(n-1)}, x_{1n}, x_{1n-1}]^{-w_{n-1,n-1,n}} = \{e\} \tag{30}$$

Using equations (25), (26),(27),(29), (30), (24) and given assumption, we get the following

$$A_3 = \prod_{i<n-1} [x_{1i}^{d(i)}, x_{1n-1}, x_{1n-1}]^{-w_{i,n-1,n-1}} \prod_{i<j<n-1} [x_{1i}^{d(i)}, x_{1j}, x_{1n-1}]^{-w_{i,j,n-1}} \tag{31}$$

Now

$$\begin{aligned}
A_4 &= \prod_{i<n} [x_{1i}^{d(i)}, x_{1n}]^{u_{in}\frac{d(n)}{d(i)}} \\
&= \prod_{1\le k\le t} [x_{2k}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)} b_{ik}} \prod_{1\le l\le u} [x_{3l}^{c_{il}}, x_{1n}]^{\sum_{i<n} u_{in}\frac{d(n)}{d(i)}} \quad \text{[from (1)]}
\end{aligned}$$

For $i=n$, $1\le k\le t$, condition (9) gives, $\sum_{i<n} u_{in}\frac{d(n)}{d(i)}b_{ik} = v_{nk}d(n)+v'_{nk}e(k)$, which implies that

$$
\begin{aligned}
A_4 &= \prod_{1\le k\le t}[x_{2k},x_{1n}]^{v_{nk}d(n)+v'_{nk}e(k)}\prod_{1\le l\le u}[x_{3l},x_{1n}]^{\sum_{i<n}u_{in}\frac{d(n)}{d(i)}c_{il}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n}^{d(n)}]^{v_{nk}}\prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{1\le k\le t}[x_{2k}^{e(k)},x_{1n}]^{v'_{nk}}\\
&\quad \prod_{1\le l\le u}[x_{3l},x_{1n}]^{\sum_{i<n}u_{in}\frac{d(n)}{d(i)}c_{il}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{1\le k\le t}\prod_{1\le l\le u}[x_{3l}^{d_{kl}},x_{1n}]^{v'_{nk}}\\
&\quad \prod_{1\le l\le u}[x_{3l},x_{1n}]^{\sum_{i<n}u_{in}\frac{d(n)}{d(i)}c_{il}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{1\le l\le u}[x_{3l},x_{1n}]^{\sum_{i<n}u_{in}\frac{d(n)}{d(i)}c_{il}+\sum_k v'_{nk}d_{kl}}
\end{aligned}
$$

For $i=n$, condition (16) becomes $\sum_{i<n}u_{in}\frac{d(n)}{d(i)}c_{il}+\sum_k v'_{nk}d_{kl} = -\sum_{i<n}u_{in}\frac{d(n)}{d(i)}\alpha_l^{(in)} - 2\sum_{i<n}w_{i,n,n}\alpha_l^{(in)} - \sum_{i<j<n}w_{i,j,n}\alpha_l^{(ij)} - \sum_{i<n}w_{i,i,n}\alpha_l^{(ii)} + \mathbb{Z}$-linear combination of $d(n)$ and $f(l)$.

Thus, we have

$$
\begin{aligned}
A_4 &= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{1\le l\le u}[x_{3l},x_{1n}]^{-\sum_{i<n}u_{in}\frac{d(n)}{d(i)}\alpha_l^{(in)}-2\sum_{i<n}w_{i,n,n}\alpha_l^{(in)}}\\
&\quad \prod_{1\le l\le u}[x_{3l},x_{1n}]^{-\sum_{i<j<n}w_{i,j,n}\alpha_l^{(ij)}-\sum_{i<n}w_{i,i,n}\alpha_l^{(ii)}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{-u_{in}\frac{d(n)}{d(i)}}\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{-2w_{i,n,n}}\\
&\quad \prod_{i<j<n}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{-w_{i,j,n}}\prod_{i<n}[x_{1i}^{d(i)},x_{1i},x_{1n}]^{-w_{i,i,n}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{i<n}\prod_{1\le k\le t}[x_{2k}^{b_{ik}},x_{1n},x_{1n}]^{-u_{in}\frac{d(n)}{d(i)}}\\
&\quad \prod_{i<n}\prod_{1\le k\le t}[x_{2k}^{b_{ik}},x_{1n},x_{1n}]^{-2w_{i,n,n}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{-w_{i,j,n}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{-w_{i,n-1,n}}\\
&= \prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}\prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-\sum_{i<n}u_{in}\frac{d(n)}{d(i)}b_{ik}}
\end{aligned}
$$

$$
\prod_{1\le k\le t}[x_{2k},x_{1n},x_{1n}]^{-2\sum_{i<n}w_{i,n,n}b_{ik}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{-w_{i,j,n}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{-w_{i,n-1,n}} \tag{32}
$$

Now we will separately solve the terms of equation (32).

For $i=n$, $1\leq k\leq t$, condition (14) implies that

$$\prod_{1\leq k\leq t}[x_{2k},x_{1n},x_{1n}]^{-v_{nk}\binom{d(n)}{2}}=\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{-w_{i,n,n}} \tag{33}$$

For $i=n$, condition (9) gives, $\sum_{i<n}u_{in}\frac{d(n)}{d(i)}b_{ik}=v_{nk}d(n)+v'_{nk}e(k)$, which implies that

$$\prod_{1\leq k\leq t}[x_{2k},x_{1n},x_{1n}]^{-\sum_{i<n}u_{in}\frac{d(n)}{d(i)}b_{ik}}=\{e\} \tag{34}$$

For $i=n$, $1\leq k\leq t$, condition (14) becomes, $\sum_{i<n}w_{i,n,n}b_{ik}=v_{nk}\binom{d(n)}{2}+\mathbb{Z}$-linear combination of $d(n)$ and $e(k)$, which implies that

$$\prod_{1\leq k\leq t}[x_{2k},x_{1n},x_{1n}]^{-2\sum_{i<n}w_{i,n,n}b_{ik}}=\{e\} \tag{35}$$

From (33), (34)and (35), equation (32) reduces to

$$A_4=\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{-w_{i,n,n}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{-w_{i,n-1,n}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{-w_{i,j,n}} \tag{36}$$

Substitute the values A_1, A_2, A_3 and A_4 from (21), (22), (31) and (36) in (20), we get the following

$$\begin{aligned}A=&\prod_{1\leq i<j\leq n-3}[x_{1i}^{d(i)},x_{1j},x_{1j}]^{-w_{i,j,j}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n-1}]^{-w_{i,n-1,n-1}}\\&\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n-1}]^{-w_{i,j,n-1}}\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{-w_{i,n,n}}\\&\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{-w_{i,n-1,n}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{-w_{i,j,n}}\end{aligned} \tag{37}$$

Now, consider

$$\begin{aligned}B=&\prod_{1\leq i<j<k\leq n}[x_{1i}^{d(i)},x_{1j},x_{1k}]^{w_{i,j,k}}\\=&\prod_{1\leq i<j<k\leq n-2}[x_{1i}^{d(i)},x_{1j},x_{1k}]^{w_{i,j,k}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n-1}]^{w_{i,j,n-1}}\\&\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{w_{i,j,n}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{w_{i,n-1,n}}\end{aligned} \tag{38}$$

For $i<j<n-2$, $d(i)=d(j)=2$, therefore, condition (13) becomes, $w_{i,j,k}\equiv 0\,(\mathrm{mod}\,(d(j)))$, which implies that

$$\prod_{1\leq i<j<k\leq n-2}[x_{1i}^{d(i)},x_{1j},x_{1k}]^{w_{i,j,k}}=\{e\}$$

Thus equation (38) reduces to

$$B=\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n-1}]^{w_{i,j,n-1}}\prod_{i<j<n-1}[x_{1i}^{d(i)},x_{1j},x_{1n}]^{w_{i,j,n}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n}]^{w_{i,n-1,n}} \tag{39}$$

Finally, consider

$$\begin{aligned}C&=\prod_{1\leq i<j\leq n}[x_{1i}^{d(i)},x_{1j},x_{1j}]^{w_{i,j,j}}\\&=\prod_{1\leq i<j\leq n-3}[x_{1i}^{d(i)},x_{1j},x_{1j}]^{w_{i,j,j}}\prod_{i<n-2}[x_{1i}^{d(i)},x_{1n-2},x_{1n-2}]^{w_{i,n-2,n-2}}\\&\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n-1}]^{w_{i,n-1,n-1}}\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{w_{i,n,n}}\end{aligned} \tag{40}$$

By using condition (11) for $j=n-2$, we get the following

$$\prod_{i<n-2}[x_{1i}^{d(i)},x_{1n-2},x_{1n-2}]^{w_{i,n-2,n-2}}=\{e\} \tag{41}$$

Thus (40) reduces to

$$\begin{aligned}C&=\prod_{1\leq i<j\leq n-3}[x_{1i}^{d(i)},x_{1j},x_{1j}]^{w_{i,j,j}}\prod_{i<n-1}[x_{1i}^{d(i)},x_{1n-1},x_{1n-1}]^{w_{i,n-1,n-1}}\\&\prod_{i<n}[x_{1i}^{d(i)},x_{1n},x_{1n}]^{w_{i,n,n}}\end{aligned} \tag{42}$$

By putting values of A, B and C from (37), (39) and (42) in (19), we get $g=\{e\}$. Thus $D_5(G)=\gamma_5(G)=\{e\}$. □

Several conditions have been obtained on metabelian 2-group G under which the exponent of $D_5(G)/\gamma_5(G)$ is 1, i.e., $D_5(G)=\gamma_5(G)$. Let us now drop the condition that G is metabelian group. It is well known that the exponent of $D_5(G)/\gamma_5(G)$ is divisible by 6. Thus, if G is finite p-group, $p\neq 2,3$, then $D_5(G)=\gamma_5(G)$. We will now discuss some conditions on finite 2-group and finite 3-group G for which $D_5(G)=\gamma_5(G)$.

Theorem 3.2. *Let G be a finite 3-group. Let $G/\gamma_2(G)\cong C_1\oplus C_2\oplus C_3$ where C_i is cyclic group of order $d(i)$, $i=1,2,3$. Let $d(i)=d(i-1)^2$, $i=2,3$ and $\gamma_2(G)/\gamma_3(G)$ be cyclic. Then the exponent of $D_5(G)/\gamma_5(G)$ is 1, i.e., $D_5(G)=\gamma_5(G)$.*

Proof: It is enough to prove the result for a 3-group G of class 4. Let g be an arbitrary element of $D_5(G)$, then by using Theorem 2.2, g is of the form

$$g = \prod_{1\le i<j\le 3}[x_{1i}^{d(i)}, x_{1j}]^{u_{ij}\frac{d(j)}{d(i)}} \prod_{1\le i\le j\le k\le 3}[x_{1i}^{d(i)}, x_{1j}, x_{1k}]^{w_{ijk}}$$

$$= [x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}}[x_{12}^{d(2)}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}}[x_{11}^{d(1)}, x_{12}, x_{12}]^{w_{122}}$$

$$[x_{11}^{d(1)}, x_{13}, x_{13}]^{w_{133}}[x_{12}^{d(2)}, x_{13}, x_{13}]^{w_{233}}[x_{11}^{d(1)}, x_{12}, x_{13}]^{w_{123}} \tag{43}$$

Since G is of class 4 and $ab = ba[a,b]\ \forall\, a,b \in G$, we have

$$g^2 = [x_{11}^{d(1)}, x_{12}]^{2u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)}, x_{13}]^{2u_{13}\frac{d(3)}{d(1)}}[x_{12}^{d(2)}, x_{13}]^{2u_{23}\frac{d(3)}{d(2)}}[x_{11}^{d(1)}, x_{12}, x_{12}]^{2w_{122}}$$

$$[x_{11}^{d(1)}, x_{13}, x_{13}]^{2w_{133}}[x_{12}^{d(2)}, x_{13}, x_{13}]^{2w_{233}}[x_{11}^{d(1)}, x_{12}, x_{13}]^{2w_{123}}$$

Let

$$A = [x_{12}^{d(2)}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}}[x_{11}^{d(1)}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}}$$

$$B = [x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}}$$

$$C = [x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}}[x_{12}^{d(2)}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}}$$

Clearly

$$ABC = [x_{11}^{d(1)}, x_{12}]^{2u_{12}\frac{d(2)}{d(1)}}[x_{11}^{d(1)}, x_{13}]^{2u_{13}\frac{d(3)}{d(1)}}[x_{12}^{d(2)}, x_{13}]^{2u_{23}\frac{d(3)}{d(2)}} \tag{44}$$

Using (44), g^2 turns out to be

$$g^2 = ABC[x_{11}^{d(1)}, x_{12}, x_{12}]^{2w_{122}}[x_{11}^{d(1)}, x_{13}, x_{13}]^{2w_{133}}[x_{12}^{d(2)}, x_{13}, x_{13}]^{2w_{233}}[x_{11}^{d(1)}, x_{12}, x_{13}]^{2w_{123}} \tag{45}$$

Since $\gamma_2(G)/\gamma_3(G)$ is cyclic, thus equation (1) reduces to

$$x_{1i}^{d(i)} = x_{21}^{b_{i1}} x_{31}^{c_{i1}} x_{32}^{c_{i2}} \dots x_{3u}^{c_{iu}} y_{4i},\ y_{4i} \in \gamma_4(G),\ 1 \le i \le 3 \tag{46}$$

Now

$$\begin{aligned}
A &= [x_{12}^{d(2)}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}}[x_{11}^{d(1)}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}} \\
&= [x_{21}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}b_{21}} \prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}}[x_{21}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}b_{11}} \\
&\prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}c_{1l}} \qquad [\text{Using (46) and Lemma 2.1}] \\
&= [x_{21}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}b_{21}+u_{13}\frac{d(3)}{d(1)}b_{11}} \prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}+u_{13}\frac{d(3)}{d(1)}c_{1l}}
\end{aligned}$$

Now, condition (9) with $i=3$ gives that $u_{23}\frac{d(3)}{d(2)}b_{21}+u_{13}\frac{d(3)}{d(1)}b_{11} = v_{31}d(3)+v'_{31}e(1)$. Thus

$$\begin{aligned}
A &= [x_{21}, x_{13}]^{v_{31}d(3)}[x_{21}, x_{13}]^{v'_{31}e(1)} \prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}+u_{13}\frac{d(3)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{13}^{d(3)}]^{v_{31}}[x_{21}, x_{13}, x_{13}]^{-v_{31}\binom{d(3)}{2}}[x_{21}^{e(1)}, x_{13}]^{v'_{31}} \\
&\prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}+u_{13}\frac{d(3)}{d(1)}c_{1l}} \\
&= [x_{21}, x_{21}^{b_{31}}]^{v_{31}}[x_{21}, x_{13}, x_{13}]^{-v_{31}\binom{d(3)}{2}} \prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}+u_{13}\frac{d(3)}{d(1)}c_{1l}+v'_{31}d_{1l}}
\end{aligned}$$

$$= [x_{21}, x_{13}, x_{13}]^{-v_{31}\binom{d(3)}{2}} \prod_{1\le l\le u}[x_{3l}, x_{13}]^{u_{23}\frac{d(3)}{d(2)}c_{2l}+u_{13}\frac{d(3)}{d(1)}c_{1l}+v'_{31}d_{1l}}$$

It follows from conditions (14) and (16) for $i=3$ that

$$v_{31}\binom{d(3)}{2} - w_{133}b_{11} - w_{233}b_{21} \equiv 0 \,(\bmod\,(d(3), e(1)))$$

and

$$-u_{13}\frac{d(3)}{d(1)}\alpha_l^{(13)} - u_{23}\frac{d(3)}{d(2)}\alpha_l^{(23)} - u_{23}\frac{d(3)}{d(2)}c_{2l} - u_{13}\frac{d(3)}{d(1)}c_{1l} - v'_{31}d_{1l} - 2w_{133}\alpha_l^{(13)}$$

$$-2w_{233}\alpha_l^{(23)} - w_{123}\alpha_l^{(12)} \equiv 0 \,(\bmod\,(d(3), f(l)))$$

Using these conditions, we get the following:

$$A = [x_{11}^{d(1)}, x_{13}, x_{13}]^{-w_{133}} [x_{12}^{d(2)}, x_{13}, x_{13}]^{-w_{233}} [x_{11}^{d(1)}, x_{13}, x_{13}]^{-u_{13}\frac{d(3)}{d(1)}}$$
$$[x_{12}^{d(2)}, x_{13}, x_{13}]^{-u_{23}\frac{d(3)}{d(2)}} [x_{11}^{d(1)}, x_{13}, x_{13}]^{-2w_{133}} [x_{12}^{d(2)}, x_{13}, x_{13}]^{-2w_{233}} [x_{11}^{d(1)}, x_{12}, x_{13}]^{-w_{123}}$$
$$= [x_{11}^{d(1)}, x_{13}, x_{13}]^{-3w_{133}} [x_{12}^{d(2)}, x_{13}, x_{13}]^{-3w_{233}} [x_{11}^{d(1)}, x_{12}, x_{13}]^{-w_{123}}$$
$$[x_{21}, x_{13}, x_{13}]^{-u_{23}\frac{d(3)}{d(2)}b_{21} - u_{13}\frac{d(3)}{d(1)}b_{11}}$$

Using condition (9) for $i = 3$, we get

$$[x_{21}, x_{13}, x_{13}]^{-u_{23}\frac{d(3)}{d(2)}b_{21} - u_{13}\frac{d(3)}{d(1)}b_{11}} = \{e\}$$

Thus,

$$A = [x_{11}^{d(1)}, x_{13}, x_{13}]^{-3w_{133}} [x_{12}^{d(2)}, x_{13}, x_{13}]^{-3w_{233}} [x_{11}^{d(1)}, x_{12}, x_{13}]^{-w_{123}} \tag{47}$$

Let

$$B = [x_{11}^{d(1)}, x_{12}]^{u_{12}\frac{d(2)}{d(1)}} [x_{11}^{d(1)}, x_{13}]^{u_{13}\frac{d(3)}{d(1)}}$$
$$= B_1 B_2 \text{ (say)} \tag{48}$$

Consider

$$B_1 = [x_{11}, x_{12}]^{u_{12}d(2)} [x_{11}, x_{12}, x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2}} [x_{11}, x_{12}, x_{11}, x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{3}}$$
$$= [x_{11}, x_{12}^{d(2)}]^{u_{12}} [x_{11}, x_{12}, x_{12}]^{-u_{12}\binom{d(2)}{2}} [x_{11}, x_{12}, x_{12}, x_{12}]^{-u_{12}\binom{d(2)}{3}}$$
$$[x_{11}, x_{12}, x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2}} [x_{11}, x_{12}, x_{11}, x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{3}} \qquad \text{[Using (46) and Lemma 2.1]} \tag{49}$$

Now we will solve separately the terms of the equation (49):

Since $\binom{d(2)}{3} \equiv 0 \,(\mathrm{mod}\,(d(1)))$ and $\frac{d(2)}{d(1)}\binom{d(1)}{3} \equiv 0 \,(\mathrm{mod}\,(d(1)))$, thus

$$[x_{11}, x_{12}, x_{12}, x_{12}]^{-u_{12}\binom{d(2)}{3}} \text{ and } [x_{11}, x_{12}, x_{11}, x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{3}} = \{e\} \tag{50}$$

$$[x_{11},x_{12},x_{11}]^{u_{12}\frac{d(2)}{d(1)}\binom{d(1)}{2}} = [x_{11},x_{12},x_{11}]^{u_{12}d(1)^2\frac{(d(1)-1)}{2}}$$

$$= [x_{11}^{d(1)},x_{12},x_{11}]^{u_{12}d(1)\frac{(d(1)-1)}{2}}$$

$$[x_{12},x_{11},x_{11},x_{11}]^{u_{12}d(1)\frac{(d(1)-1)}{2}\binom{d(1)}{2}}$$

$$= [x_{11}^{d(1)},x_{12},x_{11}^{d(1)}]^{u_{12}\frac{(d(1)-1)}{2}}$$

$$= \{e\} \tag{51}$$

Using (50) and (51), equation (49) reduces to

$$B_1 = [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{12},x_{12}]^{-u_{12}\binom{d(2)}{2}}$$

Similarly we get $B_2 = [x_{11},x_{13}^{d(3)}]^{u_{13}}[x_{11},x_{13},x_{13}]^{-u_{13}\binom{d(3)}{2}}$. Thus

$$B = [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{13}^{d(3)}]^{u_{13}}[x_{11},x_{12},x_{12}]^{-u_{12}\binom{d(2)}{2}}[x_{11},x_{13},x_{13}]^{-u_{13}\binom{d(3)}{2}}$$

Now condition (7) for $i=1, j=2$ yields that $-u_{12}\binom{d(2)}{2} + w_{122}d(1) + w'_{122}d(2) = 0$ and for $i=1, j=3$, we get $-u_{13}\binom{d(3)}{2} + w_{133}d(1) + w'_{133}d(3) = 0$. Thus

$$B = [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{13}^{d(3)}]^{u_{13}}[x_{11},x_{12},x_{12}]^{-w_{122}d(1)}[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}$$

$$[x_{11},x_{13},x_{13}]^{-w_{133}d(1)}[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)}$$

$$= [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{13}^{d(3)}]^{u_{13}}[x_{11}^{d(1)},x_{12},x_{12}]^{-w_{122}}[x_{12},x_{11},x_{11},x_{12}]^{-w_{122}\binom{d(1)}{2}}$$

$$[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}[x_{11}^{d(1)},x_{13},x_{13}]^{-w_{133}}[x_{13},x_{11},x_{11},x_{13}]^{-w_{133}\binom{d(1)}{2}}$$

$$[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)}$$

Since $\binom{d(1)}{2} \equiv 0 \pmod{\mathrm{d}(1)}$, thus terms $[x_{12},x_{11},x_{11},x_{12}]^{-w_{122}\binom{d(1)}{2}}$ and $[x_{13},x_{11},x_{11},x_{13}]^{-w_{133}\binom{d(1)}{2}}$ reduce to $\{e\}$. Hence

$$B = [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{13}^{d(3)}]^{u_{13}}[x_{11}^{d(1)},x_{13},x_{13}]^{-w_{133}}[x_{11}^{d(1)},x_{12},x_{12}]^{-w_{122}}$$

$$[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)}$$

$$= [x_{11},x_{12}^{d(2)}]^{u_{12}}[x_{11},x_{13}^{d(3)}]^{u_{13}}\, D \text{ (say)}$$

$$= [x_{11},x_{21}]^{u_{12}b_{21}+u_{13}b_{31}} \prod_{1\le l\le u}[x_{11},x_{3l}]^{u_{12}c_{2l}+u_{13}c_{3l}}\, D$$

For $i=1$ condition (9) becomes $u_{12}b_{21}+u_{13}b_{31}+v_{11}d(1)+v'_{11}e(1)=0$. Thus

$$B = [x_{11},x_{21}]^{-v_{11}d(1)}[x_{11},x_{21}]^{-v'_{11}e(1)} \prod_{1\le l\le u}[x_{11},x_{3l}]^{u_{12}c_{2l}+u_{13}c_{3l}} D$$
$$= [x_{11}^{d(1)},x_{21}]^{-v_{11}}[x_{11},x_{21},x_{11}]^{v_{11}\binom{d(1)}{2}} \prod_{1\le l\le u}[x_{11},x_{3l}]^{-v'_{11}d_{1l}+u_{12}c_{2l}+u_{13}c_{3l}} D$$
$$= [x_{21}^{b_{11}},x_{21}]^{-v_{11}}[x_{11},x_{21},x_{11}]^{v_{11}\binom{d(1)}{2}} \prod_{1\le l\le u}[x_{11},x_{3l}]^{-v'_{11}d_{1l}+u_{12}c_{2l}+u_{13}c_{3l}} D$$
$$= [x_{11},x_{21},x_{11}]^{v_{11}\binom{d(1)}{2}} \prod_{1\le l\le u}[x_{3l},x_{11}]^{v'_{11}d_{1l}-u_{12}c_{2l}-u_{13}c_{3l}} D$$

Conditions (14) and (16) with $i=1$ yield that

$$v_{11}\binom{d(1)}{2} - w''_{112}b_{21} - w''_{113}b_{31} \equiv 0\,(\mathrm{mod}\,(d(1),e(1)))$$

and

$$u_{12}c_{2l}+u_{13}c_{3l}-v'_{11}d_{1l}-w_{112}\alpha_l^{(12)}-w_{113}\alpha_l^{(13)}-w'_{122}\alpha_l^{(22)}$$
$$-w'_{123}\alpha_l^{(23)}-w'_{133}\alpha_l^{(33)} \equiv 0\,(\mathrm{mod}\,(d(1),f(l)))$$

Thus

$$B = [x_{11},x_{21},x_{11}]^{w''_{112}b_{21}+w''_{113}b_{31}} \prod_{1\le l\le u}[x_{3l},x_{11}]^{-w_{112}\alpha_l^{(12)}-w_{113}\alpha_l^{(13)}-w'_{122}\alpha_l^{(22)}}$$
$$\prod_{1\le l\le u}[x_{3l},x_{11}]^{-w'_{123}\alpha_l^{(23)}-w'_{133}\alpha_l^{(33)}} D$$
$$= [x_{11},x_{12}^{d(2)},x_{11}]^{w''_{112}}[x_{11},x_{13}^{d(3)},x_{11}]^{w''_{113}}[x_{11}^{d(1)},x_{12},x_{11}]^{-w_{112}}[x_{11}^{d(1)},x_{13},x_{11}]^{-w_{113}}$$
$$[x_{12}^{d(2)},x_{12},x_{11}]^{-w'_{122}}[x_{12}^{d(2)},x_{13},x_{11}]^{-w'_{123}}[x_{13}^{d(3)},x_{13},x_{11}]^{-w'_{133}} D$$

By substituting the value of D we get,

$$B = [x_{11},x_{12}^{d(2)},x_{11}]^{w''_{112}}[x_{11},x_{13}^{d(3)},x_{11}]^{w''_{113}}[x_{11}^{d(1)},x_{12},x_{11}]^{-w_{112}}$$
$$[x_{11}^{d(1)},x_{13},x_{11}]^{-w_{113}}[x_{12}^{d(2)},x_{13},x_{11}]^{-w'_{123}}[x_{11}^{d(1)},x_{12},x_{12}]^{-w_{122}}$$
$$[x_{11}^{d(1)},x_{13},x_{13}]^{-w_{133}}[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)} \tag{52}$$

It is easy to see that

$$[x_{11},x_{12}^{d(2)},x_{11}]^{w''_{112}} \text{ and } [x_{11},x_{13}^{d(3)},x_{11}]^{w''_{113}} = \{e\} \tag{53}$$

With the help of condition (6), it can be seen easily that

$$[x_{11}^{d(1)},x_{12},x_{11}]^{-w_{112}},[x_{11}^{d(1)},x_{13},x_{11}]^{-w_{113}} \text{ and } [x_{12}^{d(2)},x_{13},x_{11}]^{-w'_{123}}=\{e\} \tag{54}$$

Now, consider

$$[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)}=[x_{11},x_{12}^{d(2)},x_{12}]^{-w'_{122}}$$

$$\begin{aligned}&[x_{12},x_{11},x_{12},x_{12}]^{-w'_{122}\binom{d(2)}{2}}[x_{11},x_{13}^{d(3)},x_{13}]^{-w'_{133}}\\&[x_{13},x_{11},x_{13},x_{13}]^{-w'_{133}\binom{d(3)}{2}}\\&=[x_{11},x_{21},x_{12}]^{-w'_{122}b_{21}}[x_{11},x_{31},x_{13}]^{-w'_{133}b_{31}}\end{aligned}$$

Using conditions (8) and (15), we obtain that

$$[x_{11},x_{21},x_{12}]^{-w'_{122}b_{21}}[x_{11},x_{31},x_{13}]^{-w'_{133}b_{31}}=[x_{11}^{d(1)},x_{12},x_{13}]^{-w_{123}}$$

Thus we get the following:

$$[x_{11},x_{12},x_{12}]^{-w'_{122}d(2)}[x_{11},x_{13},x_{13}]^{-w'_{133}d(3)}=[x_{11}^{d(1)},x_{12},x_{13}]^{-w_{123}} \tag{55}$$

Substituting the values from (53), (54) and (55) in (52), we get that

$$B=[x_{11}^{d(1)},x_{12},x_{12}]^{-w_{122}}[x_{11}^{d(1)},x_{13},x_{13}]^{-w_{133}}[x_{11}^{d(1)},x_{12},x_{13}]^{-w_{123}} \tag{56}$$

Similarly, it can be shown that

$$C=[x_{11}^{d(1)},x_{12},x_{12}]^{-3w_{122}}[x_{12}^{d(2)},x_{13},x_{13}]^{-w_{233}} \tag{57}$$

Substituting the values of A,B and C from (47), (56) and (57) in equation (45), we get that

$$\begin{aligned}g^2&=[x_{11}^{d(1)},x_{12},x_{12}]^{-2w_{122}}[x_{11}^{d(1)},x_{13},x_{13}]^{-2w_{133}}[x_{12}^{d(2)},x_{13},x_{13}]^{-2w_{233}}\\&=[x_{21},x_{12},x_{12}]^{-2w_{122}b_{11}}[x_{21},x_{13},x_{13}]^{-2w_{133}b_{11}}[x_{12}^{d(2)},x_{13},x_{13}]^{-2w_{233}}\end{aligned}$$

Using condition (14) for $i=2$ and $i=3$, we get that $-2w_{122}b_{11}=-v_{21}d(2)(d(2)-1)+2w''_{223}b_{31}$ and $-2w_{133}b_{11}=-v_{31}d(3)(d(3)-1)+2w_{233}b_{21}$. Thus,

$$\begin{aligned} g^2 &= [x_{21},x_{12},x_{12}]^{-v_{21}d(2)(d(2)-1)}[x_{21},x_{12},x_{12}]^{2w''_{223}b_{31}} \\ &[x_{21},x_{13},x_{13}]^{-v_{31}d(3)(d(3)-1)}[x_{21},x_{13},x_{13}]^{2w_{233}b_{21}}[x_{12}^{d(2)},x_{13},x_{13}]^{-2w_{233}} \\ &= [x_{21},x_{12},x_{12}^{d(2)}]^{-v_{21}(d(2)-1)}[x_{13}^{d(3)},x_{12},x_{12}]^{2w''_{223}} \\ &[x_{21},x_{13},x_{13}^{d(3)}]^{-v_{31}(d(3)-1)}[x_{12}^{d(2)},x_{13},x_{13}]^{2w_{233}}[x_{12}^{d(2)},x_{13},x_{13}]^{-2w_{233}} \\ &= [x_{13}^{d(3)},x_{12},x_{12}]^{2w''_{223}} = [x_{13}^{d(2)^2},x_{12},x_{12}]^{2w''_{223}} = [x_{13},x_{12}^{d(2)},x_{12}^{d(2)}]^{2w''_{223}} = \{e\} \end{aligned}$$

Since G is 3-group, thus we get that $g=\{e\}$. Hence $D_5(G)=\gamma_5(G)$. □

Theorem 3.3. *Let G be a finite 2-group. Let $G/\gamma_2(G)$ be cyclic group of order 2 and $\gamma_2(G)/\gamma_3(G)\cong C_1\oplus C_2\oplus\cdots\oplus C_n$, where C_i, is a cyclic group of order $e(i)$, with $e(i)=2, 1\le i\le n-1$. Then $D_5(G)=\gamma_5(G)$.*

Proof: Let g be an arbitrary element of $D_5(G)$, then by using Theorem 2.2

$$g=\prod_{\substack{1\le k,l\le n,\\ k<l}}[x_{2l},x_{2k}]^{v_{ik}b_{il}}$$

It follows from conditions (5) and (14) that $v_{ik}\equiv 0\,(\mathrm{mod}\,(2,e(k)))$, $1\le k\le n$. Let $v_{ik}=2\alpha+e(k)\beta, \alpha,\beta\in\mathbb{Z}$.

$$g=\prod_{\substack{1\le k,l\le n,\\ k<l}}[x_{2l},x_{2k}]^{(2\alpha+e(k)\beta)b_{il}}$$

Since $e(k)=2$ for $1\le k\le n-1$. Thus

$$g=\prod_{\substack{1\le k,l\le n,\\ k<l}}[x_{2l},x_{2k}]^{e(k)(\alpha+\beta)b_{il}}=\prod_{\substack{1\le k,l\le n,\\ k<l}}[x_{2l},x_{2k}^{e(k)}]^{(\alpha+\beta)b_{il}}=\{e\}$$ □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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