Available online at http://scik.org

J. Math. Comput. Sci. 11 (2021), No. 5, 5614-5628

https://doi.org/10.28919/jmcs/5884

ISSN: 1927-5307

NEIGHBORHOOD DEGREES OF m-BIPOLAR FUZZY GRAPH

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Abstract: In this article, neighborhood, open and closed neighborhood degrees of the vertices in an m-bipolar fuzzy graph (m-BPFG) are discussed. Also, strongly regular and biregular m-BPFG are defined with some basic theorems and examples.

Keywords: m-bipolar fuzzy graph; strongly regular m-BPFG; biregular m-BPFG.

2010 AMS Subject Classification: 05C72.

1. INTRODUCTION

Fuzzy sets are initiated for the parameters to solve problems related to vague and uncertain in

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Received April 16, 2021

real life situations are demonstrated by Zadeh [15] in 1965. The limitations of traditional model were overcome by the introduction of bipolar fuzzy concept in 1994 by Zhang [16, 17]. This was further improved by Chen et al. [7] to m-polar fuzzy set theory.

Free body diagrams using set of nodes connected by lines representing pairs are good problem solving tools in non-deterministic real life situations. Thus, Kaufmann [11] was first set up the thought of fuzzy graph is extracted from Zadeh fuzzy relation. Rosenfeld [12] gave the concept of fuzzy vertex, fuzzy edges and fuzzy cycle etc. Akram et al. [1-5] played a crucial role in studying some major properties of bipolar fuzzy graphs, interval-valued fuzzy graphs and m-polar fuzzy graphs which paved way for the decision making in resolving social problems with fuzzy environment. Later Rashmanlou et al. [14] studied the categorical properties of bipolar fuzzy graphs. Ghorai and Pal [8-10] studied the concept of m-polar fuzzy graphs and studied some of its properties. Ramprasad et al. [13] gave the idea of product m-polar fuzzy line and intersection graphs. Bera and pal [6] introduced the concept of m-polar interval-valued fuzzy graph and studied some algebraic properties like density, regularity and irregularity etc. on m-PIVFG.

This paper attempts to develop theory to analyze parameters combining concepts from m-polar fuzzy graphs and bipolar fuzzy graphs as a unique effort. The resultant graph is turned m-BPFG and studied properties on it.

2. PRELIMINARIES

Every vertex and edge of an m-polar fuzzy graph has m elements and those elements are fixed. But these elements may be bipolar. By this arrangement, m-BPFG has been initiated.

Before defining m-bipolar fuzzy graph, we suppose the following:

For a supposed set V, classify an equivalence relation $\leftrightarrow$ on $V \times V - \{(u, u) : u \in V\}$ as follows: $(u_1, v_1) \leftrightarrow (u_2, v_2) \Leftrightarrow$ either $(u_1, v_1) = (u_2, v_2)$ or $u_1 = v_2$, $v_1 = u_2$. The quotient set got in this way is represented by $\overleftrightarrow{V^2}$.

Throughout this research paper, we assume G^* as a crisp graph $G^* = (V, E)$.

Definition 2.1. An m-bipolar fuzzy graph (m-BPFG) of G^* is a pair $G = (V, S, T)$ where $S = \left\langle \left[p_j \circ \psi_S^p, p_j \circ \psi_S^n \right]_{j=1}^m \right\rangle$, $p_j \circ \psi_S^p : V \to [0,1]$ and $p_j \circ \psi_S^n : V \to [-1, 0]$ is an m-BPFS on V; and $T = \left\langle \left[p_j \circ \psi_T^p, p_j \circ \psi_T^n \right]_{j=1}^m \right\rangle$, $p_j \circ \psi_T^p : \overleftrightarrow{V^2} \to [0, 1]$ and $p_j \circ \psi_T^n : \overleftrightarrow{V^2} \to [-1, 0]$ is an m-BPFS in $\overleftrightarrow{V^2}$ such that $p_j \circ \psi_T^p(k, l) \le \min\{p_j \circ \psi_S^p(k), p_j \circ \psi_S^p(l)\}$, $p_j \circ \psi_T^n(k, l) \ge \max\{p_j \circ \psi_S^n(k), p_j \circ \psi_S^n(l)\}$ for all $(k, l) \in \overleftrightarrow{V^2}$, $j = 1, 2, \cdots, m$ and $p_j \circ \psi_T^p(k, l) = p_j \circ \psi_T^n(k, l) = 0$ for all $(k, l) \in \overleftrightarrow{V^2} - E$.

Definition 2.2. An m-BPFG $G = (V, S, T)$ of G^* is complete if for every $s, t \in V$ and $j = 1, 2, \cdots m$ satisfying $p_j \circ \psi_T^p(s, t) = \min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}$, $p_j \circ \psi_T^n(s, t) = \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$.

Definition 2.3. An m-BPFG $G = (V, S, T)$ of G^* is strong if for every $(s, t) \in E$ and $j = 1, 2, \cdots, m$ satisfying $p_j \circ \psi_T^p(s, t) = \min\{p_j \circ \psi_S^p(s), p_j \circ \psi_S^p(t)\}$, $p_j \circ \psi_T^n(s, t) = \max\{p_j \circ \psi_S^n(s), p_j \circ \psi_S^n(t)\}$.

Definition 2.4. Let $G = (V, S, T)$ be an m-BPFG of G^*. The complement of G is an m-BPFG $\overline{G} = \left(V, \overline{S}, \overline{T}\right)$ of $\overline{G^*} = \left(V, \overleftrightarrow{V^2}\right)$ such that $\overline{S} = S$ and $\overline{T}$ is defined by $p_j \circ \psi_{\overline{T}}(s, t) = \left[p_j \circ \psi_{\overline{T}}^p(s, t), p_j \circ \psi_{\overline{T}}^n(s, t) \right]$, $p_j \circ \psi_{\overline{T}}^p(s, t) = \{p_j \circ \psi_S^p(s) \wedge p_j \circ \psi_S^p(t)\} - p_j \circ \psi_T^p(s, t)$, $p_j \circ \psi_{\overline{T}}^n(s, t) = \{p_j \circ \psi_S^n(s) \vee p_j \circ \psi_S^n(t)\} - p_j \circ \psi_T^n(s, t)$ for every $(s, t) \in \overleftrightarrow{V^2}$ and $j = 1, 2, \cdots m$.

3. REGULARITY ON M-BPFG

In this section, neighborhood degree of a vertex, open and closed neighborhood degree of vertices are defined and studied some of its properties.

Definition 3.1. The neighborhood degree of a vertex $r \in V$ in an m-BPFG $G=(V,\, S,\, T)$ is defined as $d_N(r)=\left\langle\left[p_j \circ d_N^p(r),\, p_j \circ d_N^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\sum_{t\in N(r)} p_j \circ \psi_S^p(t),\, \sum_{t\in N(r)} p_j \circ \psi_S^n(t)\right]_{j=1}^m\right\rangle.$

Definition 3.2. The open neighborhood degree of a vertex $r \in V$ in an m-BPFG $G=(V,\, S,\, T)$ is defined as $d_G(r)=\left\langle\left[p_j \circ d_G^p(r),\, p_j \circ d_G^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\sum_{\substack{r\neq s\\(r,s)\in E}} p_j \circ \psi_T^p(r,\, s),\, \sum_{\substack{r\neq s\\(r,s)\in E}} p_j \circ \psi_T^n(r,\, s)\right]_{j=1}^m\right\rangle.$

Definition 3.3. The closed neighborhood degree of a vertex $r \in V$ in an m-BPFG $G=(V,\, S,\, T)$ is defined as $d_G[r]=\left\langle\left[p_j \circ d_G^p[r],\, p_j \circ d_G^n[r]\right]_{j=1}^m\right\rangle$

$$=\left\langle\left[\sum_{\substack{r\neq s\\(r,s)\in E}} p_j \circ \psi_T^p(r,\, s),\, \sum_{\substack{r\neq s\\(r,s)\in E}} p_j \circ \psi_T^n(r,\, s)\right]_{j=1}^m\right\rangle+\left\langle\left[p_j \circ \psi_S^p(r),\, p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle.$$

Definition 3.4. An m-BPFG $G=(V,\, S,\, T)$ of G^* is said to be $\left\langle\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right\rangle-$regular if all the vertices in G have same open neighborhood degrees $\left\langle\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right\rangle$.

Definition 3.5. An m-BPFG $G=(V,\, S,\, T)$ of G^* is said to be $\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle-$ totally regular if all the vertices in G have same closed neighborhood degrees $\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle$.

Definition 3.6. Let $G=(V,\, S,\, T)$ be an m-BPFG of G^*

Then the order of G is

$$O(G)=\left\langle\left[p_j \circ O^p(G),\, p_j \circ O^n(G)\right]_{j=1}^m\right\rangle=\left\langle\left[\sum_{r\in V} p_j \circ \psi_S^p(r),\, \sum_{r\in V} p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle,$$

and the size of G is

$$S(G)=\left\langle\left[p_j \circ S^p(G),\, p_j \circ S^n(G)\right]_{j=1}^m\right\rangle=\left\langle\left[\sum_{(r,s)\in E} p_j \circ \psi_T^p(r,\, s),\, \sum_{(r,s)\in E} p_j \circ \psi_T^p(r,\, s)\right]_{j=1}^m\right\rangle.$$

Proposition 3.1. Let $G=(V,\, S,\, T)$ be a $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ -regular m-BPFG of G^*.

Then $S(G)=\frac{n}{2}\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ where $|V|=n.$

Proof. Suppose G is a $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ - regular m-BPFG.

Then $d_G(r)=\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ for all $r \in V$.

This implies that $\left\langle \left[\sum_{\substack{r \neq s \\ (r,s)\in E}} p_j \circ \psi_T^p(r,\, s),\ \sum_{\substack{r \neq s \\ (r,s)\in E}} p_j \circ \psi_T^n(r,\, s) \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ for all $r \in V$.

$$\sum_{r\in V}\left\langle \left[\sum_{\substack{r \neq s \\ (r,s)\in E}} p_j \circ \psi_T^p(r,\, s),\ \sum_{\substack{r \neq s \\ (r,s)\in E}} p_j \circ \psi_T^n(r,\, s) \right]_{j=1}^m \right\rangle = \sum_{r\in V}\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle. \text{ i.e. } \quad 2S(G)=n\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle.$$

Hence $S(G)=\frac{n}{2}\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$. □

Proposition 3.2. Let $G=(V,\, S,\, T)$ be a $\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ - totally regular m-BPFG of G^*.

Then $2S(G)+O(G)=n\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ where $|V|=n.$

Proof. Suppose G is a $\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ - totally regular m-BPFG. Then $d_G[r]=\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ for all $r \in V$. This implies that $d_G(r)+\left\langle \left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r) \right]_{j=1}^m \right\rangle = \left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ for all $r \in V$.

Therefore $\sum_{r\in V} d_G(r)+\sum_{r\in V}\left\langle \left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r) \right]_{j=1}^m \right\rangle = \sum_{r\in V}\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle,$

i.e. $2S(G)+O(G)=n\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle.$ □

Proposition 3.3. Let $G=(V,\, S,\, T)$ be a $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ -regular and $\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle$ - totally regular m-BPFG of G^*. Then $O(G)=n\left\langle \left[\gamma_j^p-\eta_j^p, \gamma_j^n-\eta_j^n \right]_{j=1}^m \right\rangle$ where $|V|=n.$

Proof. From Proposition 3.2, we get $2S(G)+O(G)=n\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle,$

i.e. $O(G)=n\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle - 2S(G) = n\left\langle \left[\gamma_j^p, \gamma_j^n \right]_{j=1}^m \right\rangle - 2\frac{n}{2}\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle = n\left\langle \left[\gamma_j^p-\eta_j^p, \gamma_j^n-\eta_j^n \right]_{j=1}^m \right\rangle.$ □

Theorem 3.1. Let $G=(V, S, T)$ be an m-BPFG of G^*. Then $S=\left\langle\left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m\right\rangle$ is a constant function if and only if the subsequent conditions are equivalent.

(i) G is $\left\langle\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right\rangle$ - regular m-BPFG,

(ii) G is $\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle$-totally regular m-BPFG.

Proof. Suppose $S=\left\langle\left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m\right\rangle$ is a constant function.

Then $\left\langle\left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\tau_j^p, \tau_j^n\right]_{j=1}^m\right\rangle\ \forall r \in V,$ where $\tau_j^p \in[0, 1],\ \tau_j^n \in[-1, 0]$ for all $j=1, 2, \cdots, m.$ Let G be a $\left\langle\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right\rangle$-regular m-BPFG.

Then for all $r \in V,\ d_G(r)=\left\langle\left[\eta_j^p, \eta_j^n\right]_{j=1}^m\right\rangle,$

$$d_G[r]=\left\langle\left[\sum_{\substack{r \neq s \\ (r, s) \in E}} p_j \circ \psi_T^p(r, s),\ \sum_{\substack{r \neq s \\ (r, s) \in E}} p_j \circ \psi_T^n(r, s)\right]_{j=1}^m\right\rangle+\left\langle\left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\eta_j^p+\tau_j^p, \eta_j^n+\tau_j^n\right]_{j=1}^m\right\rangle.$$

Then G is a $\left\langle\left[\eta_j^p+\tau_j^p, \eta_j^n+\tau_j^p\right]_{j=1}^m\right\rangle$-totally regular m-BPFG.

Let G be a $\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle$- totally regular m-BPFG. Then $d_G[r]=\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle$ for all $r \in V$.

So for all $r \in V$, we have $d_G[r]=\ d_G(r)+\left\langle\left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle,$

$$d_G(r)=\left\langle\left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m\right\rangle-\left\langle\left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m\right\rangle=\left\langle\left[\gamma_j^p-\tau_j^p, \gamma_j^n-\tau_j^n\right]_{j=1}^m\right\rangle.$$

Hence, G is $\left\langle\left[\gamma_j^p-\tau_j^p, \gamma_j^n-\tau_j^n\right]_{j=1}^m\right\rangle$-regular m-BPFG.

Conversely, suppose that conditions (i) and (ii) are equivalent. Now we have to prove that $\left\langle\left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m\right\rangle$ is a constant function.

In a contrary way, we suppose that $\left\langle\left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m\right\rangle$ is not a constant function.

Then $\left\langle\left[p_j \circ \psi_S^p(r_1),\ p_j \circ \psi_S^n(r_1)\right]_{j=1}^m\right\rangle \neq\left\langle\left[p_j \circ \psi_S^p(s_1),\ p_j \circ \psi_S^n(s_1)\right]_{j=1}^m\right\rangle$ for at least one pair of vertices $r_1,\ s_1 \in V.$

Let G be a $\left\langle \left[\eta_j^p, \eta_j^n\right]_{j=1}^m \right\rangle$-regular m-BPFG. Then $d_G(r_1) = d_G(s_1) = \left\langle \left[\eta_j^p, \eta_j^n\right]_{j=1}^m \right\rangle$.

So for all $r_1, s_1 \in V$,

$$d_G[r_1] = d_G(r_1) + \left\langle \left[p_j \circ \psi_S^p(r_1),\ p_j \circ \psi_S^n(r_1)\right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p + p_j \circ \psi_S^p(r_1),\ \eta_j^n + p_j \circ \psi_S^n(r_1)\right]_{j=1}^m \right\rangle,$$

$$d_G[s_1] = d_G(s_1) + \left\langle \left[p_j \circ \psi_S^p(s_1),\ p_j \circ \psi_S^n(s_1)\right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p + p_j \circ \psi_S^p(s_1),\ \eta_j^n + p_j \circ \psi_S^n(s_1)\right]_{j=1}^m \right\rangle$$

and $d_G[r_1] \neq d_G[s_1]$ since $\left\langle \left[p_j \circ \psi_S^p(r_1),\ p_j \circ \psi_S^n(r_1)\right]_{j=1}^m \right\rangle \neq \left\langle \left[p_j \circ \psi_S^p(s_1),\ p_j \circ \psi_S^n(s_1)\right]_{j=1}^m \right\rangle$.

Thus, G is not a totally regular m-BPFG. This contradicts our assumption. Hence $\left\langle \left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m \right\rangle$ is a constant function .

Similarly, $\left\langle \left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m \right\rangle$ is a constant function for totally regular m-BPFG. □

Proposition 3.4. Let $G = (V, S, T)$ be an m-BPFG of $G*$ and G is both regular and totally regular. Then $S = \left\langle \left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m \right\rangle$ is constant.

Proof. Let G be a $\left\langle \left[\eta_j^p, \eta_j^n\right]_{j=1}^m \right\rangle$-regular and $\left\langle \left[\gamma_j^p, \gamma_j^n\right]_{j=1}^m \right\rangle$-totally regular m-BPFG. Then

$$d_G[r] = d_G(r) + \left\langle \left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m \right\rangle, \left\langle \left[p_j \circ \psi_S^p(r),\ p_j \circ \psi_S^n(r)\right]_{j=1}^m \right\rangle = \left\langle \left[\gamma_j^p - \eta_j^p, \gamma_j^n - \eta_j^p\right]_{j=1}^m \right\rangle$$

for all $r \in V$. This shows that $S = \left\langle \left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n\right]_{j=1}^m \right\rangle$ is constant. □

Example 3.1. The converse of the above proposition need not be true. This can be proved with an example given below. The open and closed neighborhood degree of the vertices for the 2-BPFG G of $G*$ shown in Figure 1. are $d_G(A) = \langle [1.3, -1.1], [0.2, -0.3] \rangle$,

$d_G(B) = \langle [1.1, -0.8], [0.25, -0.45] \rangle$, $d_G(C) = \langle [1.4, -1.1], [0.25, -0.35] \rangle$,

$d_G[A] = \langle [2.2, -1.9], [0.4, -0.6] \rangle$, $d_G[B] = \langle [2.0, -1.6], [0.45, -0.75] \rangle$,

$d_G[C] = \langle [2.3, -1.9], [0.45, -0.65] \rangle$.

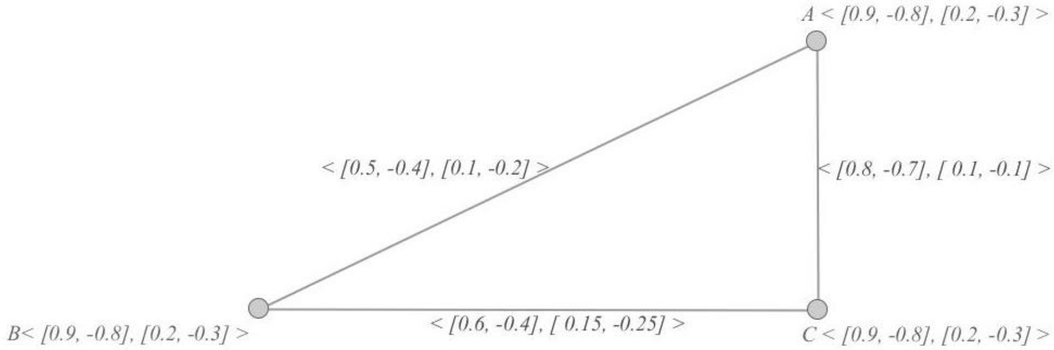

Figure 1. $S=\left\langle \left[p_j \circ \psi_S^p,\ p_j \circ \psi_S^n \right]_{j=1}^m \right\rangle$ **is constant but** G **is neither regular nor totally regular m-BPFG**

Hence, it shows that S is constant but G is neither regular and nor totally regular m-BPFG.

Theorem 3.2. Let $G=(V,\ S,\ T)$ be an m-BPFG of an odd cycle of $G*$. Then G is regular m-BPFG if and only $T=\left\langle \left[p_j \circ \psi_T^p,\ p_j \circ \psi_T^n \right]_{j=1}^m \right\rangle$ is constant.

Proof. Suppose G is a $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular m-BPFG. Let $t_1,\ t_2,\ t_3,\ \cdots,\ t_{2n+1}$ be the edges of $G*$ such that $t_i=(r_{i-1},\ r_i)\in E,\ r_0,\ r_i\in V,\ i=1,\ 2,\ \cdots,\ 2n+1$ and $r_0=r_{2n+1}$.

Let $\left\langle \left[p_j \circ \psi_T^p(t_1),\ p_j \circ \psi_T^n(t_1) \right]_{j=1}^m \right\rangle = \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle$ where $a_j^p\in[0,\ 1],\ a_j^n\in[-1,\ 0]$ for all $j=1,\ 2,\ \cdots,m$. Since G is $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular, we have $d_G(r_1)=\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$.

This means,

$$d_G(r_1)=\left\langle \left[p_j \circ \psi_T^p(t_1),\ p_j \circ \psi_T^n(t_1) \right]_{j=1}^m \right\rangle + \left\langle \left[p_j \circ \psi_T^p(t_2),\ p_j \circ \psi_T^n(t_2) \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle,$$

i.e. $\left\langle \left[p_j \circ \psi_T^p(t_2),\ p_j \circ \psi_T^n(t_2) \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle - \left\langle \left[p_j \circ \psi_T^p(t_1),\ p_i \circ \psi_T^n(t_1) \right]_{j=1}^m \right\rangle,$

i.e. $\left\langle \left[p_j \circ \psi_T^p(t_2),\ p_j \circ \psi_T^n(t_2) \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle - \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p - a_j^p, \eta_j^n - a_j^n \right]_{j=1}^m \right\rangle.$

Again, $d_G(r_2)=\left\langle \left[p_j \circ \psi_T^p(t_2),\ p_j \circ \psi_T^n(t_2) \right]_{j=1}^m \right\rangle + \left\langle \left[p_j \circ \psi_T^p(t_3),\ p_j \circ \psi_T^n(t_3) \right]_{j=1}^m \right\rangle$

$=\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$. i.e. $\left\langle \left[p_j \circ \psi_T^p(t_3),\ p_j \circ \psi_T^n(t_3) \right]_{j=1}^m \right\rangle = \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle$ and so on.

Therefore, $\left\langle \left[p_j \circ \psi_T^p(t_i),\ p_j \circ \psi_T^n(t_i) \right]_{j=1}^m \right\rangle = \begin{cases} \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle & \text{if } i \text{ is odd} \\ \left\langle \left[\eta_j^p - a_j^p, \eta_j^n - a_j^n \right]_{i=1}^m \right\rangle & \text{if } i \text{ is even} \end{cases}$

Hence, $\left\langle \left[p_j \circ \psi_T^p(t_1),\ p_j \circ \psi_T^p(t_1) \right]_{j=1}^m \right\rangle = \left\langle \left[p_j \circ \psi_T^p(t_{2n+1}),\ p_j \circ \psi_T^p(t_{2n+1}) \right]_{j=1}^m \right\rangle = \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle$.

Since t_1 and t_{2n+1} are incident on the vertex r_0 and $d_G(r_0) = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$, we have $\left\langle \left[p_j \circ \psi_T^p(t_1),\ p_j \circ \psi_T^n(t_1) \right]_{j=1}^m \right\rangle + \left\langle \left[p_j \circ \psi_T^p(t_{2n+1}),\ p_j \circ \psi_T^n(t_{2n+1}) \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$

i.e. $\left\langle \left[2a_j^p, 2a_j^n \right]_{j=1}^m \right\rangle = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle, \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle = \left\langle \left[\frac{\eta_j^p}{2}, \frac{\eta_j^n}{2} \right]_{j=1}^m \right\rangle$.

Therefore, $\left\langle \left[p_j \circ \psi_T^p(t_i),\ p_j \circ \psi_T^n(t_i) \right]_{j=1}^m \right\rangle = \left\langle \left[\frac{\eta_j^p}{2}, \frac{\eta_j^n}{2} \right]_{j=1}^m \right\rangle$ for all $i = 1, 2, \cdots, 2n+1$.

Hence, $T = \left\langle \left[p_j \circ \psi_T^p,\ p_j \circ \psi_T^n \right]_{j=1}^m \right\rangle$ is constant.

Conversely, let $\left\langle \left[p_j \circ \psi_T^p,\ p_j \circ \psi_T^n \right]_{j=1}^m \right\rangle$ be a constant function.

Let $\left\langle \left[p_j \circ \psi_T^p(r, s),\ p_j \circ \psi_T^n(r, s) \right]_{j=1}^m \right\rangle = \left\langle \left[a_j^p, a_j^n \right]_{j=1}^m \right\rangle$, for all $(r, s) \in E$ where

$a_j^p \in [0, 1],\ a_j^n \in [-1, 0]$ for all $j = 1, 2, \cdots, m$.

Then $d_G(r) = \left\langle \left[\sum_{\substack{r \neq s \\ (r,s) \in E}} p_j \circ \psi_T^p(r, s),\ \sum_{\substack{r \neq s \\ (r,s) \in E}} p_j \circ \psi_T^n(r, s) \right]_{j=1}^m \right\rangle = \left\langle \left[2a_j^p, 2a_j^n \right]_{j=1}^m \right\rangle$ for all $r \in V$.

Consequently, G is a $\left\langle \left[2a_j^p, 2a_j^n \right]_{j=1}^m \right\rangle$ -regular m-BPFG. □

4. STRONGLY REGULAR BIPOLAR FUZZY GRAPH

In this section, we initiated the concept of strongly regular and biregular m-BPFGS.

Definition 4.1. A finite m-BPFG $G = (V, S, T)$ is said to be strongly regular m-BPFG if

(i) G is $\eta = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular m-BPFG,

(ii) The sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of adjacent vertices and non-adjacent vertices of G has the same weight and is denoted by $\lambda = \left\langle \left[\lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\rangle$, $\delta = \left\langle \left[\delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle$ respectively.

A strongly regular m-BPFG G is denoted by $G = (n, \eta, \lambda, \delta)$ where $n = |V|$.

Example 4.1.

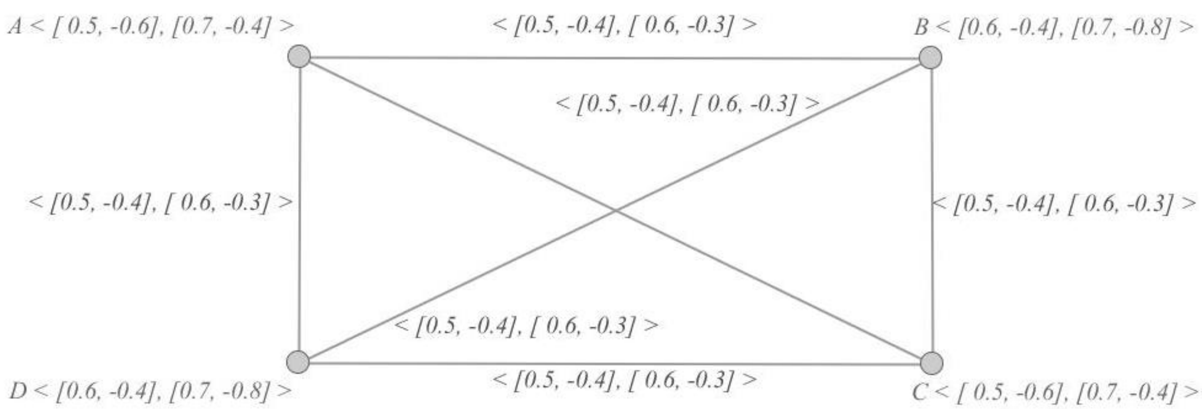

Figure 2: Strongly regular m-BPFG

Let us consider the 2-BPFG $G = (V, S, T)$ of $G^* = (V, E)$ shown in Figure 2.

Here, $n = 4$, $\eta = \langle [1.5, -1.2], [1.8, -0.9] \rangle$, $\lambda = \langle [1.1, -1.0], [1.4, -1.2] \rangle$ and $\delta = \langle [0, 0], [0, 0] \rangle$. Hence G is a strongly regular 2-BPFG.

Definition 4.2. An m-BPFG $G = (V, S, T)$ of G^* is said to be a biregular m-BPFG if G is $\eta = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular m-BPFG and V can be partitioned into $V_1 \cup V_2$ such that each vertex in V_1 has the same neighborhood degree $M = \left\langle \left[M_j^p, M_j^n \right]_{j=1}^m \right\rangle$ and each vertex in V_2 has the same neighborhood degree $N = \left\langle \left[N_j^p, N_j^n \right]_{j=1}^m \right\rangle$, where M and N are constants.

Example 4.2.

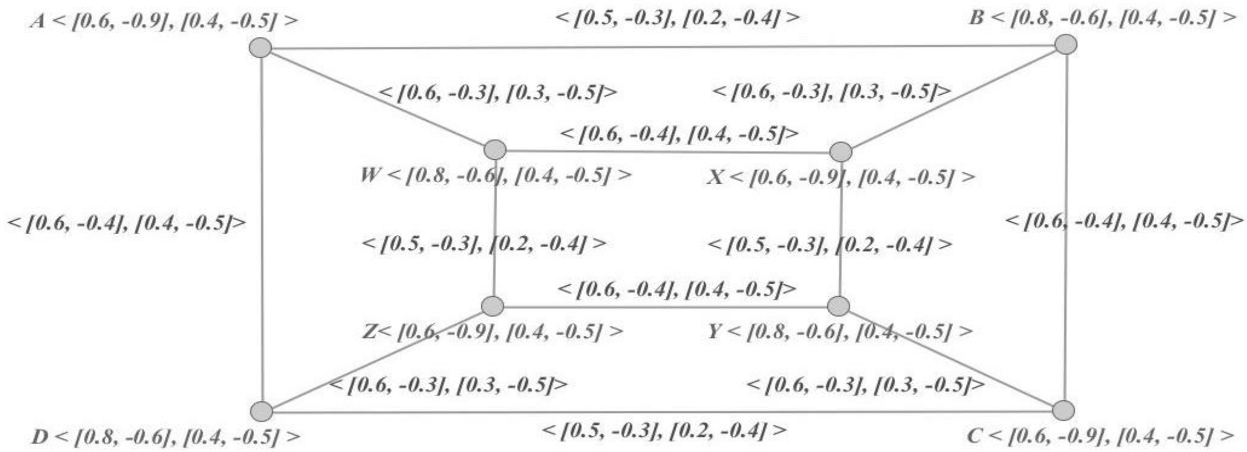

Figure 3: Biregular m-BPFG

Let us consider the 2-BPFG $G=(V,\ S,\ T)$ of $G^*=(V,\ E)$ shown in Figure 3.

Here $n=8$, $\eta=\langle[1.7,\ -1],\ [0.9,\ -1.4]\rangle$, $V_1=\{A,\ C,\ X,\ Z\}$, $V_2=\{B,\ D,\ W,\ Y\}$,

$M=\langle[2.4,\ -1.8],\ [1.2,\ -1.5]\rangle$ and $N=\langle[1.8,\ -2.7],\ [1.2,\ -1.5]\rangle$. Hence G is a biregular 2-BPFG.

Theorem 4.1. Let $G=(V,\ S,\ T)$ be a complete m-BPFG of G^* in which S and T are constant functions. Then G is strongly regular m-BPFG.

Proof. Let $G=(V,\ S,\ T)$ be a complete bipolar fuzzy graph where $V=\{t_1,\ t_2,\ ...,\ t_n\}$.

Let $S(t_k)=\left\langle\left[a_j^p,\ a_j^n\right]_{j=1}^m\right\rangle$ for all $t_k\in V$ and $T(t_p,\ t_l)=\left\langle\left[b_j^p,\ b_j^n\right]_{j=1}^m\right\rangle$ for all $(t_p,\ t_l)\in E$ where $a_j^p,\ a_j^n,\ b_j^p,\ b_j^n$ are constants. Since G is complete, we have G is $\left\langle\left[(n-1)b_j^p,\ (n-1)b_j^n\right]_{j=1}^m\right\rangle$ -regular m-BPFG. Again G is complete, therefore the sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of adjacent vertices has the same weight $\lambda=\left\langle\left[(n-2)a_j^p,\ (n-2)a_j^n\right]_{j=1}^m\right\rangle$ and the sum of the positive membership values and negative membership values of the common neighborhood vertices of any pair of non adjacent vertices has the same weight

$\delta = \langle [0,\ 0],\ [0,\ 0],\ \cdots,\ [0,\ 0] \rangle$. So G is strongly regular m-BPFG. □

Theorem 4.2. If $G = (V,\ S,\ T)$ is a strongly regular m-BPFG which is strong, then $\overline{G}$ is a $\eta = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular.

Proof. Let $G = (V,\ S,\ T)$ be a strongly regular m-BPFG. Then by definition, G is $\eta = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle -$regular. Since G is strong and for all $j = 1,\ 2,\ \cdots,\ m$, we have

$$p_j \circ \psi_{\overline{T}}^p (t_k,\ t_l) = \begin{cases} 0 & \text{for all } (t_k,\ t_l) \in E \\ \left\{ p_j \circ \psi_S^p (t_k) \wedge p_j \circ \psi_S^p (t_l) \right\} & \text{for all } (t_k,\ t_l) \notin E \end{cases}$$

$$p_j \circ \psi_{\overline{T}}^n (t_k,\ t_l) = \begin{cases} 0 & \text{for all } (t_k,\ t_l) \in E \\ \left\{ p_j \circ \psi_S^n (t_k) \vee p_j \circ \psi_S^n (t_l) \right\} & \text{for all } (t_k,\ t_l) \notin E \end{cases}$$

Since G is strong, we have the degree of a vertex t_k in $\overline{G}$ is

$$d_{\overline{G}}(t_k) = \left\langle \left[p_j \circ d_{\overline{G}}^p (t_k),\ p_j \circ d_{\overline{G}}^n (t_k) \right]_{j=1}^m \right\rangle$$

where $p_j \circ d_{\overline{G}}^p (t_k) = \sum_{\substack{t_k \neq t_l \\ (t_k,\ t_l) \in E}} p_j \circ \psi_{\overline{T}}^p (t_k,\ t_l) = \sum_{\substack{t_k \neq t_l \\ (t_k,\ t_l) \in E}} \left\{ p_j \circ \psi_S^p (t_k) \wedge p_j \circ \psi_S^p (t_l) \right\} = \eta_j^p$,

$$p_j \circ d_{\overline{G}}^n (t_k) = \sum_{\substack{t_k \neq t_l \\ (t_k,\ t_l) \in E}} p_j \circ \psi_{\overline{T}}^n (t_k,\ t_l) = \sum_{\substack{t_k \neq t_l \\ (t_k,\ t_l) \in E}} \left\{ p_j \circ \psi_S^n (t_k) \vee p_j \circ \psi_S^n (t_l) \right\} = \eta_j^n, \quad \forall\ t_k \in V,\ j = 1,\ 2,\ \cdots,\ m.$$

Hence $d_{\overline{G}}(t_k) = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ $\forall\ t_k \in V$. So $\overline{G}$ is $\eta = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular m-BPFG. □

Theorem 4.3. Let $G = (V,\ S,\ T)$ be a strong m-BPFG. Then, G is a strongly regular if and only if $\overline{G}$ is a strongly regular.

Proof. Suppose that $G = (V,\ S,\ T)$ is a strongly regular m-BPFG. Then G is $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular and the adjacent vertices and the non-adjacent vertices have the same common neighborhood weight $\left\langle \left[\lambda_j^p,\ \lambda_j^n \right]_{j=1}^m \right\rangle$ and $\left\langle \left[\delta_j^p,\ \delta_j^n \right]_{j=1}^m \right\rangle$ respectively. Now we have to prove that $\overline{G}$ is strongly regular m-BPFG. If G is strongly regular m-BPFG and which is strong then

$\bar{G}$ is $\left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$-regular m-BPFG by Theorem 4.2. Next, let F_1 and F_2 be the set of all adjacent vertices and non-adjacent vertices of G; $\overline{F_1}$ and $\overline{F_2}$ denote set of all adjacent vertices and non-adjacent vertices of $\bar{G}$.

i.e. $F_1 = \{(t_k, t_l) | (t_k, t_l) \in E\}$, where t_k and t_l have same common neighborhood weight $\lambda = \left\langle \left[\lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\rangle$ and $F_2 = \{(t_k, t_l) | (t_k, t_l) \notin E\}$ where t_k and t_l have same common neighborhood weight $\delta = \left\langle \left[\delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle$. Then, $\overline{F}_1 = \left\{ (t_k, t_l) \middle| (t_k, t_l) \in \overline{E} \right\}$ where t_k and t_l have same common neighborhood weight $\delta = \left\langle \left[\delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle$ and $\overline{F_2} = \left\{ (t_k, t_l) \middle| (t_k, t_l) \notin \overline{E} \right\}$, where t_k and t_l have a same common neighborhood weight $\lambda = \left\langle \left[\lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\rangle$. This implies $\bar{G}$ is a strongly regular. Similarly, we can prove G is strongly regular if $\overline{G}$ is strongly regular. □

Theorem 4.4. A strongly regular m-BPFG $G = (V, S, T)$ is a biregular m-BPFG if the adjacent vertices have the same common neighborhood weight $\lambda = \left\langle \left[\lambda_j^p, \lambda_j^n \right]_{j=1}^m \right\rangle \neq \langle [0, 0], [0, 0], \cdots, [0, 0] \rangle$ and the non-adjacent vertices have the same common neighborhood weight $\delta = \left\langle \left[\delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle \neq \langle [0, 0], [0, 0], \cdots, [0, 0] \rangle$.

Proof. Let $G = (V, S, T)$ be a strongly regular m-BPFG. Then we have $d_G(t_k) = \left\langle \left[\eta_j^p, \eta_j^n \right]_{j=1}^m \right\rangle$ for all $t_k \in V$. Let F_1 be the set of all non-adjacent vertices of G . Then F_1 is a non empty subset of V since non adjacent vertices have the same common neighborhood weight $\delta = \left\langle \left[\delta_j^p, \delta_j^n \right]_{j=1}^m \right\rangle \neq \langle [0, 0], [0, 0], \cdots, [0, 0] \rangle$.

So, $F_1 = \{t_k, t_l | t_k \text{ is not adjacent to } t_l, k \neq l, t_k, t_l \in V\}$. Then the vertex partition of G is $V_1 = \{t_k | t_k \in F_1\}$ and $V_2 = \{t_l | t_l \in F_1\}$. Hence, G is a biregular m-BPFG. □

CONCLUSIONS

In this paper, we proved some properties of open and closed neighborhood degree of the vertices in an m-BPFG. Also, strongly regular and biregular m-BPFGs are described with sustaining illustrations and theorems. In future, we intend our investigations to the other properties of m-BPFG and extend them to solve different decision making problems in fuzzy environment.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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