Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 5095-5105
https://doi.org/10.28919/jmcs/5885
ISSN: 1927-5307

ON $\widehat{D}$-HOMEOMORPHISM IN TOPOLOGICAL SPACES

K. DASS, G. SURESH[†,*]

PG & Research Department of Mathematics, The M.D.T Hindu College, Tirunelveli, India
(Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India)

Abstract. In this paper we introduce and investigate new class of maps called $\widehat{D}$-homeomorphism, $\widehat{D}$-quotient map and several characterization and some of their properties. Also we investigate its relationship with other types of functions.

Keywords: $\widehat{D}$-open set; $\widehat{D}$-closed set; $\widehat{D}$-homeomorphism; $\widehat{D}$-quotient map.
2010 AMS Subject Classification: 54A05, 54C10.

1. INTRODUCTION

The notion homeomorphism plays a very important role in topology. By definition a homeomorphism between two topological spaces X and Y is a bijective map $f : (X,\tau) \to (Y,\sigma)$ when both f and f^{-1} are continuous map. K. Dass and G. Suresh [10] introduced $\widehat{D}$-closed set in topological spaces. K. Dass and G. Suresh [3] introduced $\widehat{D}$-continuous map, in topological spaces. In this paper we introduce the concept of $\widehat{D}$-open maps, quasi $\widehat{D}$-open maps and strongly $\widehat{D}$-open maps in topological spaces and also $\widehat{D}$-homeomorphism, strongly $\widehat{D}$-homeomorphism and $\widehat{D}$-quotient map are obtained.

*Corresponding author
E-mail address: sureshgamrs22@gmail.com
†Reg. No: 18211072091007, Research Scholar
Received April 16, 2021

2. PRELIMINARIES

Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f:(X,\tau)\to(Y,\sigma)$ (or simply $f:X\to Y$) denotes a function f of a space (X,τ) into a space (Y,σ). Let A be a subset of a space X. The closure, the interior and complement of A are denoted by $cl(A)$, $int(A)$ and A^c respectively.

Definition 2.1. *A subset A of a topological space (X,τ) is called*

i) a pre-open set [5] *if $A\subset int(cl(A))$ and a pre-closed set if $cl(int(A))\subset A$,*

ii) a semi-open set [2] *if $A\subset cl(int(A))$ and a semi-closed set if $int(cl(A))\subset A$,*

iii) a semi-pre-open set [7] *(β-open* [1]*) if $A\subset cl(int(cl(A)))$ and a semi-preclosed set ($=\beta$-closed) if $int(cl(int(A)))\subset A$.*

Definition 2.2. *Let (X,τ) be a topological space and $A\subset X$*

i) an ω-closed set [8] *($=\widehat{g}$-closed* [9]*) if $cl(A)\subset U$ whenever $A\subset U$ and U is semi-open in (X,τ),*

ii) a D-closed set [4] *if $pcl(A)\subset int(U)$ whenever $A\subset U$ and U is ω-open in (X,τ).*

Complements of the above mentioned sets are called their respectively open sets

Definition 2.3. *A subset A of (X,τ) is called an $\widehat{D}$-closed* [10] *set if $spcl(A)\subset U$ whenever $A\subset U$ and U is D-open in (X,τ). The class of all $\widehat{D}$-closed sets in (X,τ) is denoted by $\widehat{D}c(\tau)$. That is, $\widehat{D}c(\tau)=\{A\subset X : A \text{ is } \widehat{D}-\text{closed in } (X,\tau)\}$.*

Definition 2.4. *Let (X,τ) be a topological space and $A\subset X$*

(1) semi-pre interior of A denoted by $spint(A)$ is the union of all semi-pre open subsets of A

(2) semi-pre closure of A denoted by $spcl(A)$ is the intersection of all semi-pre closed subsets of A

Definition 2.5. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be $\widehat{D}$-continuous* [3] *if $f^{-1}(H)$ is $\widehat{D}$-closed in (X,τ) for every closed set H in Y.*

Definition 2.6. *A map $f:X\to Y$ is called $\widehat{D}$-irresolute* [6] *if $f^{-1}(F)$ is $\widehat{D}$-closed in X for every $\widehat{D}$-closed set F of Y.*

Proposition 2.7. [6] *If $f : X \to Y$ is $\widehat{D}$-irresolute, then f is $\widehat{D}$-continuous but not conversely.*

Proposition 2.8. *Let $f : X \to Y$ and $g : Y \to Z$ be any two maps. Then*

(a) $g \circ f$ is $\widehat{D}$-irresolute if both f and g are $\widehat{D}$-irresolute.

(b) $g \circ f$ is $\widehat{D}$-continuous if g is $\widehat{D}$-continuous and f is $\widehat{D}$-irresolute.

Proposition 2.9. *Let X be a topological space, Y be a $T_{\widehat{D}}$-space and $f : X \to Y$ be a map. Then the following are equivalent:*

(i) f is $\widehat{D}$-irresolute,

(ii) f is $\widehat{D}$-continuous.

3. $\widehat{D}$-HOMEOMORPHISM

Definition 3.1. *A map $f : X \to Y$ is said to be an $\widehat{D}$-open map if the image $f(A)$ is $\widehat{D}$-open in Y for each open set A in X.*

Example 3.2. *Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\phi, \{a,b\}, Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be an identity map. Here $\widehat{D}o(\sigma) = P(X) - \{c\}$. Then f is an $\widehat{D}$-open map.*

Theorem 3.3. *A surjective map $f : X \to Y$ is $\widehat{D}$ - open if and only if for any subset S of Y and for any closed F containing $f^{-1}(S)$, there exists an $\widehat{D}$ - closed set K of Y containing S such that $f^{-1}(K) \subset F$*

Theorem 3.4. *For any bijection $f : X \to Y$, the following conditions are equivalent.*

i) $f^{-1} : Y \to X$ is $\widehat{D}$ - continuous.

ii) f is an $\widehat{D}$ - open map.

iii) f is an $\widehat{D}$ - closed map.

Proof. **(i) $\Longrightarrow$ (ii) :** Let U be an open set of X. By assumption $(f^{-1})^{-1}(U) = f(U)$ is $\widehat{D}$ - open in Y and so f is $\widehat{D}$ - open.

(ii) $\Longrightarrow$ (iii) : Let F be a closed set of X. Then F^c is open in X. By (ii), $f(F^c)$ is $\widehat{D}$ - open in Y and therefore $f(F^c) = (f(F))^c$ is $\widehat{D}$ - open in Y. Thus $f(F)$ is $\widehat{D}$ - closed in Y implies f is $\widehat{D}$ - closed.

(iii) $\Longrightarrow$ (i) : Let F be a closed set of X. By (iii), $f(F)$ is $\widehat{D}$ - closed in Y. But $f(F) = (f^{-1})^{-1}(F)$ and therefore f^{-1} is $\widehat{D}$ - continuous. □

Definition 3.5. *A map $f : X \to Y$ is said to be strongly $\widehat{D}$ -open if the image of every $\widehat{D}$-open set in X is $\widehat{D}$ - open in Y.*

Definition 3.6. *A map $f : X \to Y$ is said to be quasi - $\widehat{D}$ -open if the image every $\widehat{D}$-open set in X is open in Y.*

Theorem 3.7. *A surjective map $f : X \to Y$ is quasi - $\widehat{D}$ - open if and only if for any subset B of Y and any $\widehat{D}$ - closed set F of X containing $f^{-1}(B)$, there exists a closed set G of Y containing B such that $f^{-1}(G) \subset F$.*

Proof. Suppose f is quasi - $\widehat{D}$ - open. Let $B \subset Y$ and F be an $\widehat{D}$ - closed set of X containing $f^{-1}(B)$. Now, put $G = (f(F^c))^c$. Then G is a closed set of Y containing B such that $f^{-1}(G) \subset F$.

Conversely, let U be an $\widehat{D}$ - open set of X and put $B = (f(U))^c$. Then U^c is an $\widehat{D}$ - closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset U^c$. Hence, we obtain $f(U) \subset F^c$. On the otherhand it follows that $B \subset F$, $F^c \subset B^c = f(U)$. Thus we obtain $f(u) = F^c$ which is open in Y and hence f is quasi - $\widehat{D}$ - open map. □

Remark 3.8. *From the above definitions we obtain the following implications.*

quasi - $\widehat{D}$ - open strongly - $\widehat{D}$ - open $\Longrightarrow$ $\widehat{D}$ - open. However the reverse implications are not true by the following examples.

Example 3.9. *Let $X = \{p,q,r\}$, $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma = \{\phi,\{p,q\},Y\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is strongly $\widehat{D}$-open map but not quasi $\widehat{D}$-open map, since $\{q\}$ is $\widehat{D}$-open in X but $f(\{q\}) = \{q\}$ is not open in Y.*

Example 3.10. *Let $X = \{p,q,r\}$, $Y = \{p,q,r\}$, $\tau = \{\phi,\{p,q\},X\}$ and $\sigma = \{\phi,\{r\},\{q,r\},X\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is $\widehat{D}$-open map but not strongly $\widehat{D}$-open, Since $\{q\}$ is $\widehat{D}$-open in X but $f(\{q\}) = \{q\}$ is not $\widehat{D}$-open in Y.*

Theorem 3.11. *For any bijection $f : X \to Y$, the following conditions are equivalent:*

i) $f^{-1}: Y \to X$ *is* $\widehat{D}$ *- irresolute,*

ii) f *is a strongly* $\widehat{D}$ *- open map,*

iii) f *is a strongly* $\widehat{D}$ *- closed map.*

Proof. Similar to that of Theorem 3.4. □

Definition 3.12. *A bijection* $f : X \to Y$ *is called* $\widehat{D}$ *- homeomorphisms if* f *is both* $\widehat{D}$ *- continuous and* $\widehat{D}$ *- open.*

Proposition 3.13. *Every homeomorphism is an* $\widehat{D}$ *- homeomorphism but not conversely.*

Proof. Follows from Definitions. □

Example 3.14. *Let* $X = \{p,q,r\}$ *and* $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q\},\{p,q\},X\}$ *and* $\sigma =$ $\{\phi,\{p\},$
$\{p,q\},Y\}$. *Clearly identity map* $f : (X,\tau) \to (Y,\sigma)$ *is* $\widehat{D}$ *- homeomorphisms but not homeomorphisms, Since* $f(\{q\}) = \{q\}$ *is open in* X *but* $f(\{p\}) = \{q\}$ *is not open in* Y*, hence* f *is not an open map.*

Theorem 3.15. *Let* $f : X \to Y$ *be a bijective,* $\widehat{D}$ *- continuous map. Then the following conditions are equivalent:*

i) f *is an* $\widehat{D}$ *- open map,*

ii) f *is an* $\widehat{D}$ *- homeomorphism,*

iii) f *is an* $\widehat{D}$ *- closed map.*

Proof. **(i)** $\Longrightarrow$ **(ii) :** Obvious from definition.

(ii) $\Longrightarrow$ **(iii) :** Suppose f is an $\widehat{D}$ - open map and let F be a closed set in X. Then F^c is open in X, hence $f(F^c) = (f(F))^c$ is $\widehat{D}$ - open in Y implies f is a closed map Converse follows by the same technique. □

Remark 3.16. *The composition of two* $\widehat{D}$ *- homeomorphisms need not be an* $\widehat{D}$ *- homeomorphisms as seen from the following example.*

Example 3.17. *Let* $X = Y = Z = \{p,q,r\}$, $\tau = \{\phi,\{p\},X\}$, $\sigma = \{\phi,\{p,q\},Y\}$ *and* $\eta = \{\phi,\{p\},$ $\{q\},\{p,q\},Z\}$. *Let* $f : (X,\tau) \to (Y,\sigma)$ *and* $g : (Y,\sigma) \to (Z,\eta)$ *be two identity map. Then* f *and*

g are $\widehat{D}$ - homeomorphisms. Let $A = \{p,r\}$ be a closed in Z. Then $(g \circ f)^{-1}(A) = f^{-1}g^{-1}(A) = \{p,r\}$ which is not $\widehat{D}$-closed in (X,τ). Therefore composition $g \circ f : (X,\tau) \to (Z,\eta)$ is not an $\widehat{D}$ - homeomorphisms.

Definition 3.18. *A bijection $f : X \to Y$ is said to be strongly - $\widehat{D}$ - homeomorphisms if both f and f^{-1} and $\widehat{D}$ - irresolute.*

Example 3.19. *Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma = \{\phi,\{p,q\},$
$Y\}$. Let $f : (X,\tau) \to (Y,\phi)$ be an identity map. Then f is strongly $\widehat{D}$ - homeomorphism.*

We denote the family of all $\widehat{D}$ - homeomorphisms (resp, strongly $\widehat{D}$ - homeomorphism) of a topological space X onto itself by $\widehat{D} - h(X)$ (resp. $S\widehat{D} - h(X)$).

Proposition 3.20. *Every strongly $\widehat{D}$ - homeomorphism is an $\widehat{D}$ - homeomorphism but not conversely. In otherwards for any space X, $S\widehat{D} - h(X) \subset \widehat{D} - h(X)$.*

Proof. Since every $\widehat{D}$ - irresolute map is $\widehat{D}$ - continuous and also from remark 3.8, we get the proof. □

Example 3.21. *Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma = \{\phi,\{p\},$
$\{p,q\},Y\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is $\widehat{D}$ - homeomorphisms but not strongly $\widehat{D}$ - homeomorphisms, Since $\{r\}$ is $\widehat{D}$ - open in Y but $f^{-1}(\{r\}) = \{r\}$ is not $\widehat{D}$ - open in X. Hence f is $\widehat{D}$ - irresolute and so f is not strongly $\widehat{D}$ - homeomorphisms.*

Proposition 3.22. *If $f : X \to Y$ and $f : Y \to Z$ are two strongly $\widehat{D}$ - homeomorphisms then their composition $g \circ f : X \to Z$ is also a strongly $\widehat{D}$ - homeomorphism.*

Proof. Let U be an $\widehat{D}$ - open set in Z. Now $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$ where $V = g^{-1}(U)$. By hypothesis, V is $\widehat{D}$ - open in Y and so again by hypothesis $f^{-1}(V)$ is $\widehat{D}$ - open in X. Thus $g \circ f$ is $\widehat{D}$ - irresolute. Also for an $\widehat{D}$ - open set G in X, we have $(g \circ f)(G) = g(D)$ where $D = f(G)$, by hypothesis $f(G)$ is $\widehat{D}$ - open in Y and so again by hypothesis, $g(f(G))$ is $\widehat{D}$ - open in Z. Thus $(g \circ f)^{-1}$ is $\widehat{D}$ - irresolute. Hence $(g \circ f)$ is strongly $\widehat{D}$ - homeomorphism. □

Proposition 3.23. *The set $S\widehat{D}-h(X)$ is a group under the composition of maps.*

Proof. Define a binary operation $\circ : S\widehat{D}-h(X)\times S\widehat{D}-h(X)\to S\widehat{D}-h(X)$ by $f\circ g=g\circ f$ for all f and g in $S\widehat{D}-h(X)$ and $\circ$ is the usual operation of composition of maps. Then by Proposition 3.22, $g\circ f\in S\widehat{D}-h(X)$. We know that the composition of maps is associative and the identity map $i:X\to X$ belonging to $S\widehat{D}-h(X)$ serves as the identity element. If $f\in S\widehat{D}-h(X)$ that $f^{-1}\in S\widehat{D}-h(X)$ such that $f\circ f^{-1}=f^{-1}\circ f=i$ and so inverse exist for each element of $S\widehat{D}-h(X)$. Therefore, $(S\widehat{D}-h(X),\circ)$ is a group under the composition of maps. □

Theorem 3.24. *Let $f:X\to Y$ be an $S\widehat{D}$ - homeomorphism. Then f induces an isomorphism from the group $S\widehat{D}-h(X)$ onto the group $S\widehat{D}-h(Y)$.*

Proof. Using the map f, we define a map $\psi_f : S\widehat{D}-h(X)\to S\widehat{D}-h(Y)$ by $\psi_f(h)=f\circ h\circ f^{-1}$ for each $h\in S\widehat{D}-h(X)$. Then ψ_f is a bijection, further for $h_1,h_2\in S\widehat{D}-h(X)$. $\psi_f(h_1\circ h_2)=f\circ(h_1\circ h_2)\circ f^{-1}=(f\circ h_1\circ f^{-1})\circ(f\circ h_2\circ f^{-1})=\psi_f(h_1)\circ\psi_f(h_2)$. Therefore, ψ_f is a homomorphism and so it induces an isomorphism induced by f. □

Theorem 3.25. *$S\widehat{D}$ - homeomorphism is an equivalence relation on the collection of all topological spaces.*

Proof. Reflexivity and symmetry are immediate and transitivity follows from Proposition 3.22. □

4. $\widehat{D}$ - Quotient Map

Definition 4.1. *A surjective map $f:X\to Y$ is said to be an $\widehat{D}$ - quotient map if f is $\widehat{D}$ - continuous and $f^{-1}(V)$ is open in X implies V is $\widehat{D}$ - open in Y.*

The following proposition is an easy consequence from the definitions.

Proposition 4.2. *Every quotient map is $\widehat{D}$ - quotient but not conversely.*

Proof. The proof follows from the Definitions. □

Example 4.3. *Let $X=\{p,q,r\}$ and $Y=\{p,q,r\}$, $\tau=\{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma=\{\phi,\{p,q\},$*

$Y\}$. Clearly identity map $f:(X,\tau)\to(Y,\sigma)$ is an $\widehat{D}$ - quotient map but not a quotient map, Since $\{q\}$ is open in X but $f^{-1}(\{q\})=\{q\}$ is not open in Y.

Proposition 4.4. *If a map $f:X\to Y$ is surjective, $\widehat{D}$ - continuous and $\widehat{D}$ - open, then f is an $\widehat{D}$ - quotient map.*

Proof. We only need to prove that $f^{-1}(V)$ is open in X implies V is an $\widehat{D}$ - open set in Y. Let $f^{-1}(V)$ be open in X. Then $f(f^{-1}(V))$ is an $\widehat{D}$ - open set, since f is $\widehat{D}$ - open. Hence, V is an $\widehat{D}$ - open set, as f is surjective and $f(f^{-1}(V))=V$. Thus f is an $\widehat{D}$ - quotient map. □

Proposition 4.5. *If a map $f:X\to Y$ is a homeomorphism, then f is a quotient map but not conversely.*

Proof. Clearly follows from the definition. □

Example 4.6. *Let $X=\{p,q,r\}$ and $Y=\{p,q,r\}$, $\tau=\{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma=\{\phi,\{p,q\},$
$Y\}$. Clearly identity map $f:(X,\tau)\to(Y,\sigma)$ is an $\widehat{D}$ - quotient map but not homeomorphism, Since $\{q\}$ is open in X but $f^{-1}(\{q\})=\{q\}$ is not open in Y.*

Proposition 4.7. *Let $f:X\to Y$ be an open surjective, $\widehat{D}$ - irresolute map and $g:Y\to Z$ be an $\widehat{D}$ - quotient map. Then the composition $g\circ f:X\to Z$ is an $\widehat{D}$ - quotient map.*

Proof. Let V be any open set in Z. Then $g^{-1}(V)$ is an $\widehat{D}$ - open set, since g is an $\widehat{D}$ - quotient map. Since f is $\widehat{D}$ - irresolute, $f^{-1}(g^{-1}(V))=(g\circ f)^{-1}(V)$ is an $\widehat{D}$ - open in X, which implies $(g\circ f)^{-1}(V)$ is an $\widehat{D}$ - open set. This shows that $g\circ f$ is $\widehat{D}$ - continuous. Also, assume that $(g\circ f)^{-1}(V)$ is open in X for $V\subset Z$, that is, $f^{-1}(g^{-1}(V))$ is open set in X. Since f is open $f(f^{-1}(V))$ is open in Y. It follows that $g^{-1}(V)$ is open in Y, because f is surjective. Since g is a $\widehat{D}$ - quotient map, V is an $\widehat{D}$ - open set. Thus $g\circ f:X\to Z$ is an $\widehat{D}$ - quotient map. □

Proposition 4.8. *Let $h:X\to Y$ is an $\widehat{D}$ - quotient map and $g:X\to Z$ is a continuous map where Z is a space that is constant on each set $h^{-1}(\{y\})$ for each $y\in Y$, then g induces an $\widehat{D}$ - continuous an $\widehat{D}$ - continuous map $f:Y\to Z$ such that $f\circ h=g$.*

Proof. Since g is constant on $h^{-1}(\{y\})$, for each $y \in Y$, the set $g(h^{-1}(\{y\}))$ is an one point set in Z. If we let $f(y)$ to denote this point then it is clear that f is well defined and for each $x \in X$, $f(h(X)) = g(X)$. We claim that f is $\widehat{D}$ - continuous. For if we let V be any open set in Z, then $g^{-1}(V)$ is open set as g is continuous. But $g^{-1}(V) = h^{-1}(f^{-1}(V))$ is open in X. Since h is an $\widehat{D}$ - quotient map, $f^{-1}(V)$ is an $\widehat{D}$ - open in Y. □

Definition 4.9. *A surjective map $f : X \to Y$ is said to be a strongly $\widehat{D}$ - quotient map if f is $\widehat{D}$ - continuous and $f^{-1}(V)$ is $\widehat{D}$ - open in X implies V is $\widehat{D}$ - open in Y.*

Proposition 4.10. *Every strongly $\widehat{D}$ - quotient map is an $\widehat{D}$ - quotient map.*

Proof. Let $f : X \to Y$ be a strongly $\widehat{D}$ - quotient map. Let $f^{-1}(V)$ be an open in X. Then $f^{-1}(V)$ be an $\widehat{D}$ - open in X. Since f is a strongly $\widehat{D}$ - quotient map, V is $\widehat{D}$ - open in Y. This shows that f is an $\widehat{D}$ - quotient map. □

Remark 4.11. *The converse of the above proposition need not be true in general as shown in the following example.*

Example 4.12. *Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q,r\},X\}$ and $\sigma =$ $\{\phi,\{p\},\{q\},$
$\{p,q\},Y\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is an $\widehat{D}$ - quotient map but not strongly $\widehat{D}$ - quotient map, since $\{q\}$ is $\widehat{D}$ - open in X but $f^{-1}(\{r\}) = \{r\}$ is not $\widehat{D}$ - open in Y.*

Definition 4.13. *Let $f : X \to Y$ be a surjective map. Then f is called a completenly $\widehat{D}$ - quotient map if f is $\widehat{D}$ - irresolute and $f^{-1}(V)$ is $\widehat{D}$ - open in X implies U is open in Y.*

Theorem 4.14. *Let $f : X \to Y$ be a surjective map. Strongly $\widehat{D}$ - open and $\widehat{D}$ - irresolute map and $g : Y \to Z$ be a completely $\widehat{D}$ - quotient map. Then $g \circ f$ is a completely $\widehat{D}$ - quotient map.*

Proof. Since f and g are $\widehat{D}$ - irresolute, $g \circ f$ is $\widehat{D}$ - irresolute, by Proposition 2.8. Suppose $(g \circ f)^{-1}(V)$ is an $\widehat{D}$ - open in X for $V \subset Z$, that is, $f^{-1}(g^{-1}(V))$ is an $\widehat{D}$ - open in X. Since f is surjective and strongly $\widehat{D}$ - open, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is $\widehat{D}$ - open in Y. Also g is completely $\widehat{D}$ - quotient implies V is open in Z. Thus $g \circ f$ is a completely $\widehat{D}$ - quotient map. □

Proposition 4.15. *Every completely $\widehat{D}$ - quotient map is strongly $\widehat{D}$ - quotient map.*

Proof. Let $f : X \to Y$ be a completely $\widehat{D}$ - quotient map. By Proposition 2.7, f is $\widehat{D}$ - irresolute implies f is $\widehat{D}$ - continuous. Hence the proof follows. □

Remark 4.16. *The converse of the above proposition need not be true in general as shown in the following example.*

Example 4.17. *Let $X = \{p,q,r\}$ and $Y = \{p,q,r\}$, $\tau = \{\phi,\{p\},\{q\},\{p,q\},X\}$ and $\sigma =$ $\{\phi,\{p\},$
$\{q,r\},Y\}$. Clearly identity map $f : (X,\tau) \to (Y,\sigma)$ is an strongly $\widehat{D}$ - quotient map but not a completely $\widehat{D}$ - quotient map, since $\{r\}$ is $\widehat{D}$ - open in X but $f^{-1}(\{r\}) = \{r\}$ is not $\widehat{D}$ - open in X, implies that f is not $\widehat{D}$ - irresolute.*

Theorem 4.18. *Let $f : X \to Y$ be a surjective map and both X and Y be $T_{\widehat{D}}$ - spaces. Then the following are equivalent.*

(i) f is a completely $\widehat{D}$ - quotient map;
(ii) f is a strongly $\widehat{D}$ - quotient map;
(iii) f is a $\widehat{D}$ - quotient map;

Proof. **(i) $\Longrightarrow$ (ii) :** Follows by Proposition 4.15.

(ii) $\Longrightarrow$ (iii) : Follows by Proposition 4.10.

(iii) $\Longrightarrow$ (i) : Since Y is a $T_{\widehat{D}}$ - space, f is $\widehat{D}$ - irresolute, by Proposition 2.9. Suppose $f^{-1}(V)$ is $\widehat{D}$ - open in X. Since X is a $T_{\widehat{D}}$, $f^{-1}(V)$ is open in X. By (iii), V is $\widehat{D}$ - open in Y. Since Y is a $T_{\widehat{D}}$ - space, V is open in Y. Hence, we get (i). □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ. 12 (1983), 77-90.

[2] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Mon. 70 (1963), 36-41.

[3] K. Dass, G. Suresh, $\widehat{D}$-Continuous functions and its some characterizations, Int. J. Anal. Experiment. Modal Anal. 12 (2020), 1606-1618.

[4] J. Antony Rex Rodrigo, K. Dass, A New type of generalized closed sets, Int. J. Math. Arch. 3(4) (2012), 1517-1523.

[5] S. Mashhour, M. E. Abd. El-Monsef, S. N. El-Deeb, On pre continuous mappings and weak pre-continuous mappings, Proc. Math, Phys. Soc. Egypt. 53 (1982), 47-53.

[6] K. Dass, G. Suresh, $\widehat{D}$-irresolute functions and its properties, Int. J. Mod. Agric. 9(4) (2020), 701-706.

[7] D. Andrijevie, Semi-preopen sets, Mat. Vesnik, 38 (1986), 24-32.

[8] P. Sundaram and M. Sheik john, On ω-closed sets in topology, Acta Ciencia Indica, 4 (2000), 389-392.

[9] M. K. R. S. Veera Kumar, μ_p-closed sets in Topological spaces, Antarctica J. Math. 2(1) (2005), 31-52.

[10] K. Dass, G. Suresh, $\widehat{D}$-closed sets in Topological spaces, Mayala J. Mat. S(1) (2021), 265-269.