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(a,d)-TOTAL EDGE IRREGULARITY STRENGTH OF GRAPHS

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Abstract. A new graph characteristic, (a,d)-total edge irregularity strength of graphs is introduced. (a,d)-edge irregular evaluations of some families of graphs has been made, upper and lower bounds of the above parameter are determined.

Keywords: irregular labeling; (a,d)-irregular labeling; irregularity strength.

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1. Introduction

A graph *labeling* is a mapping $\sigma : \mathscr{D} \to \{1,2,\ldots,h\}$ subject to certain conditions,if the domain $\mathscr{D}$ is the set of vertices (or edges), then σ is called a *vertex labeling* (or an *edge labeling*). If $\mathscr{D}$ is the set of vertices and edges, then σ is called a *total labeling*. For an edge h-labeling $\phi : E(G) \to \{1,2,\ldots,h\}$, the associated weight of a vertex $x \in V(G)$ is $w_\phi(x) = \sum \phi(xy)$, where the sum is taken over all vertices y adjacent to x.

In 1988, Chartrand *et al.* [6] introduced edge h-labeling ϕ of a graph G such that $w_\phi(x) \neq w_\phi(y)$ for all vertices $x,y \in V(G)$ with $x \neq y$. Such labelings were called irregular assignments and the irregularity strength $s(G)$ of a graph G is known as the minimum h for which G has an

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irregular assignment using labels atmost h. Many authors were much attracted by this parameter and investigated the bounds of $s(G)$ [1, 2, 3, 5, 7, 8]. Baca *et al.* [4] modified this irregularity strength and introduced the concept of total edge irregularity strength for a graph G. A total h-labeling $\psi : V \cup E \to \{1,2,\dots,h\}$ of a graph G is said to be an edge irregular total h-labeling if for each two distinct edges xy and $x^{'}y^{'}$ their weights $\psi(x)+\psi(xy)+\psi(y)$ and $\psi(x')+\psi(x'y')+\psi(y')$ are distinct. The minimum h for which the graph G has an edge irregular total h-labeling is called the total edge irregularity strength of G, denoted by $tes(G)$.

In [4], they have given the bounds of the total edge irregularity strength for all graphs and the result is as follows:

$$\left\lceil \frac{|E|+2}{3} \right\rceil \geq tes(G) \geq |E|,$$

where $|E|$ is the cardinality of the edgeset of a graph G. Ivanco and Jendrol [10] proved that

$$tes(T) = \max\{\left\lceil \frac{|E(T)|+2}{3} \right\rceil, \frac{\Delta(T)+1}{2}\}, \text{ where } T \text{ is a tree.}$$

Motivated by this parameter,Indra Rajasingh and Teresa Arockiamary Santiago were investigated and determined the exact value of this parameter for uniform theta graph in [9] and F.Salama determined the same for polar grid graph in [13]. Recently, Lucia Ratnasari *et al.*[11] found that the exact value of tes of an odd arithmetic book graph $B_n(C_{3,5,7,\dots,2n+1})$ of n sheets is equal to $\left\lceil \frac{n^2+n+3}{3} \right\rceil$ and tes of an even arithmetic book graph $B_n(C_{4,6,8,\dots,2n+2})$ is equal to $\left\lceil \frac{n^2+2n+3}{3} \right\rceil$. Also, Yeni Susanti *et al.* [14] determined the exact values of tes of staircase graphs and its related graphs.

Due to the involvement on the total irregularity strength of graphs, we introduce a new parameter, namely (a,d)-total edge irregularity strength of graphs.

Let $G = (V,E)$ be a graph of order n and size m. A total h-labeling $\psi : V(G) \cup E(G) \to \{1,2,\dots,h\}$ is called (a,d)-edge irregular labeling if there exists a bijective function $\sigma : E(G) \to \{a, a+d, a+2d, \dots, a+(m-1)d\}$ defined by $\sigma(xy) = \psi(x)+\psi(y)+\psi(xy)$ called arithmetic progression edge weight of the edge xy, where $a \geq 3, d > 1$.

We define the (a,d)-total edge irregularity strength of a graph G, denoted by $(a,d)-tes(G)$, as the minimum h for which G has a (a,d)-edge irregular h-labeling.Also, we define another parameter called (a,d)-total vertex irregularity strength of a graph G, denoted by $(a,d)-tvs(G)$ in [12]

The main aim of this paper is to show the bounds of the (a,d)-total edge irregularity strength and to determine the precise value of this parameter for some families of graphs.

2. (a,d)-Edge Irregular Labeling of Graphs

The following theorem provides the upper and lower bounds of $(a,d)-tes(G)$.

Lemma 2.1. *Let $G=(V,E)$ be a graph of order n and size q. For integers $a \geq 3$ and $d \geq 2$, $\left\lceil \frac{a+(m-1)d}{3} \right\rceil \leq (a,d)-tes(G) \leq a-2+(m-1)d$.*

Proof. The upper bound of $(a,d)-tes(G)$ can be obtained by assigning label 1 to all the vertices of G, further assign labels $a-2, a-2+d, a-2+2d, \ldots, a-2+(m-1)d$ to the edges of G at random.

Assume that the graph G has $(a,d)-$*edge irregular labeling* τ. Thus the edge weights are $a, a+d, a+2d, \ldots, a+(m-1)d$. Since the heaviest weight $a+(m-1)d$ is the sum of three labels, $(a,d)-tes(G) \geq \left\lceil \frac{a+(m-1)d}{3} \right\rceil$. □

Theorem 2.2. *Let P_n be a path of order $n \geq 3$. Then $(3,2)-tes(P_n) = \left\lceil \frac{2n-1}{3} \right\rceil$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n. Define total labeling $\tau_1 : V(P_n) \cup E(P_n) \to \{1, 2, \ldots, \left\lceil \frac{2n-1}{3} \right\rceil\}$ as follows:

$$\tau_1(v_{3i+1}) = 2i+1, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n-1}{3} \right\rfloor,$$

$$\tau_1(v_{3i+2}) = 2i+1, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n-2}{3} \right\rfloor,$$

$$\tau_1(v_{3i}) = 2i, \quad \text{if } 1 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor,$$

$$\tau_1(v_{3i+1}v_{3i+2}) = 2i+1, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n-2}{3} \right\rfloor,$$

$$\tau_1(v_{3i+2}v_{3i+3}) = 2i+2, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n-3}{3} \right\rfloor,$$

$$\tau_1(v_{3i+3}v_{3i+4}) = 2i+2, \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n-4}{3} \right\rfloor.$$

Under the labeling τ_1 the edge weights are as follows:

$$w_{\tau_1}(v_iv_{i+1}) = 2i+1, \quad \text{if } 1 \leq i \leq n-1.$$

The weights of the edges of P_n forms an arithmetic progression with common difference 2 and hence $(3,2)-tes(P_n) \leq \lceil \frac{2n-1}{3} \rceil$. Lemma 2.1 shows that $(3,2)-tes(P_n) \geq \lceil \frac{2n-1}{3} \rceil$, this concludes the proof. □

Definition 2.3. *The corona product $G_1 \odot G_2$ of two graphs G_1 and G_2 is a graph G obtained by taking one copy G_1 which has n vertices and n copies of G_2 and then joining i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.*

Definition 2.4. *A special type of graph $C(n,t)$ is defined by the corona product of the path P_n by tK_1 i.e., $C(n,t) = P_n \odot tK_1$.*

Theorem 2.5. *Let P_n be the path on n vertices, then $(3,2)-tes(C(n,t)) = \left\lceil \frac{n(2t+2)-1}{3} \right\rceil$, $n \geq 2$.*

Proof. Let $C(n,t) = P_n \odot tK_1$ be the corona product of path P_n by tK_1. Let $V(C(n,t)) = \{v_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq t\}$ and $E(C(n,t)) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_iu_{i,j} : 1 \leq i \leq n, 1 \leq j \leq t\}$ be the vertex set and edge set of $C(n,t)$. Define total labeling $\tau_2 : V(G) \cup E(G) \to \{1,2,\ldots,\left\lceil \frac{n(2t+2)-1}{3} \right\rceil\}$ as follows:

$$\tau_2(v_i) = 1+(i-1)t, \quad \text{if } 1 \leq i \leq n.$$

$$\tau_2(v_iv_{i+1}) = t+2i-1, \quad \text{if } 1 \leq i \leq n-1.$$

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq j \leq t$,

$$\tau_2(v_{2i}u_{2i,j}) = \begin{cases} \frac{t+3}{2} + (i-1)(t+2) + (j-1), & \text{if t is odd,} \\ \frac{t+4}{2} + (i-1)(t+2) + (j-1), & \text{if t is even.} \end{cases}$$

For $1 \leq i \leq \lceil \frac{n}{2} \rceil$, $1 \leq j \leq t$,

$$\tau_2(v_{2i-1}u_{2i-1,j}) = 1+(i-1)(t+2)+(j-1).$$

For $1 \leq i \leq n$, $1 \leq j \leq t$,

$$\tau_2(u_{i,j}) = \begin{cases} \tau_2(v_i u_{i,j}), & \text{if i is odd,} \\ \tau_2(v_i u_{i,j})+1, & \text{if i is even.} \end{cases}$$

The labeling τ_2 induces edge weight function $\sigma : E(G) \to \{3,5,\ldots,n(2t+2)-1\}$ is as follows:

$$\sigma(v_i v_{i+1}) = (2t+2)i+1, \quad \text{if } 1 \leq i \leq n-1.$$

$$\sigma(v_i u_{i,j}) = (i-1)(2t+2)+2j+1, \quad \text{if } 1 \leq i \leq n, \quad 1 \leq j \leq t.$$

Thus weights of the edges of $C(n,t)$ forms an arithmetic progression sequence and hence $(3,2)-tes(C(n,t)) \leq \left\lceil \frac{n(2t+2)-1}{3} \right\rceil$. Lemma 2.1 shows that $(3,2)-tes(C(n,t)) \geq \left\lceil \frac{n(2t+2)-1}{3} \right\rceil$. Hence the proof. □

Definition 2.6. *A friendship graph F_n is a graph which consists of n triangles sharing a common vertex.*

Theorem 2.7. *If F_n is a friendship graph of order $2n+1$, then $(3,2)-tes(F_n) = \lceil \frac{6n+1}{3} \rceil$, $n \geq 3$.*

Proof. Let F_n be a friendship graph of $2n+1$ vertices and $3n$ edges. Let $V(F_n) = \{u, v_i : 1 \leq i \leq 2n\}$ and $E(F_n) = \{v_{2i-1}v_{2i} : 1 \leq i \leq n\} \cup \{uv_i : 1 \leq i \leq 2n\}$ be the vertex set and edge set of F_n. Define total labeling $\tau_3 : V(F_n) \cup E(F_n) \to \{1,2,\ldots,\lceil \frac{6n+1}{3} \rceil\}$ as follows:

$$\tau_3(v_{2i-1}v_{2i}) = \begin{cases} 1, & \text{if i=1,} \\ \lceil \frac{6n+1}{3} \rceil, & \text{if } 2 \leq i \leq n. \end{cases}$$

$$\tau_3(u) = 3$$

For $1 \leq i \leq 2n$,

$$\tau_3(v_i) = \begin{cases} i, & \text{if i is odd,} \\ 1, & \text{if i=2,} \\ \lceil \frac{6n+1}{3} \rceil, & \text{if i is even and } i \neq 2. \end{cases}$$

For $1 \leq i \leq 2n$,

$$\tau_3(uv_i) = \begin{cases} 1, & \text{if i is odd,} \\ \left\lceil \frac{6n+1}{3} \right\rceil, & \text{if i=2,} \\ i-1, & \text{if i is even and } i \neq 2. \end{cases}$$

Then the edge weight function $\sigma : E(F_n) \to \{3, 5, \ldots, 6n+1\}$ is as follows:

$$\sigma(v_{2i-1}v_{2i}) = \begin{cases} 3, & \text{if i=1 ,} \\ 2\left\lceil \frac{6n+1}{3} \right\rceil + 2i - 1, & \text{if } 2 \leq i \leq n. \end{cases}$$

For $1 \leq i \leq 2n$

$$\sigma(uv_i) = \begin{cases} i+4, & \text{if i is odd ,} \\ \left\lceil \frac{6n+1}{3} \right\rceil + i + 2, & \text{if i is even.} \end{cases}$$

Thus weights of the edges of F_n forms an arithmetic progression with common difference 2 and hence $(3,2)-tes(F_n) \leq \left\lceil \frac{6n+1}{3} \right\rceil$. Lemma 2.1 shows that $(3,2)-tes(F_n) \geq \left\lceil \frac{6n+1}{3} \right\rceil$. Hence the theorem. □

Theorem 2.8. *If $K_{1,n}$ is a star graph of order $n+1$, then $(3,2)-tes(K_{1,n}) = n$, $n \geq 2$.*

Proof. Let $V(K_{1,n}) = \{u, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{uv_i : 1 \leq i \leq n\}$ be the vertex set and edge set of $K_{1,n}$ respectively. Define total labeling $\tau_4 : V(K_{1,n}) \cup E(K_{1,n}) \to \{1, 2, \ldots, n\}$ as follows:

$$\tau_4(u) = 1.$$

$$\tau_4(v_i) = i, 1 \leq i \leq n.$$

$$\tau_4(uv_i) = i, 1 \leq i \leq n.$$

Then the edge weight function $\sigma : E(K_{1,n}) \to \{3, 5, \ldots, 2n+1\}$ is as follows:

$$\sigma(uv_i) = 2i + 1, \ \ 1 \leq i \leq n.$$

Since $\tau_4(u) = 1$, the remaining labels of edges and vertices are from $\frac{3-1}{2}, \frac{5-1}{2}, \ldots, \frac{2n}{2}$ and forms an arithmetic progression.

$\therefore (3,2) - tes(K_{1,n}) \leq n$. On the otherhand, to obtain the weight 3 it is essential to label the vertex u to 1. Thus, $(3,2) - tes(K_{1,n}) \leq n$. This concludes the theorem.

□

Theorem 2.9. *If L_n is a ladder graph of order $2n$, then $(3,2) - tes(L_n) = \lceil \frac{6n-3}{3} \rceil$, $n \geq 2$.*

Proof. Let $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$ be the vertex set and edge set of L_n respectively. Define total labeling $\tau_5 : V(L_n) \cup E(L_n) \to \{1, 2, \ldots, \lceil \frac{6n-3}{3} \rceil\}$ as follows:

$$\tau_5(u_i) = \tau_5(v_i) = \tau_5(u_i v_i) = 2i - 1, \ \ 1 \leq i \leq n.$$
$$\tau_5(u_i u_{i+1}) = 2i - 1, \ \ 1 \leq i \leq n-1.$$
$$\tau_5(v_i v_{i+1}) = 2i + 1, \ \ 1 \leq i \leq n-1.$$

Thus the edge weight function $\sigma : E(L_n) \to \{3, 5, \ldots, 6n-3\}$ is as follows:

$$\sigma(u_i v_i) = 6i - 3, \ \ 1 \leq i \leq n.$$
$$\sigma(u_i u_{i+1}) = 6i - 1, \ \ 1 \leq i \leq n-1.$$
$$\sigma(v_i v_{i+1}) = 6i + 1, \ \ 1 \leq i \leq n-1.$$

Thus weights of the edges of L_n forms an arithmetic progression and hence $(3,2) - tes(L_n) \leq \lceil \frac{6n-3}{3} \rceil$. Lemma 2.1 shows that $(3,2) - tes(L_n) \geq \lceil \frac{6n-3}{3} \rceil$. □

Theorem 2.10. *If f_n is a fan graph of order $n+1$, then $(3,2) - tes(f_n) = \lceil \frac{4n-1}{3} \rceil$, $n \geq 3$.*

Proof. Fan graph f_n is defined as $P_n + K_1$. Let $V(f_n) = \{u, v_i : 1 \leq i \leq n\}$ and $E(f_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u v_i : 1 \leq i \leq n\}$ be the vertex set and edge set of f_n respectively. Define total labeling $\tau_6 : V(f_n) \cup E(f_n) \to \{1, 2, \ldots, \lceil \frac{4n-1}{3} \rceil\}$ as follows:

$$\tau_6(u) = \left\lfloor \frac{4n-1}{3} \right\rfloor.$$

$$\tau_6(v_i) = \begin{cases} i, & \text{if i is odd,} \\ i-1, & \text{if i is even,} \end{cases}, \ \ 1 \leq i \leq \left\lceil \frac{4n-1}{6} \right\rceil.$$

$$\tau_6(v_i) = \left\lceil \frac{4n-1}{3} \right\rceil, \quad \left\lceil \frac{4n-1}{6} \right\rceil + 1 \le i \le n.$$

$$\tau_6(v_i v_{i+1}) = 1, \quad 1 \le i \le \left\lceil \frac{4n-1}{6} \right\rceil - 1.$$

$$\tau_6(v_{\lceil \frac{4n-1}{6} \rceil} v_{\lceil \frac{4n-1}{6} \rceil + 1}) = \begin{cases} \lceil \frac{2n}{3} \rceil + 2, & n \equiv 0,1 \pmod 3, \\ \lceil \frac{2n}{3} \rceil + 3, & n \equiv 2 \pmod 3. \end{cases}.$$

$$\tau_6(v_{n-i} v_{n+1-i}) = \begin{cases} \lfloor \frac{4n-1}{3} \rfloor - 2i, & n \equiv 0,1 \pmod 3, \\ \lfloor \frac{4n-1}{3} \rfloor - 2i - 1, & n \equiv 2 \pmod 3, \end{cases} \quad 1 \le i \le \left\lceil \frac{n-5}{3} \right\rceil$$

$$\tau_6(uv_n) = \begin{cases} \lceil \frac{4n-1}{3} \rceil, & n \equiv 0,1 \pmod 3, \\ \lfloor \frac{4n-1}{3} \rfloor, & n \equiv 2 \pmod 3, \end{cases}.$$

$$\tau_6(uv_{\lceil \frac{4n-1}{6} \rceil + i}) = \begin{cases} 2i+2, & n \equiv 0 \pmod 3, \\ 2i+3, & n \equiv 1 \pmod 3), \\ 2i+4, & n \equiv 2 \pmod 3), \end{cases}, \quad 1 \le i \le \left\lceil \frac{n-5}{3} \right\rceil.$$

For $n \equiv 0,1 \pmod 3$

$$\tau_6(uv_i) = \begin{cases} i, & \text{if i is odd,} \\ i+1, & \text{if i is even,} \end{cases}, \quad 1 \le i \le \left\lceil \frac{4n-1}{6} \right\rceil.$$

For $n \equiv 2 \pmod 3$

$$\tau_6(uv_i) = \begin{cases} i+1, & \text{if i is odd,} \\ i+2, & \text{if i is even,} \end{cases}, \quad 1 \le i \le \left\lceil \frac{4n-1}{6} \right\rceil.$$

Then the edge weight function $\sigma : E(f_n) \to \{3, 5, \dots, 4n-1\}$ is as follows:

$$\sigma(v_i v_{i+1}) = 2i + 1, \quad 1 \le i \le \left\lceil \frac{4n-1}{6} \right\rceil - 1.$$

$$\sigma(uv_i) = \begin{cases} \lfloor \frac{4n-1}{3} \rfloor + 2i, & \text{if } n \equiv 0,1 \pmod 3 \\ \lfloor \frac{4n-1}{3} \rfloor + 2i+1, & \text{if } n \equiv 2 \pmod 3), \end{cases}, \ 1 \le i \le \left\lceil \frac{4n-1}{6} \right\rceil.$$

$$\sigma(v_{\lceil \frac{4n-1}{6} \rceil} v_{\lceil \frac{4n-1}{6} \rceil + 1}) = \begin{cases} 2\lceil \frac{4n-1}{3} \rceil + 1, & \text{if } n \equiv 0 \pmod 3) \\ 2\lceil \frac{4n-1}{3} \rceil + 3, & \text{if } n \equiv 1,2 \pmod 3), \end{cases}.$$

$$\sigma(uv_{\lceil \frac{4n-1}{6} \rceil + i}) = \begin{cases} 2\lceil \frac{4n-1}{3} \rceil + 1 + 2i, & \text{if } n \equiv 0 \pmod 3) \\ 2\lceil \frac{4n-1}{3} \rceil + 3 + 2i, & \text{if } n \equiv 1,2 \pmod 3), \end{cases}, \ 1 \le i \le \left\lceil \frac{n-5}{3} \right\rceil.$$

$$\sigma(v_{n-i}v_{n+1-i}) = 4n-1-2i, \ 1 \le i \le \left\lceil \frac{n-5}{3} \right\rceil.$$

$$\sigma(uv_n) = 4n-1.$$

Thus weights of the edges of the fan graph f_n are $3,5,\ldots,4n-1$, which forms an arithmetic progression and hence $(3,2)-tes(f_n) \le \lceil \frac{4n-1}{3} \rceil$. Lemma 2.1 shows that $(3,2)-tes(f_n) \ge \lceil \frac{4n-1}{3} \rceil$, this concludes the proof. □

Definition 2.11. *Let $v_1, v_2, \ldots, v_n$ be the consecutive vertices of P_n. Then the graph P_n^2 can be obtained by adding an edge from every i^{th} vertex to $(i+2)^{th}$ vertex.*

Theorem 2.12. *Let P_n be the path on n vertices, then $(3,2)$-tes $(P_n^2) = \lfloor \frac{4n}{3} \rfloor - 1, n > 3$.*

Proof. Let $V = \{v_i / 1 \le i \le n\}$ be the vertex set and let $E = \{v_iv_{i+1} / 1 \le i \le n-1\} \cup \{v_iv_{i+2} / 1 \le i \le n-2\}$ be the edge set of $P_n^2, n > 3$.

Define total labeling $\tau_7 : V \cup E \to \{1, 2, \ldots, \lfloor \frac{4n}{3} \rfloor - 1\}$ as follows:

For $1 \le i \le n-1$,

$$\tau_7(v_i) = \begin{cases} \frac{4i-3}{3}, & \text{if } i \equiv 0 \pmod 3), \\ \frac{4i-1}{3}, & \text{if } i \equiv 1 \pmod 3), \\ \frac{4i-5}{3}, & \text{if } i \equiv 2 \pmod 3). \end{cases}$$

$$\tau_7(v_i) = \begin{cases} \frac{4n-3}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{4n-4}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{4n-5}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

$$\tau_7(v_2v_3) = 1$$

For $1 \leq i \leq n-2$, $i \neq 2$

$$\tau_7(v_iv_{i+1}) = \begin{cases} \frac{4i-3}{3}, & \text{if } i \equiv 0 \pmod 3, \\ \frac{4i-1}{3}, & \text{if } i \equiv 1 \pmod 3, \\ \frac{4i+1}{3}, & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_7(v_{n-1}v_n) = \left\lfloor \frac{4n}{3} \right\rfloor - 1 \text{ and } \tau_7(v_1v_3) = 3.$$

For $2 \leq i \leq n-3$,

$$\tau_7(v_iv_{i+2}) = \begin{cases} \frac{4i+3}{3}, & \text{if } i \equiv 0 \pmod 3, \\ \frac{4i-1}{3}, & \text{if } i \equiv 1 \pmod 3, \\ \frac{4i+1}{3}, & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_7(v_{n-2}v_n) = \begin{cases} \frac{4n-9}{3}, & \text{if } n \equiv 0 \pmod 3, \\ \frac{4n-4}{3}, & \text{if } n \equiv 1 \pmod 3, \\ \frac{4n-5}{3}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Under the labeling τ_7, edge weights of P_n^2 are $3, 5, \ldots, 2m+1$ where $m = 2n-3$, which are in arithmetic progression with $a = 3$ and $d = 2$. Thus, τ_7 is a $(3,2)$-labeling of P_n^2 and hence $(3,2)$-tes $(P_n^2) \leq \left\lfloor \frac{4n}{3} \right\rfloor - 1$. The lower bound of $(3,2)$-tes (P_n^2) can be obtained by the lemma 2.1 (ie) $(3,2)$-tes $(P_n^2) \geq \left\lceil \frac{4n-5}{3} \right\rceil = \left\lfloor \frac{4n}{3} \right\rfloor - 1$ and hence $(3,2)$-tes $(P_n^2) = \left\lfloor \frac{4n}{3} \right\rfloor - 1$. □

Theorem 2.13. *If $C_n \times K_2$ is the Cartesian product of the cycle C_n and K_2, then $(3,2)$-tes $(C_n \times K_2) = \left\lceil \frac{6n+1}{3} \right\rceil$, $n \geq 3$.*

Proof. Let $V = \{u_i v_i / 1 \leq i \leq n\}$ be the vertex set and let $E = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i / 1 \leq i \leq n\}$ be the edge set of $C_n \times K_2$, where $n \geq 3$.

Define total labeling $\tau_8 : V \cup E \to \{1, 2, \ldots, \lceil \frac{6n+1}{3} \rceil\}$ as follows:

For $1 \leq i \leq \lceil \frac{n}{2} \rceil + 1$,

$$\tau_8(u_i) = \begin{cases} 4\lfloor \frac{i-1}{3} \rfloor + 1, & \text{if } i \equiv 1,2 \pmod 3, \\ \frac{4i-3}{3}, & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$$\tau_8(u_{n-i+1}) = \begin{cases} 4\lfloor \frac{i-1}{3} \rfloor + 3, & \text{if } i \equiv 1,2 \pmod 3, \\ \frac{4i+3}{3}, & \text{if } i \equiv 0 \pmod 3. \end{cases}$$

For $1 \leq i \leq \lceil \frac{n}{2} \rceil$,

$$\tau_8(v_i) = n + 2i.$$

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$\tau_8(v_{n-i+1}) = n + 2(i+1).$$

For $1 \leq i \leq \lceil \frac{n}{2} \rceil + 1$,

$$\tau_8(u_i u_{i+1}) = \begin{cases} \lfloor \frac{4i-1}{3} \rfloor, & \text{if } i \equiv 0,1 \pmod 3, \\ \lceil \frac{4i-1}{3} \rceil & \text{if } i \equiv 2 \pmod 3. \end{cases}$$

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2$,

$$\tau_8(u_{n-i+1} u_{n-i}) = \begin{cases} \lfloor \frac{4i-1}{3} \rfloor + 2, & \text{if } i \equiv 0,1 \pmod 3, \\ \lceil \frac{4i-1}{3} \rceil + 2, & \text{if } i \equiv 2 \pmod 3 \end{cases}$$

and $\tau_8(u_n u_1) = 1$.

For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$$\tau_8(v_i v_{i+1}) = 2n - 3.$$

For $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$$\tau_8(v_{n+1-i}v_{n-i}) = 2n-1 \text{ and } \tau_8(v_nv_1) = 2n-1.$$
$$\tau_8(u_1v_1) = \tau_8(u_nv_n) = n.$$

For $2 \le i \le \lceil \frac{n}{2} \rceil$,

$$\tau_8(u_iv_i) = 2\left\lceil \frac{i-1}{3} \right\rceil + n + 2.$$

For $1 \le i \le \lceil \frac{n}{2} \rceil - 2$,

$$\tau_8(u_{n-i}v_{n-i}) = 2\left\lceil \frac{i}{3} \right\rceil + n.$$

Then the edge weight function $\sigma : E(C_n \times K_2) \to \{3, 5, \dots, 6n+1\}$ is as follows.

$\sigma(u_iu_{i+1}) = 4i-1$, $1 \le i \le \lceil \frac{n}{2} \rceil$

$\sigma(u_{n+1-i}u_{n-i}) = 4i+5$, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$

$\sigma(u_1u_n) = 5$, for $n \ge 3$

$\sigma(u_1v_1) = 2n+3$,

$\sigma(u_iv_i) = 2n+(4i-3)$, $2 \le i \le \lceil \frac{n}{2} \rceil$

$\sigma(u_{n-i+1}v_{n-i+1}) = 2n+(4i+3)$, $1 \le i \le \lfloor \frac{n}{2} \rfloor$

$\sigma(v_iv_{i+1}) = 4n+(4i-1)$, $1 \le i \le \lceil \frac{n}{2} \rceil$

$\sigma(v_nv_{n-i}) = 4n+(8i+1)$, $1 \le i \le \lfloor \frac{n}{2} \rfloor - 1$

$\sigma(v_1v_n) = 4n+5$, for $n \ge 3$.

Thus, the weights of the edges of $C_n \times K_2$ forms an arithmetic progression and hence $(3,2)$-tes $(C_n \times K_2) \le \lfloor \frac{6n+1}{3} \rfloor$. Lemma 2.1 shows that $(3,2)-tes(C_n \times K_2) \ge \lceil \frac{6n+1}{3} \rceil$, this concludes the proof. □

Theorem 2.14. $(3,2)$-*tes* $[CP_n(m)] = \left\lceil \frac{2nm+2n-1}{3} \right\rceil$.

Proof. A Caterpillar graph $CP_n(m)$ is a tree in which the removal of all pendant vertices results in a chordless path P_n. The m edges from each vertex of P_n to the pendant vertices are called leaves. Let $V[CP_n(m)] = \{u_i, v_{i,j}/1 \le i \le n, 1 \le j \le m\}$ be the vertex set and let $E[CP_n(m)] = \{u_iu_{i+1}/1 \le i \le n-1\} \cup \{u_iv_{i,j}/1 \le i \le n, 1 \le j \le m\}$ be the edge set of the caterpillar $CP_n(m)$ respectively.

Define total labeling $\tau_9 : V \cup E \to \{1, 2, \dots, \left\lceil \frac{2nm+2n-1}{3} \right\rceil\}$ as follows:

Case 1: For any $n \geq 2$ and $m = 1$, $1 \leq i \leq n$

$$\tau_9(u_i) = \begin{cases} i + \left\lceil \frac{i-3}{3} \right\rceil, & i \equiv 0 \pmod 3, \\ i + \left\lceil \frac{i-1}{3} \right\rceil, & i \equiv 1 \pmod 3, \\ i + \left\lceil \frac{i}{3} \right\rceil, & i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_9(v_{1,1}) = 1 \text{ and } \tau_9(v_{i,1}) = i + \left\lceil \frac{i-2}{3} \right\rceil, 2 \leq i \leq n.$$

For $1 \leq i \leq n-1$,

$$\tau_9(u_i u_{i+1}) = \begin{cases} i + \left\lceil \frac{i+1}{3} \right\rceil, & i \equiv 0 \pmod 3, \\ i + \left\lceil \frac{i-1}{3} \right\rceil, & i \equiv 1 \pmod 3, \\ i + \left\lceil \frac{i}{3} \right\rceil, & i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_9(u_1 v_{1,1}) = 1 \text{ and } \tau_9(u_i v_{i,1}) = i + \left\lceil \frac{i-2}{3} \right\rceil, 2 \leq i \leq n.$$

Case 2: Suppose $m \equiv 0 \pmod 3$ and $n \geq 2$. Let $m = 3k$ for some integer $k > 0$, then define τ_9 as follows:

$$\tau_9(u_1) = 1.$$

For $2 \leq i \leq n$,

$$\tau_9(u_i) = \begin{cases} (6k+2)\left\lceil \frac{i}{3} \right\rceil - 1, & i \equiv 0 \pmod 3, \\ (6k+2)\left\lceil \frac{i-1}{3} \right\rceil + (2k-1), & i \equiv 1 \pmod 3, \\ (6k+2)\left\lceil \frac{i}{3} \right\rceil - (2k+1), & i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_9(u_1 u_2) = 2k+1.$$

For $2 \leq i \leq n-1$,

$$\tau_9(u_i u_{i+1}) = \begin{cases} (6k+2)\left\lceil \frac{i}{3} \right\rceil - (2k-3), & i \equiv 0 \pmod 3, \\ (6k+2)\left\lceil \frac{i-1}{3} \right\rceil + 3, & i \equiv 1 \pmod 3, \\ (6k+2)\left\lceil \frac{i-2}{3} \right\rceil + (2k+3), & i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_9(v_{1,j}) = \tau_9(u_1 v_{1,j}) = j,\ 1 \leq j \leq m.$$

For $2 \leq i \leq n$ and $1 \leq j \leq m$,

$$\tau_9(v_{i,j}) = \tau_9(u_i v_{i,j}) = \begin{cases} (6k+2)\left\lceil \frac{i}{3} \right\rceil - (3k+j), & i \equiv 0 \pmod 3 \forall j \\ (6k+2)\left\lceil \frac{i-1}{3} \right\rceil + j, & i \equiv 1 \pmod 3 \forall j \\ (6k+2)\left\lceil \frac{i}{3} \right\rceil - (5k+1) + j, & i \equiv 2 \pmod 3 \forall j. \end{cases}$$

Case 3: Suppose $m \equiv 1 \pmod 3$, $m > 1$ and for any $n \geq 2$. Let $m = 3k+1$ for some integer $k > 0$, then define τ_9 as follows:

$$\tau_9(u_1) = 1.$$

For $2 \leq i \leq n$,

$$\tau_9(u_i) = \begin{cases} (6k+4)\left\lceil \frac{i-1}{3} \right\rceil + (2k+1), & i \equiv 1 \pmod 3, \\ (6k+4)\left\lceil \frac{i}{3} \right\rceil - (2k+3), & i \equiv 2 \pmod 3, \\ (6k+4)\left\lceil \frac{i}{3} \right\rceil - 1, & i \equiv 0 \pmod 3). \end{cases}$$

$$\tau_9(u_1 u_2) = 2k+3.$$

For $2 \leq i \leq n-1$,

$$\tau_9(u_i u_{i+1}) = \begin{cases} (6k+4)\left\lceil \frac{i}{3} \right\rceil - (2k-1), & i \equiv 0 \pmod 3, \\ (6k+4)\left\lceil \frac{i-1}{3} \right\rceil + 3, & i \equiv 1 \pmod 3, \\ (6k+4)\left\lceil \frac{i}{3} \right\rceil - (4k-1), & i \equiv 2 \pmod 3. \end{cases}$$

$$\tau_9(v_{1,j}) = \tau_9(u_1 v_{1,j}) = j,\ 1 \leq j \leq m.$$

For $2 \leq i \leq n$ and $1 \leq j \leq m$,

$$\tau_9(v_{i,j}) = \tau_9(u_i v_{i,j}) = \begin{cases} (6k+4)\left\lceil \frac{i}{3} \right\rceil - (3k+1) + j, & i \equiv 0 \pmod 3 \forall j \\ (6k+4)\left\lceil \frac{i-1}{3} \right\rceil - k + j, & i \equiv 1 \pmod 3 \forall j \\ (6k+4)\left\lceil \frac{i}{3} \right\rceil - (5k+2) + j, & i \equiv 2 \pmod 3 \forall j. \end{cases}$$

Case 4: Let $n \geq 2$ and $m \equiv 2 \pmod 3$. Take $m = 3k+2$, for some integer $k \geq 0$. Define $\tau_9 : V \cup E \to \{1, 2, \ldots, \left\lceil \frac{2nm+2n-1}{3} \right\rceil\}$ as follows:

$\tau_9(u_1) = 1$ and $\tau_9(u_i) = (2k+2)i - 1, 2 \leq i \leq n$.

$\tau_9(u_1 u_2) = (2k+3)$ and $\tau_9(u_i u_{i+1}) = (2k+2)i - (2k-1), 2 \leq i \leq n-1$.

$\tau_9(v_{1,j}) = \tau_9(u_1v_{1,j}) = j, 1 \leq j \leq m.$

For $2 \leq i \leq n$ and $1 \leq j \leq m$,

$\tau_9(v_{i,j}) = \tau_9(u_iv_{i,j}) = (2k+2)i-(3k+2)+j.$

Then the edge weight function $\sigma : E[CP_n(m)] \to \{3,5,\ldots,2nm+2n+1\}$ is as follows.

$$\sigma(u_iu_{i+1}) = 2(m+1)i+1, 1 \leq i \leq n-1$$
$$\sigma(u_iv_{i,j}) = 2(m+1)i-2m+2j-1, 1 \leq i \leq n,\ 1 \leq j \leq m.$$

The weights of the edges of $CP_n(m)$ forms an arithmetic progression and hence $(3,2)-tesCP_n(m) \leq \lceil \frac{2nm+2n-1}{3} \rceil$. Lemma 2.1 shows that $(3,2)-tesCP_n(m) \geq \lceil \frac{2nm+2n-1}{3} \rceil$, this concludes the proof. □

Theorem 2.15. $(3,2)$-*tes* $[CP_n(m_1,m_2,\ldots m_n)] = \left\lceil \frac{2(m_1+m_2+\ldots+m_n)+2n-1}{3} \right\rceil$, $n \geq 2, m_i \neq 0, 1 \leq i \leq n$.

Proof. The Caterpillar graph $CP_n(m_1,m_2,\ldots m_n)$ is a tree in which m_i are the leaves on the i^{th} vertex of P_n , $1 \leq i \leq n$. Let $V = \{u_i, v_{i,j}/1 \leq i \leq n, 1 \leq j \leq m_n\}$ be the vertex set and $E = \{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_{i,j}/1 \leq i \leq n, 1 \leq j \leq m_n\}$ be the edge set of the caterpillar $CP_n(m_1,m_2,\ldots m_n)$ respectively.

Define total labeling $\tau_{10} : V \cup E \to \{1,2,\ldots,\left\lceil \frac{2(m_1+m_2+\ldots+m_n)+2n-1}{3} \right\rceil\}$ is as follows:

$\tau_{10}(u_1) = 1$

$\tau_{10}(u_i) = \left\lceil \frac{2(m_1+m_2+\ldots+m_i)+2i-1}{3} \right\rceil, 2 \leq i \leq n.$

$\tau_{10}(v_{i,j}) = \left\lceil \frac{2(m_1+m_2+\ldots+m_i)+2i-1}{3} \right\rceil - m_i + j, 1 \leq i \leq n, 1 \leq j \leq m_i$ & $m_i \neq 0.$

$\tau_{10}(u_1u_2) = 2m_1+2-\left\lceil \frac{2(m_1+m_2)+3}{3} \right\rceil.$

For $2 \leq i \leq n-1$,

$\tau_{10}(u_iu_{i+1}) = 2(m_1+m_2+\ldots+m_i)+2i+1-\left\lceil \frac{2(m_1+m_2+\ldots+m_i)+2i-1}{3} \right\rceil - \left\lceil \frac{2(m_1+m_2+\ldots+m_{i+1})+2i+1}{3} \right\rceil.$

For $1 \leq i \leq n$, $1 \leq j \leq m_i$ and $m_i \neq 0$,

$$\tau_{10}(u_iv_{i,j}) = \begin{cases} \left\lceil \frac{2(m_1+m_2+\ldots+m_i)+2i-1}{3} \right\rceil - m_i + j, & \text{if } 2(m_1+\ldots+m_i)+2i-1 \equiv 0 \pmod 3, \\ \left\lceil \frac{2(m_1+m_2+\ldots+m_i)+2i-1}{3} \right\rceil - m_i + j - 1, & \text{if } 2(m_1+\ldots+m_i)+2i-1 \equiv 1,2 \pmod 3. \end{cases}$$

Then the edge weight function $\sigma : E[CP_n(m_1,m_2,\ldots m_n)] \to \{3,5,\ldots 2(m_1+m_2+\ldots+m_n)+2n-1\}$ is as follows:

$\sigma(u_iu_{i+1}) = 2(m_1+m_2+\ldots+m_i)+2i+1, 1 \leq i \leq n-1$

$\sigma(u_iv_{i,j}) = 2(m_1+m_2+\ldots+m_i)+2(i+j)-2m_i-1, 1 \leq i \leq n$, $1 \leq j \leq m_i$ and $m_i \neq 0$.

The weights of the edges of $CP_n(m_1,m_2,\ldots,m_n)$ forms an arithmetic progression and hence $(3,2)-tes[CP_n(m_1,m_2,\ldots,m_n)] \leq \left\lceil \frac{2(m_1+m_2+\ldots+m_n)+2n-1}{3} \right\rceil$. Lemma 2.1 shows that $(3,2)-tes[CP_n(m_1+m_2+\ldots+m_n)] \geq \left\lceil \frac{2(m_1+m_2+\ldots+m_n)+2n-1}{3} \right\rceil$, this concludes the proof. □

Theorem 2.16. $(3,2)$-*tes* $\{G(n,2)\} = 2n+1$*, for* $n \geq 5$.

Proof. The generalized Petersen graph on n vertices with skip 2, denoted by $G(n,2)$ is defined to be a graph with $V = \{u_i, v_i / 1 \leq i \leq n\}$ as the vertex set and $E = \{u_iv_i, v_iv_{i+1}, u_iu_{i+2} / 1 \leq i \leq n\}$ as the edge set respectively.It has 2n vertices and 3n edges.

Define total labeling $\tau_{11} : V \cup E \to \{1,2,\ldots,2n+1\}$ as follows:

Case 1: When n is odd,

$$\tau_{11}(u_1) = \tau_{11}(u_3) = 1.$$

For $1 \leq i \leq n$,

$$\tau_{11}(u_i) = \begin{cases} 3, & i \text{ is even,} \\ 5, & i \text{ is odd, } i \neq 1,3. \end{cases}$$

$\tau_{11}(v_i) = 2n+1, 1 \leq i \leq n.$

$\tau_{11}(v_iv_{i+1}) = 2i-1, 1 \leq i \leq n.$

$\tau_{11}(u_1v_1) = \tau_{11}(u_2v_2) = 1$ and $\tau_{11}(u_3v_3) = 5$.

$\tau_{11}(u_iv_i) = 4\left\lceil \frac{i-3}{2} \right\rceil + 1, 4 \leq i \leq n.$

$\tau_{11}(u_{2i-1}u_{2i+1}) = 1$ if $i = 1,2$ and $\tau_{11}(u_{n-1}u_1) = 1$.

$\tau_{11}(u_{2(\lfloor \frac{n}{2} \rfloor - 1)}u_2) = 4(\lfloor \frac{n}{2} \rfloor - 1) - 1$

$\tau_{11}(u_2u_4) = 4(\lfloor \frac{n}{2} \rfloor - 1) - 1$

$\tau_{11}(u_{2i+3}u_{2i+5}) = 4i-3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2.$

$\tau_{11}(u_{n+1-2i}u_{n-1-2i}) = 4i-1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2.$

Case 2: When n is even,

$$\tau_{11}(u_1) = \tau_{11}(u_3) = 1.$$

For $1 \leq i \leq n$,

$$\tau_{11}(u_i) = \begin{cases} 5, & i \text{ is even,} \\ 3, & i \text{ is odd, } i \neq 1,3. \end{cases}$$

$\tau_{11}(v_i) = 2n+1, 1 \leq i \leq n$.

$\tau_{11}(v_i v_{i+1}) = 2i-1, 1 \leq i \leq n$.

$\tau_{11}(u_1 v_1) = \tau_{11}(u_2 v_2) = 1$ and

$$\tau_{11}(u_3 v_3) = \begin{cases} 5, & \text{when } n = 6, \\ 3, & \text{when } n \neq 6. \end{cases}$$

$\tau_{11}(u_i v_i) = 4\left\lceil \frac{i-2}{2} \right\rceil - 1, 4 \leq i \leq n$.

$\tau_{11}(u_1 u_3) = \tau_{11}(u_{n-1} u_1) = 1$ and $\tau_{11}(u_3 u_5) = \tau_{11}(u_{n-3} u_{n-1}) = 3$.

$\tau_{11}(u_{2i+3} u_{2i+5}) = 4i+1, 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 2$.

$\tau_{11}(u_{n+1-2i} u_{n-1-2i}) = 4i-1, 2 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 1$.

$\tau_{11}(u_{2i} u_{2i+2}) = n+4i-11, 2 \leq i \leq \left\lceil \frac{n}{4} \right\rceil$.

$$\tau_{11}(u_2 u_4) = \begin{cases} 1, & n = 6, \\ n-7, & n \neq 6. \end{cases}$$

$$\tau_{11}(u_{n+2-2i} u_{n+4-2i}) = n+4i-9, 2 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor.$$

$$\tau_{11}(u_n u_2) = \begin{cases} 3, & n = 6, \\ n-5, & n \neq 6. \end{cases}$$

From the above labeling, the upper bound of $G(n,2)$ is obtained.

(ie) $(3,2)$-tes $\{G(n,2)\} \leq 2n+1$.

The lower bound of $G(n,2)$ is obtained by using the lemma 2.1

(ie) $(3,2)$-tes $\{G(n,2)\} \geq 2n+1$. Hence the proof. □

Open Problem 1. *Determine the precise value for* $(3,2)-tes(P_n^n)$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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