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PROOF OF THE GALE-NIKAIDO LEMMA FOR SEQUENTIALLY LOCALLY NON-CONSTANT MULTI-FUNCTIONS: A CONSTRUCTIVE ANALYSIS

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Abstract. In this paper we constructively prove the Gale-Nikaido lemma for sequentially locally non-constant multi-functions (multi-valued functions or correspondences), which is the basis of a proof of the existence of an equilibrium in a competitive economy, and also we will show that our Gale-Nikaido lemma leads to Sperner's lemma. We follow the Bishop style constructive mathematics.

Keywords: sequentially locally non-constant multi-functions; equilibrium in a competitive economy; Gale-Nikaido lemma; constructive mathematics.

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1. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved[1].

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[1] [5] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive (See [3] or [8]).

Thus, Kakutani's fixed point theorem for multi-functions (multi-valued functions or correspondences) and the existence of an equilibrium in a competitive economy with multi-valued excess demand functions also can not be constructively proved. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors presented an approximate version of Brouwer's theorem using Sperner's lemma (See [8] and [9]). Also Dalen in [8] states a conjecture that a function f from a simplex into itself, with property that each open set contains a point x such that x is not equal to $f(x)$ ($x \neq f(x)$) and on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently Berger and Ishihara[2] showed that the following theorem is equivalent to Brouwer's fan theorem, and so it is non-constructive.

> Each uniformly continuous function from a compact metric space into itself with at most one fixed point has a fixed point.

By reference to the notion of *sequentially at most one maximum* in Berger, Bridges and Schuster[1] we require a more general and somewhat stronger condition of *sequential local non-constancy* for functions, and in [6] we have shown the following result.

> If each uniformly continuous function from a compact metric space into itself is *sequentially locally non-constant*, then it has a fixed point,

without the fan theorem. It is a partial answer to Dalen's conjecture. In another paper [7] we have shown that we can constructively prove Kakutani's fixed point theorem for a multi-function if it has sequentially at most one fixed point and uniformly closed graph. The condition that a multi-function has sequentially at most one fixed point is stronger than the condition of sequential local non-constancy in this paper.

In this paper we constructively prove the Gale-Nikaido lemma for sequentially locally non-constant multi-functions, which is the basis of a proof of the existence of an equilibrium in a competitive economy, and also we will show that our Gale-Nikaido lemma leads to Sperner's lemma. The Gale-Nikaido lemma states the following result. Let Δ be an n-dimensional simplex and Z be a totally bounded and complete, that is, compact and convex set in $n+1$-dimensional Euclidian space. Suppose that a multi-function F from Δ to the collection of inhabited (nonempty) subsets of Z satisfies some conditions including

Weak Walras Law and has a closed graph. Then, for some $\mathbf{p}^* \in \Delta$ there exists $\mathbf{z}^* \in Z$ which satisfies

$$\mathbf{z}^* \in F(\mathbf{p}^*), \text{ and } \mathbf{z}^* \leq 0.$$

We will constructively show that the Gale-Nikaido lemma holds for sequentially locally non-constant multi-functions with uniformly closed graph. The condition of uniformly closed graph is a uniform version of the closed graph condition.

2. Kakutani's fixed point theorem for sequentially locally non-constant multi-functions

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [4]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $\mathbf{p} \in S$ there exists $\mathbf{q}$ in that ε-approximation with $|\mathbf{p}-\mathbf{q}| < \varepsilon(|\mathbf{p}-\mathbf{q}|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

Let $\mathbf{p}$ be a point in a compact metric space X, and f be a uniformly continuous function from X into itself. According to [8] and [9] f has an approximate fixed point. It means

$$\text{For each } \varepsilon > 0 \text{ there exists } \mathbf{p} \in X \text{ such that } |\mathbf{p} - f(\mathbf{p})| < \varepsilon.$$

Now consider an n-dimensional simplex Δ as a metric space. The notion that a function f has at most one fixed point in [2] is as follows;

Definition 2.1. *For all* $\mathbf{p}, \mathbf{q} \in \Delta$, *if* $\mathbf{p} \neq \mathbf{q}$, *then* $f(\mathbf{p}) \neq \mathbf{p}$ *or* $f(\mathbf{q}) \neq \mathbf{q}$.

By reference to the notion of *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy* for functions and multi-functions. About a compact set in Δ, according to Corollary 2.2.12 of [4], we have the following result.

Lemma 2.1. *For each $\varepsilon > 0$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$.*

The definition of sequential local non-constancy of functions is as follows;

Definition 2.2.(sequential local non-constancy of functions) *There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$ in each H_i, $|f(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|f(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.*

Let F be a compact and convex valued multi-function from Δ to the collection of its inhabited subsets. Since Δ and $F(\mathbf{p})$ for $\mathbf{p} \in \Delta$ are compact, $F(\mathbf{p})$ is located (see Proposition 2.2.9 in [4]), that is, $|F(\mathbf{p}) - \mathbf{q}| = \inf_{\mathbf{r}\in F(\mathbf{p})} |\mathbf{r} - \mathbf{q}|$ exists.

The definition of sequential local non-constancy of multi-functions is as follows;

Definition 2.3.(sequential local non-constancy of multi-functions) *There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$ in each H_i, $|F(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|F(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, then $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$.*

A graph of a multi-function F from Δ to the collection of its inhabited subsets is

$$G(F) = \cup_{\mathbf{p}\in\Delta}\{\mathbf{p}\} \times F(\mathbf{p}).$$

If $G(F)$ is a closed set, we say that F has a closed graph. It implies the following fact.

> For sequences $(\mathbf{p}_n)_{n\geq 1}$ and $(\mathbf{q}_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$, if $\mathbf{p}_n \longrightarrow \mathbf{p}$, then for some $\mathbf{q} \in F(\mathbf{p})$ we have $\mathbf{q}_n \longrightarrow \mathbf{q}$.

On the other hand, if the following condition is satisfied, we say that F has a uniformly closed graph.

For sequences $(\mathbf{p}_n)_{n\geq 1}$, $(\mathbf{q}_n)_{n\geq 1}$, $(\mathbf{p}'_n)_{n\geq 1}$, $(\mathbf{q}'_n)_{n\geq 1}$ such that $\mathbf{q}_n \in F(\mathbf{p}_n)$, $\mathbf{q}'_n \in F(\mathbf{p}'_n)$, if $|\mathbf{p}_n - \mathbf{p}'_n| \longrightarrow 0$, then for any $\mathbf{q}_n$ and some $\mathbf{q}'_n$, we have $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$, and for any $\mathbf{q}'_n$ and some $\mathbf{q}_n$, we have $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$.

Let $\mathbf{q} \in F(\mathbf{p})$, $(\mathbf{p}'_n)_{n\geq 1} = \{\mathbf{p}, \mathbf{p}, \dots\}$ and $(\mathbf{q}'_n)_{n\geq 1} = \{\mathbf{q}, \mathbf{q}, \dots\}$ be sequences with a constant points $\mathbf{p}$ and $\mathbf{q}$. If $|\mathbf{p}_n - \mathbf{p}'_n| = |\mathbf{p}_n - \mathbf{p}| \longrightarrow 0$, then $|\mathbf{q}_n - \mathbf{q}'_n| = |\mathbf{q}_n - \mathbf{q}| \longrightarrow 0$, that is, if $\mathbf{p}_n \longrightarrow \mathbf{p}$, then $\mathbf{q}_n \longrightarrow \mathbf{q}$, and so uniform closed graph property implies closed graph property.

In this definition

$|\mathbf{p}_n - \mathbf{p}'_n| \longrightarrow 0$ means that for any $\delta > 0$ there exists n_0 such that when $n \geq n_0$ we have $|\mathbf{p}_n - \mathbf{p}'_n| < \delta$, and $|\mathbf{q}_n - \mathbf{q}'_n| \longrightarrow 0$ means that for any $\varepsilon > 0$ there exists n'_0 such that when $n \geq n'_0$, we have $|\mathbf{q}_n - \mathbf{q}'_n| < \varepsilon$.

Now we show the following lemma.

Lemma 2.2. *Let F be a convex and compact valued multi-function with uniformly closed graph from Δ to the collection of its inhabited subsets. Assume* $\inf_{\mathbf{p}\in H_i} |F(\mathbf{p}) - \mathbf{p}| = 0$ *in some H_i such that $\Delta = \cup_{i=1}^m H_i$. If the following property holds:*

For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $\mathbf{p}, \mathbf{q} \in H_i$, $|F(\mathbf{p}) - \mathbf{p}| < \eta$ and $|F(\mathbf{q}) - \mathbf{q}| < \eta$, then $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$.

Then, there exists a point $\mathbf{r} \in H_i$ such that $\mathbf{r} \in F(\mathbf{r})$.

Proof. See Appendix A

A fixed point of a multi-function is defined as follows;

Definition 2.4. $\mathbf{p}$ *is a fixed point of a multi-function F if $\mathbf{p} \in F(\mathbf{p})$.*

We define an approximate fixed point of a multi-function F as follows;

Definition 2.5. *For each $\varepsilon > 0$ $\mathbf{p}$ is an approximate fixed point of a multi-function F if $|\mathbf{p} - F(\mathbf{p})| < \varepsilon$.*

We will constructively show that if the value of a sequentially locally non-constant multi-function F from Δ to the set of inhabited subsets of Δ with uniformly closed graph is compact and convex, it has a fixed point. If a set X is homeomorphic to Δ (so X is also compact), we can show the same result for a multi-function from X to the collection of inhabited subsets of X.

Our Kakutani's fixed point theorem is expressed as follows;

Theorem 2.1. *If F is a compact and convex valued sequentially locally non-constant multi-function with uniformly closed graph from an n-dimensional simplex Δ to the collection of its inhabited subsets, then it has a fixed point.*

Proof. See Appendix B.

3. The Gale-Nikaido lemma

The contents of the classical Gale-Nikaido lemma are as follows.

Gale-Nikaido lemma. Let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ and

$$\Delta = \{\mathbf{p}|p_i \geq 0, \sum_{i=0}^{n} p_i = 1\},$$

and let Z be a compact and convex set in $n+1$-dimensional Euclidian space. Assume that a multi-function $F(\mathbf{p})$ from Δ to the set of inhabited subsets of Z satisfies the following conditions.

[1]. $F(\mathbf{p})$ is a compact and convex set of Z for each $\mathbf{p}$.

[2]. $F(\mathbf{p})$ has a closed graph.

[3]. (Weak Walras Law) For any $\mathbf{p} \in \Delta$ and $\mathbf{z} \in Z$, $\mathbf{pz} \leq 0$ holds.

Then, for some $\mathbf{p}^* \in \Delta$ there exists $\mathbf{z}^*$ which satisfies

$$\mathbf{z}^* \in F(\mathbf{p}^*),\ \mathbf{z}^* \leq 0.$$

We will constructively show that the Gale-Nikaido lemma holds if F is a sequentially locally non-constant multi-function, and it has a uniformly closed graph.

If we interpret p_i and z_i be the price and excess demand of each good, and $F(\mathbf{p})$ be a multi-valued excess demand function, then this lemma implies the existence of an equilibrium in a competitive exchange economy with sequentially locally non-constant multi-valued excess demand functions at which excess demand for each good is zero or negative (not positive).

Let $\mathbf{z} = (z_0, z_1, \ldots, z_n)$, and consider the following function.

$$\varphi(\mathbf{p}, \mathbf{z}) = (\varphi_0, \varphi_1, \ldots, \varphi_n),\ \varphi_i(\mathbf{p}, \mathbf{z}) = \frac{p_i + \max(z_i, 0)}{1 + \sum_{j=0}^{n} \max(z_j, 0)}. \tag{1}$$

Since we have $\varphi_i \geq 0$, $\sum_{i=0}^{n} \varphi_i = 1$, and φ_i is a uniformly continuous function of $(\mathbf{p}, \mathbf{z})$, $\varphi(\mathbf{p}, \mathbf{z})$ is a uniformly continuous function from $\Delta \times Z$ to Δ. Next we define the following multi-function,

$$g(\mathbf{p}, \mathbf{z}) = \varphi(\mathbf{p}, \mathbf{z}) \times F(\mathbf{p}). \tag{2}$$

$g(\mathbf{p}, \mathbf{z})$ is a multi-function from $\Delta \times Z$ to the set of inhabited subsets of $\Delta \times Z$. $\varphi(\mathbf{p}, \mathbf{z})$ is a single-valued function, and it is a special case of multi-function. Δ itself is a subset of the set of inhabited subsets of Δ, and so we can consider that the set of inhabited subsets of Δ is the range of $\varphi(\mathbf{p}, \mathbf{z})$. Thus, g is a multi-function from $\Delta \times Z$ to the set of inhabited subsets of $\Delta \times Z$. Since φ is a uniformly continuous function, and F is a multi-function with uniformly closed graph, g also has a uniformly closed graph.

> For sequences $(\mathbf{p}_n, \mathbf{z}_n)_{n \geq 1}$ and $(\mathbf{p}'_n, \mathbf{z}'_n)_{n \geq 1}$ assume $|(\mathbf{p}_n, \mathbf{z}_n) - (\mathbf{p}'_n, \mathbf{z}'_n)| \longrightarrow 0$. Since $\varphi(\mathbf{p}, \mathbf{z})$ is uniformly continuous, for any $\varepsilon > 0$ we can select $\delta > 0$ such that if $|(\mathbf{p}_n, \mathbf{z}_n) - (\mathbf{p}'_n, \mathbf{z}'_n)| < \delta$, then $|\varphi(\mathbf{p}_n, \mathbf{z}_n) - \varphi(\mathbf{p}'_n, \mathbf{z}'_n)| < \varepsilon$. Since ε is arbitrary, corresponding to sequences $(\mathbf{p}_n, \mathbf{z}_n)_{n \geq 1}$, $(\mathbf{p}'_n, \mathbf{z}'_n)_{n \geq 1}$ such that $|(\mathbf{p}_n, \mathbf{z}_n) - (\mathbf{p}'_n, \mathbf{z}'_n)| \longrightarrow 0$, we can construct sequences $(\varphi(\mathbf{p}_n, \mathbf{z}_n))_{n \geq 1}$, $(\varphi(\mathbf{p}'_n, \mathbf{z}'_n))_{n \geq 1}$ such that $|\varphi(\mathbf{p}_n, \mathbf{z}_n) - \varphi(\mathbf{p}'_n, \mathbf{z}'_n)| \longrightarrow 0$. Thus, φ has a uniformly closed graph. Since F also has a uniformly closed graph, g has a uniformly closed graph.

Let $\mathbf{p} \neq \mathbf{p}'$. Given $\mathbf{z}$ if $|\varphi_i(\mathbf{p}, \mathbf{z}) - \varphi_i(\mathbf{p}', \mathbf{z})| \longrightarrow 0$, clearly $|\mathbf{p} - \mathbf{p}'| \longrightarrow 0$, and so φ is sequentially locally non-constant. Since F is sequentially locally non-constant, g is also sequentially locally non-constant. Because $F(\mathbf{p})$ is convex, $\varphi(\mathbf{p}, \mathbf{z}) \times F(\mathbf{p})$ is also a convex set of $\Delta \times Z$. Since Z is homeomorphic to an $n + 1$-dimensional simplex, $\Delta \times Z$ is homeomorphic to a $2n + 1$-dimensional simplex. Now we can constructively show the following result.

Theorem 3.1. *The Gale-Nikaido lemma holds for sequentially locally non-constant multi-functions with uniformly closed graph.*

Proof. Because $\Delta \times Z$ is homeomorphic to a $2n+1$-dimensional simplex, by Kakutani's fixed point theorem for sequentially locally non-constant multi-functions $g(\mathbf{p}, \mathbf{z})$ has a fixed point. Denote the fixed points by $(\mathbf{p}^*, \mathbf{z}^*)$. Then, we have

$$\mathbf{p}^* = \varphi(\mathbf{p}^*, \mathbf{z}^*), \tag{3}$$

and

$$\mathbf{z}^* \in F(\mathbf{p}^*).$$

Let $\mathbf{p}^* = (p_0^*, p_1^*, \dots, p_n^*)$, $\mathbf{z}^* = (z_0^*, z_1^*, \dots, z_n^*)$. From (3) $p_i^* = \varphi_i$ for all i. Thus,

$$\frac{p_i^* + \max(z_i^*, 0)}{1 + \sum_{j=0}^n \max(z_j^*, 0)} - p_i^* = \frac{\max(z_i^*, 0) - p_i^* \sum_{j=0}^n \max(z_j^*, 0)}{1 + \sum_{j=0}^n \max(z_j^*, 0)} = 0$$

is derived. Let $\sum_{j=0}^n \max(z_j^*, 0) = \lambda$. Then, we have

$$\max(z_i^*, 0) = \lambda p_i^*.$$

By $\sum_{i=0}^n p_i^* = 1$ there exists k which satisfies $p_k^* > 0$. If for all k satisfying $p_k^* > 0$ we have $z_k^* > 0$, then Weak Walras Law is violated because p_i can not be negative, and $p_k^* z_k^* > 0$ can not be canceled out. Thus, $\lambda = 0$ and

$$\max(z_i^*, 0) = 0. \tag{4}$$

This holds for all i, and so $\mathbf{z}^* \leq 0$. We have completed the proof.

4. The Gale-Nikaido lemma for sequentially locally non-constant multi-functions leads to Sperner's lemma

In this section we will derive Sperner's lemma from the Gale-Nikaido lemma for sequentially locally non-constant multi-functions with uniformly closed graph. Let partition an n-dimensional simplex Δ. Denote the set of small n-dimensional simplices of Δ constructed by partition by K. Vertices of these small simplices of K are labeled with the numbers 0, 1, 2, $\dots$, n. Denote vertices of an n-dimensional simplex of K by $x^0, x^1, \dots, x^n$, the j-th component of x^i by x_j^i, and the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x^i_{l(x^i)}$ for all i, and define a function $f(x^i)$ as follows[2];

$$f(x^i) = (f_0(x^i), f_1(x^i), \dots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x^i_j - \tau & \text{for } j = l(x^i), \\ x^i_j + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \tag{5}$$

f_j denotes the j-th component of f. From the labeling rules $x^i_{l(x^i)} > 0$ for all x^i, and so $\tau > 0$ is well defined. Since $\sum_{j=0}^n f_j(x^i) = \sum_{j=0}^n x^i_j = 1$, we have

$$f(x^i) \in \Delta.$$

We extend f to all points in the simplex by convex combinations of its values on the vertices of the simplex. Let y be a point in the n-dimensional simplex of K whose vertices are $x^0, x^1, \dots, x^n$. Then, y and $f(y)$ are represented as follows;

$$y = \sum_{i=0}^n \lambda_i x^i, \text{ and } f(y) = \sum_{i=0}^n \lambda_i f(x^i),\ \lambda_i \geq 0,\ \sum_{i=0}^n \lambda_i = 1.$$

It is clear that f is uniformly continuous. We verify that f is sequentially locally non-constant.

[1]. Assume that a point z is contained in an $n-1$-dimensional small simplex δ^{n-1} constructed by partition of an $n-1$-dimensional face of Δ such that its i-th coordinate is $z_i = 0$. Denote the vertices of δ^{n-1} by $z^j,\ j = 0, 1, \dots, n-1$ and their i-th coordinate by z^j_i. Then, we have

$$f_i(z) = \sum_{j=0}^{n-1} \lambda_j f_i(z^j),\ \lambda_j \geq 0,\ \sum_{j=0}^{n-1} \lambda_j = 1.$$

Since all vertices of δ^{n-1} are not labeled with i, (5) means $f_i(z^j) > z^j_i$ for all $j = \{0, 1, \dots, n-1\}$. Then, there exists no sequence $(z_m)_{m\geq 1}$ such that $|f(z_m) - z_m| \longrightarrow 0$ in an $n-1$-dimensional face of Δ.

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[2]We refer to [10] about the definition of this function.

[2]. Let z be a point in an n-dimensional simplex δ^n. Assume that no vertex of δ^n is labeled with i. Then

$$f_i(z) = \sum_{j=0}^{n} \lambda_j f_i(x^j) = z_i + \left(1 + \frac{1}{n}\right)\tau, \tag{6}$$

and so $z \neq f(z)$. Then, there exists no sequence $(z_m)_{m\geq 1}$ such that $|f(z_m) - z_m| \longrightarrow 0$ in δ^n.

[3]. Assume that z is contained in a fully labeled n-dimensional simplex δ^n, and rename vertices of δ^n so that a vertex x^i is labeled with i for each i. Then,

$$f_i(z) = \sum_{j=0}^{n} \lambda_j f_i(x^j) = \sum_{j=0}^{n} \lambda_j x_i^j + \sum_{j\neq i} \lambda_j \frac{\tau}{n} - \lambda_i \tau = z_i + \left(\frac{1}{n}\sum_{j\neq i}\lambda_j - \lambda_i\right)\tau \text{ for each } i.$$

Consider sequences $(z_m)_{m\geq 1} = (z_1, z_2, \dots)$, $(z'_m)_{m\geq 1} = (z'_1, z'_2, \dots)$ such that $|f(z_m) - z_m| \longrightarrow 0$ and $|f(z'_m) - z'_m| \longrightarrow 0$.

Let $z_m = \sum_{i=0}^{n} \lambda(m)_i x^i$ and $z'_m = \sum_{i=0}^{n} \lambda'(m)_i x^i$. Then, we have

$$\frac{1}{n}\sum_{j\neq i}\lambda(m)_j - \lambda(m)_i \longrightarrow 0, \text{ and } \frac{1}{n}\sum_{j\neq i}\lambda'(m)j - \lambda'(m)_i \longrightarrow 0 \text{ for all } i.$$

Therefore, we obtain

$$\lambda(m)_i \longrightarrow \frac{1}{n+1}, \text{and } \lambda'(m)_i \longrightarrow \frac{1}{n+1} \text{ for all } i.$$

These mean

$$|z_m - z'_m| \longrightarrow 0.$$

Thus, f is sequentially locally non-constant

Now, using f, we construct a function $F(x) = \mathbf{z} = \{z_0, z_1, \dots, z_n\}$ such that

$$z_i = f_i(x) - x_i\mu(x),\ i = 0, 1, \dots, n. \tag{7}$$

$x \in \Delta^n$ and $\mu(x)$ is defined by

$$\mu(x) = \frac{\sum_{i=0}^{n} x_i f_i(x)}{\sum_{i=0}^{n} x_i^2}.$$

Each $z_i(x)$ is uniformly continuous, and satisfies the Weak Walras law as shown below. Multiplying x_i to (7) for each i, and adding them from 0 to n yields

$$\begin{aligned}(8)\qquad \sum_{i=0}^n x_i z_i &= \sum_{i=0}^n x_i f_i(x) - \mu(x)\sum_{i=0}^n x_i^2 = \sum_{i=0}^n x_i f_i(x) - \frac{\sum_{i=0}^n x_i f_i(x)}{\sum_{i=0}^n x_i^2}\sum_{i=0}^n x_i^2 \\ &= \sum_{i=0}^n x_i f_i(x) - \sum_{i=0}^n x_i f_i(x) = 0.\end{aligned}$$

Now define the following function.

$$g(x, \mathbf{z}) = \varphi(x, \mathbf{z}) \times F(x),$$

where

$$\varphi(x, \mathbf{z}) = (\varphi_0, \varphi_1, \dots, \varphi_n),\ \varphi_i(x, \mathbf{z}) = \frac{x_i + \max(z_i, 0)}{1 + \sum_{j=0}^n \max(z_j, 0)}.$$

g is a uniformly continuous function of $(x, \mathbf{z})$, and it is a special case of a compact and convex valued multi-function with uniformly closed graph. F is constructed by f and x, so it is sequentially locally non-constant by the sequential local non-constancy of f. And similarly to the sequential local non-constancy of φ in (1), φ is also sequentially locally non-constant. Therefore, g is sequentially locally non-constant, and it satisfies the conditions for the approximate version of the Gale-Nikaido lemma. Then, there exist x^* and $\mathbf{z}^*$ such that

$$F(x^*) = \mathbf{z}^*,\ \text{and } \mathbf{z}^* \leq 0.$$

From $\max(z_i, 0) = 0$ (see (4)) we have $f_i(x^*) - x_i^*\mu(x^*) \leq 0$ for all i. Since it is impossible that $z_i < 0$ for i satisfying $x_i^* > 0$ because of (8), we have $z_i = f_i(x^*) - x_i^*\mu(x^*) \geq 0$ for such i. Also for i such that $x_i^* = 0$, we have $z_i = f_i(x^*) - x_i^*\mu(x^*) \geq 0$. Therefore,

$$(9)\qquad f_i(x^*) - x_i^*\mu(x^*) = 0$$

is obtained. Adding this equality side by side from 0 to n yields

$$\sum_{i=0}^n f_i(x^*) - \mu(x^*)\sum_{i=0}^n x_i^* = 0.$$

From $\sum_{i=0}^n f_i(x^*) = \sum_{i=0}^n x_i^* = 1$ we obtain

$$(10)\qquad \mu(x^*) = 1.$$

Further from (9) and (10)

$$f_i(x^*) = x_i^*$$

is derived. This relation holds for all i.

Let $\gamma > 0$ and $\tilde{x}$ be a point in $V(x^*, \gamma)$, where $V(x^*, \gamma)$ is a γ-neighborhood of x^*. If γ is sufficiently small, uniform continuity of f means

$$|f_i(\tilde{x}) - \tilde{x}_i| < \varepsilon \tag{11}$$

for any $\varepsilon > 0$ and for all i. $\tilde{x}_i$ is the i-th component of $\tilde{x}$. Let Δ^{n^*} be a simplex of K which contains $\tilde{x}$, and $x^0, x^1, \dots, x^n$ be the vertices of Δ^{n^*}. Then, $\tilde{x}$ and $F(\tilde{x})$ are represented as

$$\tilde{x} = \sum_{i=0}^{n} \lambda_i x^i \text{ and } f(\tilde{x}) = \sum_{i=0}^{n} \lambda_i f(x^i),\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1.$$

(5) implies that if only one x^k among $x^0, x^1, \dots, x^n$ is labeled with i, we have

$$|f_i(\tilde{x}) - \tilde{x}_i| = \left| \sum_{j=0}^{n} \lambda_j x_i^j + \sum_{j=0, j\neq k}^{n} \lambda_j \frac{\tau}{n} - \lambda_k \tau - \tilde{x}_i \right| = \left| \left(\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \right) \tau \right| < \varepsilon.$$

x_i^j is the i-th component of x^j. This means

$$\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \approx 0.$$

It is satisfied with $\lambda_k \approx \frac{1}{n+1}$ for all k. On the other hand, if no x^j is labeled with i, we have

$$f_i(\tilde{x}) = \sum_{j=0}^{n} \lambda_j x_i^j = x_i^* + (1 + \frac{1}{n})\tau,$$

and then (11) can not be satisfied. Thus, for each i one and only one x^j must be labeled with i. Therefore, Δ^{n^*} must be a fully labeled simplex. We have completed the proof of Sperner's lemma.

Appendices

A. Proof of Lemma 2.2

Choose a sequence $(\mathbf{p}_n)_{n\geq 1}$ in Δ such that $|F(\mathbf{p}_n)-\mathbf{p}_n|\longrightarrow 0$. Compute N such that $|F(\mathbf{p}_n)-\mathbf{p}_n|<\eta$ for all $n\geq N$. Then, for $m,n\geq N$ we have $|\mathbf{p}_m-\mathbf{p}_n|\leq\varepsilon$. Since $\varepsilon>0$ is arbitrary, $(\mathbf{p}_n)_{n\geq 1}$ is a Cauchy sequence in Δ, and converges to a limit $\mathbf{r}\in\Delta$. The uniformly closed graph property of F yields $\mathbf{r}\in F(\mathbf{r})$.

B. Proof of Theorem 2.1

[1]. Let Δ be an n-dimensional simplex, and consider m-th subdivision of Δ. Subdivision in a case of 2-dimensional simplex is illustrated in Figure 1. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider subdivision of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the way above mentioned, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let us partition Δ sufficiently fine, and define a uniformly continuous function $f^m:\ \Delta\longrightarrow\Delta$ as follows. If $\mathbf{p}$ is a vertex of a simplex constructed by m-th subdivision of Δ, $f^m(\mathbf{p})=\mathbf{q}$ for some $\mathbf{q}\in F(\mathbf{p})$. For other $\mathbf{p}\in\Delta$ we define f^m by a convex combination of the values of F at vertices of a simplex $\mathbf{p}_0^m$, $\mathbf{p}_1^m$, …, $\mathbf{p}_n^m$. Let $\sum_{i=0}^{n}\lambda_i=1$, $\lambda_i\geq 0$,

$$f^m(\mathbf{p})=\sum_{i=0}^{n}\lambda_i f^m(\mathbf{p}_i^m)\text{ with }\mathbf{p}=\sum_{i=0}^{n}\lambda_i\mathbf{p}_i^m.$$

Since f^m is clearly uniformly continuous, it has an approximate fixed point according to [8] and [9]. Let $\mathbf{p}^*$ be an approximate fixed point of f^m, then for each $\frac{\varepsilon}{2}>0$ there exists $\mathbf{p}^*\in\Delta$ which satisfies

$$|\mathbf{p}^*-f^m(\mathbf{p}^*)|<\frac{\varepsilon}{2}.$$

About sequences of the distance between vertices of simplices constructed by partition $(|\mathbf{p}_i^m-\mathbf{p}_j^m|)_{m\geq 1},\ i\neq j)$ assume $|\mathbf{p}_i^m-\mathbf{p}_j^m|\longrightarrow 0$. Since F has a uniformly

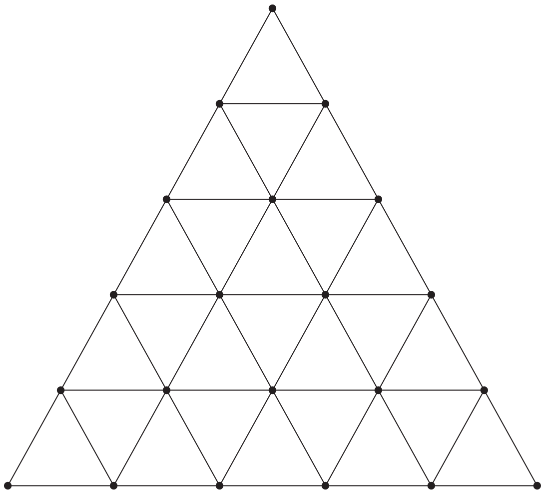

FIGURE 1. Subdivision of 2-dimensional simplex

closed graph, for any $\mathbf{q}_i^m \in F(\mathbf{p}_i^m)$ and some $\mathbf{q}_j^m \in F(\mathbf{p}_j^m)$ we have $|\mathbf{q}_i^m - \mathbf{q}_j^m| \longrightarrow 0$, and for any $\mathbf{q}_j^m \in F(\mathbf{p}_j^m)$ and some $\mathbf{q}_i^m \in F(\mathbf{p}_i^m)$ we have $|\mathbf{q}_i^m - \mathbf{q}_j^m| \longrightarrow 0$. Then, for any $\varepsilon > 0$ there exists M such that for any $m \geq M$ $|\mathbf{q}_i^m - \mathbf{q}_j^m| < \varepsilon$. Since $\mathbf{p}^* = \sum_{i=0}^n \lambda_i \mathbf{p}_i^m$, for any $\mathbf{q}_i \in F(\mathbf{p}_i^m)$ and some $\mathbf{q}_i^* \in F(\mathbf{p}^*)$, $|\mathbf{q}_i - \mathbf{q}_i^*| < \frac{\varepsilon}{2}$. $\mathbf{q}_i^*$ may be different from $\mathbf{q}_j^*$ with $i \neq j$. But, by convexity of $F(\mathbf{p}^*)$

$$\mathbf{q}^* = \sum_{i=0}^n \lambda_i \mathbf{q}_i^* \in F(\mathbf{p}^*)$$

holds. Since $|\mathbf{q}_i - \mathbf{q}_i^*| < \frac{\varepsilon}{2}$ for each i, and

$$f^m(\mathbf{p}^*) = \sum_{i=0}^n \lambda_i f^m(\mathbf{p}_i^m) = \sum_{i=0}^n \lambda_i \mathbf{q}_i,$$

we have $|f^m(\mathbf{p}^*) - \mathbf{q}^*| < \frac{\varepsilon}{2}$. From $|\mathbf{p}^* - f^m(\mathbf{p}^*)| < \frac{\varepsilon}{2}$, $|\mathbf{p}^* - \mathbf{q}^*| < \varepsilon$ is obtained. Since $\mathbf{q}^* \in F(\mathbf{p}^*)$, $\mathbf{p}^*$ is an approximate fixed point of F. ε is arbitrary, and so

$$\inf_{\mathbf{p}^* \in \Delta} |\mathbf{p}^* - F(\mathbf{p}^*)| = 0.$$

This means

$$\inf_{\mathbf{p}^* \in H_i} |\mathbf{p}^* - F(\mathbf{p}^*)| = 0$$

in some H_i.

[2]. Choose a sequence $(\mathbf{r}_m)_{m\geq 1}$ in Δ such that $|\mathbf{r}_m - F(\mathbf{r}_m)| \longrightarrow 0$. In view of Lemma 2.2 it is enough to prove that the following condition holds.

For each $\varepsilon > 0$ there exists $\eta > 0$ such that if $\mathbf{p}, \mathbf{q} \in \Delta$, $|F(\mathbf{p}) - \mathbf{p}| < \eta$ and $|F(\mathbf{q}) - \mathbf{q}| < \eta$, then $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$.

Assume that the set

$$K = \{(\mathbf{p}, \mathbf{q}) \in \Delta \times \Delta : \ |\mathbf{p} - \mathbf{q}| \geq \varepsilon\}$$

is nonempty and compact. Since the mapping $(\mathbf{p}, \mathbf{q}) \longrightarrow \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \inf_{(\mathbf{p},\mathbf{q})\in K} \max(|F(\mathbf{p}) - \mathbf{p}|, |F(\mathbf{q}) - \mathbf{q}|) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $|F(\mathbf{p}) - \mathbf{p}| < 2^{-n-1}$, $|F(\mathbf{q}) - \mathbf{q}| < 2^{-n-1}$, we have $(\mathbf{p}, \mathbf{q}) \notin K$ and $|\mathbf{p} - \mathbf{q}| \leq \varepsilon$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(\mathbf{p}_n, \mathbf{q}_n) \in K$ such that $\max(|F(\mathbf{p}_n) - \mathbf{p}_n|, |F(\mathbf{q}_n) - \mathbf{q}_n|) < 2^{-n}$, and if $\lambda_n = 1$, set $\mathbf{p}_n = \mathbf{q}_n = \mathbf{r}_n$. Then, $|F(\mathbf{p}_n) - \mathbf{p}_n| \longrightarrow 0$ and $|F(\mathbf{q}_n) - \mathbf{q}_n| \longrightarrow 0$, so $|\mathbf{p}_n - \mathbf{q}_n| \longrightarrow 0$. Computing N such that $|\mathbf{p}_N - \mathbf{q}_N| < \varepsilon$, we must have $\lambda_N = 1$. We have completed the proof.

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