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ON CERTAIN SUBCLASSES OF UNIFORMLY SPIRALLIKE FUNCTIONS ASSOCIATED WITH STRUVE FUNCTIONS

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Abstract. The main object of this paper is to find necessary and sufficient conditions for generalized Struve functions of first kind to be in the classes $\mathcal{SP}_p(\alpha,\beta)$ and $\mathcal{UCSP}(\alpha,\beta)$ of uniformly spirallike functions and also give necessary and sufficient conditions for $z(2-u_p(z))$ to be in the above classes. Furthermore, we give conditions for the integral operator $\mathcal{L}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathcal{UCSPT}(\alpha,\beta)$. Several corollaries and consequences of the main results are also considered.

Keywords: analytic functions; Hadamard product; spirallike; uniformly convex; Bessel functions; Struve functions.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathscr{T}$ be a subclass of $\mathscr{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad z \in \mathbb{U}. \tag{2}$$

A function $f \in \mathscr{A}$ is spirallike if

$$\Re\left(e^{-i\alpha}\frac{zf'(z)}{f(z)}\right) > 0,$$

for some α with $|\alpha| < \pi/2$ and for all $z \in \mathbb{U}$. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike.

In [36], Selvaraj and Geetha introduced the following subclasses of unifromly spirallike and convex spirallike functions.

Definition 1.1. *A function f of the form (1) is said to be in the class $\mathscr{S}\mathscr{P}_p(\alpha,\beta)$ if it satisfies the following condition:*

$$\Re\left\{e^{-i\alpha}\left(\frac{zf'(z)}{f(z)}\right)\right\} > \left|\frac{zf'(z)}{f'(z)} - 1\right| + \beta \qquad (z \in \mathbb{U}; |\alpha| < \pi/2\ ; 0 \leq \beta < 1)$$

and $f \in \mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha,\beta)$ if and only if $zf'(z) \in \mathscr{S}\mathscr{P}_p(\alpha,\beta)$.

Further,

$$\mathscr{S}\mathscr{P}_p\mathscr{T}(\alpha,\beta) = \mathscr{S}\mathscr{P}_p(\alpha,\beta) \cap \mathscr{T}$$

and

$$\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta) = \mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha,\beta) \cap \mathscr{T}.$$

In particular, we note that $\mathscr{S}\mathscr{P}_p(\alpha,0) = \mathscr{S}\mathscr{P}_p(\alpha)$ and $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha,0) = \mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha)$, the classes of uniformly spirallike and uniformly convex spirallike were introduced by Ravichandran et al. [33]. For $\alpha = 0$, the classes $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha)$ and $\mathscr{S}\mathscr{P}_p(\alpha)$, respectively reduces to the classes $\mathscr{U}\mathscr{C}\mathscr{V}$ and $\mathscr{S}\mathscr{P}$ introduced and studied by Rønning [35].

For more interesting developments of some related subclasses of uniformly spirallike and uniformly convex spirallike, the readers may be referred to the works of Bharati et al.[5], Frasin [12, 13], Frasin and Aldawish [17], Goodman [20, 21], Tariq Al-Hawary and Frasin [22], Kanas and Wisniowska [24] and Rønning [34, 35].

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^{\tau}(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathscr{R}^{\tau}(A,B)$ was introduced earlier by Dixit and Pal [9]. If we put

$$\tau = 1,\ A = \beta \text{ and } B = -\beta\ (0 < \beta \leq 1),$$

we obtain the class of functions $f \in \mathscr{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \quad (z \in \mathbb{U}; 0 < \beta \leq 1)$$

which was studied by (among others) Padmanabhan [32] and Caplinger and Causey [6].

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture.The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [7, 19, 25, 27, 37, 39] and the Bessel functions [1, 2, 3, 4, 26].

We recall here the Struve function of order p (see [31, 40]), denoted by $\mathscr{H}_p$ is given by

$$\mathscr{H}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\,\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C} \tag{3}$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2\omega''(z) + z\omega'^2 - p^2)\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{4}$$

where p is unrestricted real(or complex) number. The solution of the non-homogeneous differential equation

$$z^2\omega''(z) + z\omega'^2 + p^2)\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{5}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathscr{L}_p(z) = -e^{-ip\pi/2}\mathscr{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{3}{2})\,\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}. \tag{6}$$

Let the second order inhomogeneous linear differential equation [40],

$$z^2\omega''(z)+bz\omega'^2-p^2+(1-b)p]\omega(z)=\frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})} \tag{7}$$

where $b,p,c\in\mathbb{C}$ which is natural generalization of Struve equation. It is of interest to note that when $b=c=1$, then we get the Struve function (4) and for $c=-1,b=1$ the modified Struve function (5). This permit us to study Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{b+2}{2})}\left(\frac{z}{2}\right)^{2n+p+1},\forall z\in\mathbb{C} \tag{8}$$

which is the particular solution of the differential equation (7) Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. Now,consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z)=2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)\ z^{\frac{-p-1}{2}}\ \omega_{p,b,c}\ (\sqrt{z}),\ \ \sqrt{1}=1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & (n=0),\\ \\ a(a+1)(a+2)\cdots(a+n-1) & (n\in\mathbb{N}=\{1,2,3,\ldots\})\end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned}u_{p,b,c}(z)\ &=\ \sum_{n=0}^{\infty}\frac{(-c/4)^n}{(3/2)_n\ (m)_n}\ z^n\\ &=\ b_0+b_1z+b_2z^2+...+b_nz^n+...,\end{aligned} \tag{9}$$

where $m=\left(p+\frac{b+2}{2}\right)\neq 0,-1,-2,\ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2u''(z)+2(2p+b+3)zu'(z)+(cz+2p+b)u(z)=2p+b.$$

For convenience throughout in the sequel, we use the following notations

$$w_{p,b,c}(z)=w_p(z)\qquad u_{p,b,c}(z)=u_p(z),\qquad m=p+\frac{b+2}{2}$$

and for if $c<0, m>0$ let,

$$zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n=z+\sum_{n=2}^{\infty}b_{n-1}z^n \tag{10}$$

and

$$\Psi(z)=z(2-u_p(z))=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n. \tag{11}$$

Now, we consider the linear operator

$$\mathscr{I}(c,m):\mathscr{A}\to\mathscr{A}$$

defined by

$$\mathscr{I}(c,m)f(z)=zu_{p,b,c}(z)*f(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}a_n\, z^n.$$

Orhan and Yagmur [40] have determined various sufficient conditions for the parameters p,b and c such that the functions $u_{p,b,c}(z)$ or $z\to zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [7, 25, 27, 37, 39]), Struve functions (see [8, 23]), Poisson distribution series (see [10, 14, 16, 28, 30]) and Pascal distribution series (see [11, 15, 18]),, we obtain sufficient condition for function $h_\mu(z)$, given by

$$h_\mu(z)=(1-\mu)zu_p(z)+\mu z{u_p}'(z)=z+\sum_{n=2}^{\infty}(1+n\mu-\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n, \tag{12}$$

where $0\le\mu\le1$, to be in the classes $\mathscr{S}\mathscr{P}_p(\alpha,\beta)$and $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha,\beta)$ and also proved that those sufficient conditions are necessary for functions of the form (11). Furthermore, we give necessary and sufficient conditions for $\mathscr{I}(c,m)f$ to be in $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta)$ provided that the function f is in the class $\mathscr{R}^\tau(A,B)$. Finally, we give conditions for the integral operator $\mathscr{L}(m,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta)$.

To establish our main results, we need the following lemmas.

Lemma 1.2. *(see* [36]*) (i) A sufficient condition for a function f of the form (1) to be in the class $\mathscr{S}\mathscr{P}_p(\alpha,\beta)$ is that*

$$\sum_{n=2}^{\infty}(2n-\cos\alpha-\beta)\,|a_n|\le\cos\alpha-\beta\qquad(|\alpha|<\pi/2\,;0\le\beta<1) \tag{13}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{SP}_p\mathcal{T}(\alpha,\beta)$ is that the condition (13) is satisfied. In particular, when $\beta = 0$, we obtain a sufficient condition for a function f of the form (1) to be in the class $\mathcal{SP}_p(\alpha)$ is that

$$\sum_{n=2}^{\infty}(2n-\cos\alpha)\,|a_n| \leq \cos\alpha \qquad (|\alpha| < \pi/2) \tag{14}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{SP}_p\mathcal{T}(\alpha)$ is that the condition (14) is satisfied.

(ii) A sufficient condition for a function f of the form (1) to be in the class $\mathcal{UCSP}(\alpha,\beta)$ is that

$$\sum_{n=2}^{\infty}n(2n-\cos\alpha-\beta)\,|a_n| \leq \cos\alpha-\beta \qquad (|\alpha| < \pi/2\,;0\leq\beta<1) \tag{15}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{UCSPT}(\alpha,\beta)$ is that the condition (15) is satisfied. In particular, when $\beta = 0$, we obtain a sufficient condition for a function f of the form (1) to be in the class $\mathcal{UCSP}(\alpha)$ is that

$$\sum_{n=2}^{\infty}n(2n-\cos\alpha)\,|a_n| \leq \cos\alpha \qquad (|\alpha| < \pi/2) \tag{16}$$

and a necessary and sufficient condition for a function f of the form (2) to be in the class $\mathcal{UCSPT}(\alpha)$ is that the condition (16) is satisfied.

Lemma 1.3. [9] *If $f \in \mathcal{R}^\tau(A,B)$ is of form (1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n\in\mathbb{N}\setminus\{1\}. \tag{17}$$

The result is sharp for the function

$$f(z) = \int_0^z (1+(A-B)\frac{\tau t^{n-1}}{1+Bt^{n-1}})dt, \qquad (z\in\mathbb{U}; n\in\mathbb{N}-\{1\}).$$

2. Main Results

Theorem 2.1. *Let $c<0$ and $m>0$. Then $h_\mu(z) \in \mathcal{SP}_p(\alpha,\beta)$ if*

$$2\mu u_p''(1)+[2+\mu(4-\cos\alpha-\beta)]u_p'(1)+[2-\cos\alpha-\beta]u_p(1) \leq 2(1-\beta). \tag{18}$$

Proof. Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}} z^n. \tag{19}$$

then

$$u_p(1) - 1 = \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{20}$$

Differentiating $zu_p(z)$ with respect to z and taking $z = 1$ we have

$$u'_p(1) + u_p(1) - 1 = \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{21}$$

Also, differentiating $zu'_p(z) + u_p(z)$ with respect to z and taking $z = 1$, we have

$$u''_p(1) + 2u'_p(1) = \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}(3/2)_{n-1}}. \tag{22}$$

Since $h_\mu(z) \in \mathcal{S}\mathcal{P}_p(\alpha, \beta)$, by virtue of (13), it suffices to show that

$$L(\alpha, \beta, \mu) = \sum_{n=2}^{\infty} (1 + n\mu - \mu)(2n - \cos\alpha - \beta) \; \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}} \leq \cos\alpha - \beta. \tag{23}$$

Now,

$$\begin{aligned} L(\alpha, \beta, \mu) \;\; &= \;\; 2\mu \sum_{n=2}^{\infty} n^2 \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}} \\ &\quad + [(2 - \mu(\cos\alpha + \beta + 2)] \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}} \\ &\quad - (1 - \mu)(\cos\alpha + \beta) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}}. \end{aligned}$$

Writing $n^2 = n(n-1) + n$, we get

$$\begin{aligned} L(\alpha, \beta, \mu) \;\; &= \;\; 2\mu \sum_{n=2}^{\infty} n(n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}} \\ &\quad + [(2 - \mu(\cos\alpha + \beta)] \sum_{n=2}^{\infty} n \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}} \\ &\quad - (1 - \mu)(\cos\alpha + \beta) \sum_{n=2}^{\infty} \; \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (3/2)_{n-1}}. \end{aligned}$$

From (20), (21) and (22), we immediately have

$$\begin{aligned} L(\alpha,\beta,\mu) &= 2\mu[u_p''(1)+2u_p'(1)]+[2-\mu(\cos\alpha+\beta)][u_p'(1)+u_p(1)-1] \\ &\quad -(1-\mu)(\cos\alpha+\beta)[u_p(1)-1] \\ &= 2\mu u_p''(1)+[2+\mu(4-\cos\alpha-\beta)]u_p'(1)+[2-\cos\alpha-\beta][u_p(1)-1]. \end{aligned}$$

But the last expression is bounded above by $\cos\alpha-\beta$ if and only if (18) holds. □

Theorem 2.2. *Let $c<0$ and $m>0$. Then $zu_p(z)\in\mathscr{SP}_p(\alpha,\beta)$ if*

$$2u_p'(1)+(2-\cos\alpha-\beta)u_p(1)\le 2(1-\beta). \tag{24}$$

Moreover (24) is necessary and sufficient for $z(2-u_p(z))$ to be in $\mathscr{SP}_p\mathscr{T}(\alpha,\beta)$.

Proof. By virtue of (13), it suffices to show that

$$\sum_{n=2}^{\infty}[2n-\cos\alpha-\beta]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(3/2)_{n-1}}\right)\le\cos\alpha-\beta. \tag{25}$$

Since $h_0(z)=zu_p(z)$, hence by taking $\mu=0$ in (23) we get the inequality(25). Hence by taking $\mu=0$ in the Theorem 2.1, we get the desired result given in (24). □

Theorem 2.3. *Let $c<0$ and $m>0$.Then $zu_p(z)\in\mathscr{UCSP}(\alpha,\beta)$ if*

$$2u_p''(1)+(6-\cos\alpha-\beta)u_p'(1)+(2-\cos\alpha-\beta)u_p(1)\le 2(1-\beta). \tag{26}$$

Moreover (26) is necessary and sufficient for $z(2-u_p(z))$ to be in $\mathscr{UCSPT}(\alpha,\beta)$.

Proof. By virtue of (15), it suffices to show that

$$\sum_{n=2}^{\infty}n[2n-\cos\alpha-\beta]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(3/2)_{n-1}}\right)\le\cos\alpha-\beta.$$

By definition $zu_p(z)\in\mathscr{UCSP}(\alpha,\beta)\Leftrightarrow zu_p'(z)\in\mathscr{SP}_p(\alpha,\beta)$. That is by taking $\mu=1$ we have $h_1(z)=zu_p'(z)\in\mathscr{SP}_p(\alpha,\beta)$, hence by taking $\mu=1$ in the Theorem 2.1,we get the desired result given in (26). □

Putting $\beta=0$ in Theorems 2.1-2.3, we obtain the following corollaries.

Corollary 2.4. *Let $c<0$ and $m>0$. Then $h_\mu(z)\in\mathscr{SP}_p(\alpha)$ if*

$$2\mu u_p''(1)+[2+\mu(4-\cos\alpha)]u_p'(1)+[2-\cos\alpha]u_p(1)\leq 2. \tag{27}$$

Corollary 2.5. *Let $c<0$ and $m>0$. Then $zu_p(z)\in\mathscr{S}\mathscr{P}_p(\alpha)$ if*

$$2u_p'(1)+(2-\cos\alpha)u_p(1)\leq 2. \tag{28}$$

Moreover (28) is necessary and sufficient for $z(2-u_p(z))$ to be in $\mathscr{S}\mathscr{P}_p\mathscr{T}(\alpha)$.

Corollary 2.6. *Let $c<0$ and $m>0$.Then $zu_p(z)\in\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}(\alpha)$ if*

$$2u_p''(1)+(6-\cos\alpha)u_p'(1)+(2-\cos\alpha)u_p(1)\leq 2. \tag{29}$$

Moreover (29) is necessary and sufficient for $z(2-u_p(z))$ to be in $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha)$.

Remark 2.7. *The above conditions (18) and (26) are also necessary for functions $\Psi(z)$ given by (11) and of the form*

$$h_\mu^*(z)=(1-\mu)\Psi(z)+\mu\Psi'(z)=z-\sum_{n=2}^{\infty}(1+n\mu-\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n$$

is in the classes $\mathscr{S}\mathscr{P}_p\mathscr{T}(\alpha,\beta)$ and $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta)$, respectively.

3. Inclusion Properties

Making use of Lemma 1.3, we will study the action of the Struve function on the class $\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta)$.

Theorem 3.1. *Let $c<0,m>0$. If $f\in\mathscr{R}^\tau(A,B)$, and if the inequality*

$$(A-B)\,|\tau|\,[2u_p'(1)+(2-\cos\alpha-\beta)(u_p(1)-1)]\leq\cos\alpha-\beta \tag{30}$$

is satisfied, then $\mathscr{I}(c,m)(f)\in\mathscr{U}\mathscr{C}\mathscr{S}\mathscr{P}\mathscr{T}(\alpha,\beta)$.

Proof. Let f be of the form (1) belong to the class $\mathscr{R}^\tau(A,B)$. By virtue of (15), it suffices to show that

$$\sum_{n=2}^{\infty}n[2n-\cos\alpha-\beta]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n|\leq\cos\alpha-\beta.$$

Since $f \in \mathscr{R}^{\tau}(A,B)$ then by Lemma 1.3, we have

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence

$$\begin{aligned}\Upsilon(\alpha,\beta) &= \sum_{n=2}^{\infty} n[2n-\cos\alpha-\beta]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}|a_n| \\ &\leq (A-B)\,|\tau|\sum_{n=2}^{\infty}[2n-\cos\alpha-\beta]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}.\end{aligned}$$

Making use of (21) and (22), we get

$$\Upsilon(\alpha,\beta) \leq (A-B)\,|\tau|\,[2u_p'(1)+(2-\cos\alpha-\beta)(u_p(1)-1)],$$

but this last expression is bounded above by $\cos\alpha-\beta$ if and only if (30) holds. □

Theorem 3.2. *Let $c<0, m>0$. Then*

$$\mathscr{L}(m,c,z) = \int_0^z (2-u_p(t))dt$$

is in $\mathscr{UCSPT}(\alpha,\beta)$ if and only if the condition (26) is satisfied.

Proof. Since

$$\mathscr{L}(m,c,z) = z - \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}$$

then from (15), we need only to show that

$$\sum_{n=2}^{\infty} n[2n-\cos\alpha-\beta]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \leq \cos\alpha-\beta,$$

or, equivalently

$$\sum_{n=2}^{\infty}[2n^2-n(\cos\alpha+\beta)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \leq \cos\alpha-\beta.$$

By writing $n^2 = n(n-1)+n$, and proceeding as in Theorem 2.1, we get

$$\begin{aligned}&\sum_{n=2}^{\infty} n[2n-\cos\alpha-\beta]\ \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \\ &= 2(u_p''(1)+2u_p'(1))+(2-\cos\alpha-\beta)(u_p'(1)+u_p(1)-1) \\ &= 2u_p''(1)+(6-\cos\alpha-\beta)u_p'(1)+(2-\cos\alpha-\beta)(u_p(1)-1),\end{aligned}$$

which is bounded above by $\cos\alpha-\beta$ if and only if (26) holds. □

Putting $\beta=0$ in Theorems 3.1 and 3.2, we obtain the following corollaries.

Corollary 3.3. *Let $c<0, m>0$. If $f \in \mathscr{R}^{\tau}(A,B)$, and if the inequality*

$$(A-B)\,|\tau|\,[2u_p'(1)+(2-\cos\alpha)(u_p(1)-1)] \le \cos\alpha \tag{31}$$

is satisfied, then $\mathscr{I}(c,m)(f) \in \mathscr{UCSPT}(\alpha,\beta)$.

Corollary 3.4. *Let $c<0, m>0$. Then*

$$\mathscr{L}(m,c,z)=\int_0^z (2-u_p(t))dt$$

is in $\mathscr{UCSPT}(\alpha,\beta)$ if and only if the condition (29) is satisfied.

Remark 3.5. *(i) If we put $\alpha=0$ in Corollary 2.5, we obtain the necessary and sufficient condition for $z(2-u_p(z))$ to be in $\mathscr{TS}_{\mathscr{P}}(0,1)$* [23, Corollary 1, (i)].

(ii) If we put $\alpha=0$ in Corollary 2.6, we obtain the necessary and sufficient condition for $z(2-u_p(z))$ to be in $\mathscr{UST}(0,1)$ [23, Corollary 2, (ii)]. *We note that the coefficient of $u_p'(1)$ in* [23, Corollary 2, (ii)]*, should be corrected to $(3+2\beta-\alpha)$.*

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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