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COMMON FIXED POINTS FOR GENERALIZED $\psi - \emptyset -$WEAK CONTRACTION IN METRIC SPACE

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Abstract: In this paper, we first prove some common fixed point theorems for weakly compatible mappings, Point wise and $\mathfrak{R}$-weakly commuting mappings, Reciprocally continuous Mappings, $\mathfrak{R}$-weakly commuting mappings of type (A_f), type (A_g), type (P) satisfying a generalized $\emptyset -$weak contraction condition that involves cubic terms of $d(x, y)$. Secondly, we prove common fixed point theorems for weakly compatible mappings along with (E.A) property and (CLR) property. At the last of each theorems, we give corollaries and applications in support of our theorems.

Keywords and phrases: $\emptyset -$weak contraction; weakly compatible mappings; point wise and $\mathfrak{R}$-weakly commuting mappings; reciprocally continuous; reciprocally continuous $\mathfrak{R}$-weakly commuting mappings of type (A_f); $\mathfrak{R}$-weakly commuting mappings of type (A_g) and $\mathfrak{R}$-weakly commuting mappings of type (P); (E.A.) property; (CLR) property.

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1. INTRODUCTION

Banach fixed point theorem is the basic tool to study fixed point theory and show the existence and uniqueness of a fixed point under appropriate conditions. This result is known as Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering.

In 1969, Boyd and Wong [3] replaced the constant k in Banach contraction principle by a control function ψ as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$ .If $T : X \to X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point.

In 1997, Alber and Gueree-Delabriere [2] introduced the concept of weak contraction and in 2001, Rhoades [11] had shown that the results of Alber et. al. equally hold good in complete metric spaces.

A map $T : X \to X$ is said to be weak contraction if for each x, $y \in X$, there exists a function $\emptyset : [0, \infty) \to [0, \infty)$, $\emptyset\,(t) > 0$ for all $t > 0$ and $\emptyset\,(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \emptyset\,(d(x, y)).$$

2. PRELIMINARIES

It was the turning point in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [5] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. This result was further generalized, extended and unified using various types of contractions and minimal commutative mappings. The first ever attempt to relax the commutativity of mappings to a smaller subset of the domain of mappings was initiated by Sessa [16], who in 1982 gave the notion of weak commutativity. Two self mappings f and g of a metric space (X, d) are said to be weakly commuting if $d(fgx, gfx) \leq d(gx, fx)$ for all x in X.

Further, in 1986 Jungck [6] introduced more generalized commutativity so called compatibility. Clearly commuting, weakly commuting mappings are compatible, but converse need not be true (see [6]). One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence.

Definition 2.1[6] Two self mappings f and g on a metric space (X, d) are called compatible if $lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fx_n = lim_n gx_n = t$ for some t in$X$.

Definition 2.2 Two self mappings f and g on a metric space (X, d) are called point wise $\Re -$ weakly commuting on X if given $x \in X$, there exists $\Re > 0$ such that $d(fgx, gfx) \leq \Re\, d(gx, fx)$ for all x in X.

Remark 2.1 It is obvious that point wise $\Re -$ weakly commuting maps commute at their coincidence points, but maps f and g can fail to be point wise $\Re$ -weakly commuting only if there exists some x in X such that $fx = gx$ but $fgx \neq gfx$. Therefore, the notion of point wise $\Re$ -weak commutativity type mapping is equivalent to commutativity at coincidence points. Moreover since contractive conditions exclude the possibilities of the existence of a common fixed point together with existence of a coincidence fixed point at which the mappings do not commute, point wise $\Re$ -weakly commutativity is a necessary condition for the existence of common fixed points for contractive type mappings, and also it is noted compatible maps are necessarily point wise $\Re$ -weakly commuting, since compatible maps commute at their coincident points, but converse may not be true.

Definition 2.3 Two self mappings f and g on a metric space (X, d) are said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$,whenever $\{x_n\}$ is a sequence in X such that $fx_n = lim_n gx_n = t$ for some t in X.

Remark 2.2 Continuous mappings are reciprocally continuous on (X, d), but the converse is not true.

In 1998, Jungck and Rhoades [9] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but not conversely.

Definition 2.3[9] Two self mappings f and g on a metric space (X, d) are called weakly compatible if they commute at their coincidence point i.e. if $fu = gu$ for some $u \in X$ then $fgu = gfu$.

Remark 2.1 [9] Two Compatible self mappings are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.

In 1994, Pant[13] introduced the concept of $\Re$ -weakly commuting mappings in metric spaces, with the purpose of extending the scope of the study of common fixed point theorems from compatible to $\Re$ -weakly commuting mappings. Also, at the fixed points, these maps are not necessarily continuous.

Definition 2.4[13] Two self mappings f and g on a metric space (X, d) are called $\Re$-weakly commuting if there exists some $\Re > 0$ such that

$$d(fgx, gfx) \leq \Re\, d(gx, fx), \text{ for all } x \in X.$$

In 1997, Pathak and Kang[11] introduced $\Re$-weakly commuting mappings of type (A_f) and $\Re$-weakly commuting mappings of type (A_g) which is the improved notions of $\Re$-weakly commuting mappings and Kumar and Garg [18] introduced the $\Re$-weakly commuting mappings of type (P).

Definition 2.5[11,18] Two self mappings f and g on a metric space (X, d) are said to be:

(i) $\Re$-weakly commuting mappings of type (A_f) if there exists some $\Re > 0$ such that

$$d(fgx, ggx) \leq \Re\, d(fx, gx), \text{ for all } x \in X.$$

(ii) $\Re$-weakly commuting mappings of type (A_g) if there exists some $\Re > 0$ such that

$$d(gfx, ffx) \leq \Re\, d(fx, gx), \text{ for all } x \in X.$$

(iii) $\Re$-weakly commuting mappings of type (P) if there exists some $\Re > 0$ such that

$$d(ffx, ggx) \leq \Re\, d(fx, gx), \text{ for all } x \in X.$$

Remark 2.2[18] We have suitable example which show that $\Re$-weakly commuting mappings, $\Re$-weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g) and $\Re$-weakly commuting mappings of type (P) are all distinct.

Example 2.1[18] Let $X = [-1,1]$ be a usual a usual metric space with usual metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define $f(x) = x$ and $g(x) = x - 1$. Then we have $d(fx, gx = 1, d(fgx, gfx) = 2(1 - x), d(fgx, ggx) = 1,$
$d(gfx, ffx) = 1$, $(dffx, ggx) = 2x$, for all $x, y \in X$.

Now we have the following : (i) pair (f, g) is not weakly commuting

(ii) For $\Re$ =2, the pair (f, g) is $\Re$ -weakly commuting mappings, $\Re$ -weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g) and $\Re$-weakly commuting mappings of type (P).

(iii) For $\Re = \frac{3}{2}$, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f), but not $\Re$-weakly commuting mappings and $\Re$-weakly commuting mappings of type (P).

Example 2.2[18] Let $X = [0,1]$ be a usual a usual metric space with usual metric defined by $d(x,y) = |x-y|$ for all $x, y \in X$. Define $f(x) = x$ and $g(x) = x^2$, Then we have
$ffx = x, gfx = x^2, fgx = x^2, ggx = x^4$ and

$$d(fgx, gfx = 0, d(fgx, ggx) = |x^2(x-1)(x+1)|, d(gfx, ffx) = |x(x-1)|,$$

$d(ffx, ggx) = |(x^2+x+1)x(x-1)|$, $(dfx, gx) = |x(x-1)|$, for all $x, y \in X$.

Now we have the following : (i) pair (f,g) is $\Re -$ weakly commuting.

(ii) For $\Re$ =3, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g) and $\Re$-weakly commuting mappings of type (P).

(iii) For $\Re$ =2, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g), but not $\Re$-weakly commuting mappings of type (P) (for this $x = \frac{3}{4}$).

Example 2.3[18] Let $X = [\frac{1}{2}, 2]$ be a usual a usual metric space. Define self maps f and g as $f(x) = \frac{x+1}{3}$ and $g(x) = \frac{x+2}{5}$, Then we have $d(fx, gx = \frac{2x-1}{15}, d(fgx, gfx) = 0$, $, d(fgx, ggx) = \frac{2x-1}{75}, d(gfx, ffx) = \frac{2x-1}{45} (dffx, ggx) = \frac{8(2x-1)}{225}$, for all $x, y \in X$.
Now we have the following : (i) pair (f,g) is $\Re -$ weakly commuting.

(ii) For $\Re \geq \frac{8}{15}$, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g) and $\Re$-weakly commuting mappings of type (P).

(iii) For $\frac{1}{3} \leq \Re < \frac{8}{15}$, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f), $\Re$-weakly commuting mappings of type (A_g), but not $\Re$-weakly commuting mappings of type (P).

(iv) For $\frac{1}{5} \leq \Re < \frac{1}{3}$, the pair (f,g) is $\Re$-weakly commuting mappings of type (A_f) and $\Re$-weakly commuting mappings of type (P), but not $\Re$-weakly commuting mappings of type (A_g).

3. WEAKLY COMPATIBLE MAPPINGS

First we first prove a common fixed point theorems for weakly compatible mappings satisfying a generalized $\emptyset$ −weak contraction condition that involves cubic terms of $d(x,y)$.

Theorem 3.1 Let S,T, A and B be four mappings of a complete metric space (X,d) into itself satisfying the following conditions:

(C1) $S(X) \subset B(X), T(X) \subset A(X)$;

(C2) $$d^3(Sx,Ty) \leq pmax \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Ax,Sx)d(By,Ty) \\ +d(Ax,Sx)d^2(By,Ty) \end{bmatrix}, \\ d(Ax,Sx)d(Ax,Ty)d(By,Sx), \\ d(Ax,Ty)d(By,Sx)d(By,Ty) \end{Bmatrix} - \emptyset\{m(Ax,By)\}$$

where $$m(Ax,By) = \max \begin{Bmatrix} d^2(Ax,By), \\ d(Ax,Sx)d(By,Ty), \\ d(Ax,Ty)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Ty) \\ +d(By,Sx)d(By,Ty) \end{bmatrix} \end{Bmatrix}$$

for all $x,y \in X$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\ (t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.

(C3) One of subspace AX or BX or SX or TX is complete; then

(i) A and S have a point of coincidence,

(ii) B and T have a point of coincidence.

Moreover assume that the pairs (A,S) and (B,T) are weakly compatible, then S, T, A and B have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. From (C1) we can find x_1 such that $S(x_0) = B(x_1) = y_0$ for this x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$.Continuing in this way one can construct a sequence such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}), y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \text{ for each } n \geq 0. \tag{3.1}$$

For brevity, we write $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.

First we prove that $\{\alpha_{2n}\}$ is non increasing sequence and converges to zero.

Case I If n is even, taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ + d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}) \end{bmatrix}, \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{matrix}\right\} - \emptyset\{m(Ax_{2n}, Bx_{2n+1})\}$$

where, $m(Ax_{2n}, Bx_{2n+1}) = \max\left\{\begin{matrix} d^2(Ax_{2n}, Bx_{2n+1}), \\ d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1}) \\ +d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{bmatrix} \end{matrix}\right\}$.

Using (3.1), we have

$$d^3(y_{2n}, y_{2n+1}) \leq pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\ + d(y_{2n-1}, y_{2n})d^2(y_{2n}, y_{2n+1}) \end{bmatrix}, \\ d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1}) \end{matrix}\right\} - \emptyset\{m(y_{2n-1}, y_{2n})\}$$

where, $m(y_{2n-1}, y_{2n}) = \max\left\{\begin{matrix} d^2(y_{2n-1}, y_{2n}), \\ d(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}), \\ d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1}) \\ +d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1}) \end{bmatrix} \end{matrix}\right\}$. (3.2)

On using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in (3.2) , we have

$$\alpha_{2n}^3 \leq p\, max\left\{\tfrac{1}{2}[\alpha_{2n-1}^2\alpha_{2n} + \alpha_{2n-1}\alpha_{2n}^2], 0,0\right\} - \emptyset\{m(y_{2n-1}, y_{2n})\} \quad (3.3)$$

If $\alpha_{2n-1} < \alpha_{2n}$, then (3.3) reduces to

$\alpha_{2n}^3 \leq p\alpha_{2n}^3 - \emptyset\{\alpha_{2n}^2\}$, a contradiction, therefore, $\alpha_{2n} \leq \alpha_{2n-1}$.

In a similar way, if n is odd, then we can obtain $\alpha_{2n+1} < \alpha_{2n}$.

It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n\to\infty} \alpha_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$; then from inequality (C2), we have

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ + d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}) \end{array}\right], \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{array}\right\}$$

$$-\emptyset\{m(Ax_{2n}, Bx_{2n+1})\}$$

$$m(Ax_{2n}, Bx_{2n+1}) = \max\left\{\begin{array}{c} d^2(Ax_{2n}, Bx_{2n+1}), \\ d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ \frac{1}{2}\left[\begin{array}{c} d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1}) \\ +d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{array}\right] \end{array}\right\}.$$

Now by using (3.3), triangular inequality and property of $\emptyset$ and proceed limits $n \to \infty$, we get $r^3 \leq pr^3 - \emptyset(r^2)$, a contradiction, therefore we get $r = 0$, therefore

$$\lim_{n\to\infty} \alpha_{2n} = \lim_{n\to\infty} d(y_{2n}, y_{2n-1}) = r = 0. \tag{3.4}$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose we assume that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$ we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$,

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon, \qquad d\big(y_{m(k)}, y_{n(k)-1}\big) < \epsilon \tag{3.5}$$

Now $\quad \epsilon \leq d\big(y_{m(k)}, y_{n(k)}\big) \leq d\big(y_{m(k)}, y_{n(k)-1}\big) + d(y_{n(k)-1}, y_{n(k)})$

Letting $k \to \infty$, we get $\lim_{k\to\infty} d\big(y_{m(k)}, y_{n(k)}\big) = \epsilon$

Now from the triangular inequality we have,

$$\big|d\big(y_{n(k)}, y_{m(k)+1}\big) - d\big(y_{m(k)}, y_{n(k)}\big)\big| \leq d\big(y_{m(k)}, y_{m(k)+1}\big).$$

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$$\lim_{k\to\infty} d\big(y_{n(k)}, y_{m(k)+1}\big) = \epsilon.$$

Again from the triangular inequality, we have

$\left|d(y_{m(k)}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})\right| \leq d(y_{n(k)}, y_{n(k)+1})$.

Taking limits as $k \to \infty$ and using (3.4) and (3.5), we have

$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)+1}) = \epsilon$.

Similarly on using triangular inequality, we have

$$\left|d(y_{m(k)+1}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})\right| \leq d(y_{m(k)}, y_{m(k)+1}) + d(y_{n(k)}, y_{n(k)+1})$$

Taking limit as $k \to \infty$ in the above inequality and using (3.4) and (3.5), we have

$\lim_{k\to\infty} d(y_{n(k)+1}, y_{m(k)+1}) = \epsilon$.

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (C2), we get

$$d^3(Sx_{m(k)}, Tx_{n(k)}) \leq pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(Ax_{m(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}) \\ +d(Ax_{m(k)}, Sx_{m(k)})d^2(Bx_{n(k)}, Tx_{n(k)}) \end{array}\right], \\ d(Ax_{m(k)}, Sx_{m(k)})d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)}), \\ d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}) \end{array}\right\}$$

$$-\emptyset\{m(Ax_{m(k)}, Bx_{n(k)})\}$$

where $$m(Ax_{m(k)}, Bx_{n(k)}) = \max\left\{\begin{array}{c} d^2(Ax_{m(k)}, Bx_{n(k)}), \\ d(Ax_{m(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}), \\ d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)}), \\ \frac{1}{2}\left[\begin{array}{c} d(Ax_{m(k)}, Sx_{m(k)})d(Ax_{m(k)}, Tx_{n(k)}) \\ +d(Bx_{n(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}) \end{array}\right] \end{array}\right\}.$$

Using (3.1) we obtain

$$d^3(y_{m(k)}, y_{n(k)}) \leq pmax\left\{\begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}) \\ +d(y_{m(k)-1}, y_{m(k)})d^2(y_{n(k)-1}, y_{n(k)}) \end{array}\right], \\ d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}), \\ d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}) \end{array}\right\}$$

$$-\emptyset\{m(Ax_{m(k)}, Bx_{n(k)})\}$$

where $m\left(Ax_{m(k)}, Bx_{n(k)}\right) = \max\left\{\begin{matrix} d^2\left(y_{m(k)-1}, y_{n(k)-1}\right), \\ d\left(y_{m(k)-1}, y_{m(k)}\right)d\left(y_{n(k)-1}, y_{n(k)}\right), \\ d\left(y_{m(k)-1}, y_{n(k)}\right)d\left(y_{n(k)-1}, y_{m(k)}\right), \\ \frac{1}{2}\left[\begin{matrix} d\left(y_{m(k)-1}, y_{m(k)}\right)d\left(y_{m(k)-1}, y_{n(k)}\right) \\ +d\left(y_{n(k)-1}, y_{m(k)}\right)d\left(y_{n(k)-1}, y_{n(k)}\right) \end{matrix}\right] \end{matrix}\right\}$

Letting $k \to \infty$, we get $\epsilon^3 \le p\, max\left\{\frac{1}{2}[0+0], 0,0\right\} - \emptyset(\epsilon^2) = -\emptyset(\epsilon^2)$,which is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in X. Now suppose that AX is complete subspace of X, then there exist $z \in X$ such that

$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \to z \text{ as } n \to \infty.$$

Consequently we can find $w \in X$ such that $Aw = z$. Further a Cauchy sequence $\{y_n\}$ has a convergent subsequence$\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have $y_{2n} = S(x_{2n}) = B(x_{2n+1}) \to z$ as $n \to \infty$.

On setting $x = w$ and $y = z$ in (C2) we get

$$d^3(Sw, Tz) \le pmax\left\{\begin{matrix} \frac{1}{2}\left[\begin{matrix} d^2(Aw, Sw)d(Bz, Tz) + \\ d(Aw, Sw)d^2(Bz, Tz) \end{matrix}\right], \\ d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), \\ d(Aw, Tz)d(Bz, Sw)d(Bz, Tz) \end{matrix}\right\} - \emptyset\{m(Aw, Bz)\}$$

where $m(Aw, Bz) = \max\left\{\begin{matrix} d^2(Aw, Bz), d(Aw, Sw)d(Bz, Tz), \\ d(Aw, Tz)d(Bz, Sw), \\ \frac{1}{2}\left[\begin{matrix} d(Aw, Sw)d(Aw, Tz) \\ +d(Bz, Sw)d(Bz, Tz) \end{matrix}\right] \end{matrix}\right\}$

$$m(Aw, Bz) = \max\left\{\begin{matrix} d^2(z, z), d(z, Sw)d(Tz, Tz), d(z, z)d(z, Sw), \\ \frac{1}{2}[d(z, Sw)d(z, z) + d(z, Sw)d(Tz, Tz)] \end{matrix}\right\} = 0$$

Therefore, $d^3(Sw, z) \le pmax\left\{\begin{matrix} \frac{1}{2}[d^2(z, Sw)d(z, z) + d(z, Sw)d^2(z, z)], \\ d(z, Sw)d(z, z)d(z, Sw), \\ d(z, z)d(z, Sw)d(z, z) \end{matrix}\right\} - \emptyset(0)$

This implies that $Sw = z$ and hence $Sw = Aw = z$. Therefore, w is a coincidence point of A and S. Since $z = Sw \in SX \subset BX$ there exist $v \in X$ such that $z = Bv$.

Next we claim that $Tv = z$. Now putting $x = x_{2n}$ and $y = v$ in (C2)

$$d^3(Sx_{2n}, Tv) \leq pmax \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(Ax_{2n}, Sx_{2n})d(Bv, Tv) \\ +d(Ax_{2n}, Sx_{2n})d^2(Bv, Tv) \end{array} \right], \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tv)d(Bz, Sx_{2n}), \\ d(Ax_{2n}, Tv)d(Bv, Sx_{2n})d(Bv, Tv) \end{array} \right\} - \emptyset\{m(Ax_{2n}, Bv)\}$$

where $m(Ax_{2n}, Bv) = \max \left\{ \begin{array}{c} d^2(Ax_{2n}, Bv), \\ d(Ax_{2n}, Sx_{2n})d(Bv, Tv), \\ d(Ax_{2n}, Tv)d(Bv, Sx_{2n}), \\ \frac{1}{2}\left[\begin{array}{c} d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tv) \\ +d(Bv, Sx_{2n})d(Bv, Tv) \end{array} \right] \end{array} \right\} = 0.$

Therefore, $d^3(z, Tv) \leq pmax \left\{ \begin{array}{c} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{array} \right\} - \emptyset(0)$, this gives $z = Tv$ and hence $z = Tv = Bv$.

Therefore, v is a coincidence point of B and T. Since the pairs A, S and B, T are weakly compatible, we have $Sz = S(Aw) = A(Sw) = Az$. $Tz = T(Bv) = B(Tv) = Bz$.

Next we show that $Sz = z$. For this put $x = z$ and $y = x_{2n+1}$ in (C2)

$$d^3(Sz, Tx_{2n+1}) \leq pmax \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{c} d^2(Az, Sz)d(z, z) \\ +d(Az, Sz)d^2(z, z) \end{array} \right], \\ d(Az, Sz)d(Az, z)d(z, Sz), \\ d(Az, z)d(z, Sz)d(z, z) \end{array} \right\} - \emptyset\{m(Az, z)\}$$

where $m(Az, z) = \max \left\{ \begin{array}{c} d^2(Az, z), \\ d(Az, Sz)d(z, z), \\ d(Az, z)d(z, Sz), \\ \frac{1}{2}\left[\begin{array}{c} d(Az, Sz)d(Az, z) \\ +d(z, Sz)d(z, z) \end{array} \right] \end{array} \right\} = d^2(Sz, z)$

Therefore, we get $d^3(Sz, z) \leq pmax \left\{ \begin{array}{c} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{array} \right\} - \emptyset\{d^2(Sz, z)\}$

Thus we get $d^2(Sz, z) = 0$. This implies that $Sz = z$ and hence $Sz = Az = z$.

Next we claim that $Tz = z$. Now put $x = x_{2n}$ and $y = z$ in (C2)

$$d^3(Sx_{2n},Tz) \leq \; pmax \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Ax_{2n},Sx_{2n})d(Bz,Tz) \\ +d(Ax_{2n},Sx_{2n})d^2(Bz,Tz) \end{bmatrix}, \\ d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)d(Bz,Sx_{2n}), \\ d(Ax_{2n},Tz)d(Bz,Sx_{2n})d(Bz,Tz) \end{Bmatrix} - \emptyset\{m(Ax_{2n},Bz)\}$$

where $\quad m(Ax_{2n},Bz) = \max \begin{Bmatrix} d^2(Ax_{2n},Bz), \\ d(Ax_{2n},Sx_{2n})d(Bz,Tz), \\ d(Ax_{2n},Tz)d(Bz,Sx_{2n}), \\ \frac{1}{2}\begin{bmatrix} d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz) \\ +d(Bz,Sx_{2n})d(Bz,Tz) \end{bmatrix} \end{Bmatrix} = \; d^2(z,Tz).$

Hence we get $d^3(z,Tz) \leq pmax \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix} - \emptyset\{d^2(z,Tz)\}$

This gives $z = Tz$ and hence $z = Tz = Bz$. Therefore z is a common fixed point of A, B, S and T.

Similarly we can complete the proofs for the cases in which BX or SX or TX is complete.

Uniqueness: Suppose $z \neq w$ be two common fixed point of S,T, A and B.

Put $x = z$ and $y = w$ in (C_2)

$d^3(Sz,Tw) \leq \; p\, max\{0,0,0\} - \emptyset\{m(Az,Bw)\}$

$$d^3(Sz,Tw) \leq \; pmax\{0,0,0\} - \emptyset\{d^2(Sz,Tw)\}$$

$\Longrightarrow d^2(z,w) = 0 \Longrightarrow z = w.$ This completes the proof.

Application

In 2002 Branciari [4] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality. Now we give the following theorem as an application of Theorem 3.1.

Theorem 3.2 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying $(C_1), (C_3)$ the following condition:

$$\int_0^{d^3(Sx,Ty)} \varphi(t)\, dt \; \leq \int_0^{\mathcal{M}(x,y)} \varphi(t)\, dt$$

$$\mathcal{M}(x, y) = pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax,Sx)d(By,Ty) \\ +d(Ax,Sx)d^2(By,Ty) \end{bmatrix}, \\ d(Ax,Sx)d(Ax,Ty)d(By,Sx), \\ d(Ax,Ty)d(By,Sx)d(By,Ty) \end{matrix}\right\} - \emptyset\{m(Ax,By)\}$$

where $$m(Ax,By) = \max\left\{\begin{matrix} d^2(Ax,By), \\ d(Ax,Sx)d(By,Ty), \\ d(Ax,Ty)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Ty) \\ +d(By,Sx)d(By,Ty) \end{bmatrix} \end{matrix}\right\},$$

for all $x, y \in X$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Further, where $\varphi : R^+ \to R^+$ is a Lebesgue -integrable over R^+function which is summable on each compact subset of R^+, non-negative, and such that for each $\in\ > 0$, $\int_0^{\in} \varphi(t)dt > 0$. Moreover assume that the pairs (A,S) and (B,T) are weakly compatible, then S, T, A and B have a unique common fixed point.

Proof. The proof of the theorem follows on the same lines of the proof of the Theorem 3.1. on setting φ (t) = 1.

Remark 3.1. Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting φ (t) = 1.

If we put $S = T$ in theorem 3.1. Then we obtain the following Corollary

Corollary 3.1 Let S, A and B be four self-mappings of a complete metric space (X, d) satisfying the conditions

(C4) $S(X) \subset B(X), S(X) \subset A(X)$,

(C5) one of subspace AX or BX or SX is complete,

(C6) $$d^3(Sx,Sy) \le pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax,Sx)d(By,Sy) \\ +d(Ax,Sx)d^2(By,Sy) \end{bmatrix}, \\ d(Ax,Sx)d(Ax,Sy)d(By,Sx), \\ d(Ax,Sy)d(By,Sx)d(By,Sy) \end{matrix}\right\} - \emptyset\{m(Ax,By)\}$$

where $$m(Ax,By) = \max\left\{\begin{matrix} d^2(Ax,By), \\ d(Ax,Sx)d(By,Sy), \\ d(Ax,Sy)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Sy) \\ +d(By,Sx)d(By,Sy) \end{bmatrix} \end{matrix}\right\}$$

for all $x, y \in X$, $p \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Assume that the pairs (A, S) and (B, T) are weakly compatible. Then S, A and B have a unique common fixed point.

In Theorem 3.1. if we put $A = B = I$, we obtain the following result.

Corollary 3.2 Let S and T be mappings of a complete metric space (X, d) into itself satisfying the following conditions:

$$d^3(Sx, Ty) \leq p\max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(x, Sx)d(y, Ty) \\ +d(x, Sx)d^2(y, Ty) \end{bmatrix}, \\ d(x, Sx)d(x, Ty)d(y, Sx), \\ d(x, Ty)d(y, Sx)d(y, Ty) \end{matrix}\right\} - \emptyset\{m\,(x, y)\}$$

$$\text{where} \quad m(x, y) = \max\left\{\begin{matrix} d^2(x, y), \\ d(x, Sx)d(y, Ty), \\ d(x, Ty)d(y, Sx), \\ \frac{1}{2}\begin{bmatrix} d(x, Sx)d(x, Ty) \\ +d(y, Sx)d(y, Ty) \end{bmatrix} \end{matrix}\right\},$$

for all $x, y \in X$, $p \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. And one of subspace SX or TX is complete.

Then S and T have a unique common fixed point.

Also we prove Theorem 3.3 for weakly compatible mappings in a metric space by dropping the condition of completeness of subspaces as follows:

Theorem 3.3 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying (C1),(C3) and the following condition

(C7) one of subspace AX or BX or SX or TX is closed subset of X,

Assume that the pairs (A, S) and (B, T) are weakly compatible. Then S, T, A and B have a unique common fixed point.

Proof. As we know that the subspace of a complete metric space is complete if and only if it is closed. By Theorem 3.1, this conclusion holds. This completes the proof.

Theorem 3.4 Let (A, S) and (B, T) be point wise $\Re -$ weakly commuting pairs of self mappings of a complete metric space (X, d) satisfying (C1),(C3) and the following condition

(C8) Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then S, T, A and B have a unique common fixed point.

Proof. By Theorem 3.1, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, then there exist $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Also $\lim_{n\to\infty} T(x_{2n+1}) = \lim_{n\to\infty} A(x_{2n+2}) = \lim_{n\to\infty} S(x_{2n}) = \lim_{n\to\infty} B(x_{2n+1}) = z$. Suppose B and T are compatible and reciprocally continuous. Then by reciprocally continuous B and T, we have $\lim_{n\to\infty} BTx_n = Bz$ and $\lim_{n\to\infty} TBx_n = Tz$. Also by compatibility of B and T implies that $Bz = Tz$. Since $T(X) \subset A(X)$, so there exists a point $v \in X$ such that $Tz = Av$.

Next we show that $Tz = Sv$. Now putting $x = v$ and $y = z$ in (C2)

$$d^3(Sv,Tz) \le\ pmax\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Av,Sv)d(Bz,Tz) \\ +d(Av,Sv)d^2(Bz,Tz)\end{bmatrix}, \\ d(Av,Sv)d(Av,Tz)d(Bz,Sv), \\ d(Av,Tz)d(Bz,Sv)d(Bz,Tz) \end{Bmatrix} - \emptyset\{m(Av,Bz)\}$$

$$\text{where} \quad m(Av,Bz) = \max\begin{Bmatrix} d^2(Av,Bz), \\ d(Av,Sv)d(Bz,Tz), \\ d(Av,Tz)d(Bz,Sv), \\ \frac{1}{2}\begin{bmatrix} d(Av,Sv)d(Av,Tz) \\ +d(Bz,Sv)d(Bz,Tz)\end{bmatrix}\end{Bmatrix} = 0.$$

$$d^3\big((Sv,Tz)\big) \le pmax\begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix}$$

This gives $Sv = Tz$. Thus $Bz = Tz = Sv = Av$. Since B and T are $\Re-$ weak commutativity, there exists $\Re > 0$ such that $d(BTz,TBz) \le \Re\, d(Bz,Tz) = 0$, this implies that $BTz = TBz$ and $BBz = BTz = TBz = TTz$. Also A and S are $\Re-$ weak commutative, implies that $d(ASv,SAv) \le \Re\, d(Av,Sv) = 0$, then $ASv = SAv$ and so we have $ASv = SAv = SSv = AAv$. Next we show that $Tz = TTz$. Now putting $x = v$ and $y = Tz$ in (C2)

$$d^3(Sv,TTz) \le\ pmax\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Av,Sv)d(BTz,TTz) \\ +d(Av,Sv)d^2(BTz,TTz)\end{bmatrix}, \\ d(Av,Sv)d(Av,TTz)d(BTz,Sv), \\ d(Av,TTz)d(BTz,Sv)d(BTz,TTz) \end{Bmatrix} - \emptyset\{m(Av,BTz)\}$$

$$\text{where} \quad m(Av,BTz) = \max\left\{\begin{matrix} d^2(Av,BTz), \\ d(Av,Sv)d(BTz,TTz), \\ d(Av,TTz)d(BTz,Sv), \\ \frac{1}{2}\begin{bmatrix} d(Av,Sv)d(Av,TTz) \\ +d(BTz,Sx)d(BTz,TTz) \end{bmatrix} \end{matrix}\right\} = d^2(Av,BTz)$$

$$d^3\big((Tz,TTz)\big) \le pmax\begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix} - \emptyset\{d^2(Tz,TTz)\}$$

This gives $Tz = TTz$. Thus Tz is a common fixed point of B and T.

Next we claim that $SSv = Sv$. Now putting $x = Sv$ and $y = z$ in (C2)

$$d^3(SSv,Tz) \le pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(ASv,SSv)d(Bz,Tz) \\ +d(ASv,SSv)d^2(Bz,Tz) \end{bmatrix}, \\ d(ASv,SSv)d(ASv,Tz)d(Bz,SSv), \\ d(ASv,Tz)d(Bz,SSv)d(Bz,Tz) \end{matrix}\right\} - \emptyset\{m(ASv,Bz)\}$$

$$\text{where} \quad m(ASv,Bz) = \max\left\{\begin{matrix} d^2(ASv,Bz), \\ d(ASv,SSv)d(Bz,Tz), \\ d(ASv,Tz)d(Bz,SSv), \\ \frac{1}{2}\begin{bmatrix} d(ASv,SSv)d(ASv,Tz) \\ +d(Bz,SSv)d(Bz,Tz) \end{bmatrix} \end{matrix}\right\} = d^2(SSv,Tz)$$

$$d^3(SSv,Sv) \le pmax\begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix} - \emptyset\{d^2(SSv,Sv)\}$$

This gives $SSv = Sv$. Thus Sv is a common fixed point of A and S.

Hence $Sv = Tz$ is a common fixed point of S, T, A and B.

Finally, in order to prove uniqueness of Tz, Suppose that Tz and Tw, $Tz \neq Tw$ are common fixed points of S, T, A and B.

Next we claim that $Tz = Tw$. Now putting $x = Tz$ and $y = Tw$ in (C2)

$$d^3(STz,TTw) \le pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(ATz,STz)d(BTw,TTw) \\ +d(ATz,STz)d^2(BTw,TTw) \end{bmatrix}, \\ d(ATz,STz)d(ATz,TTw)d(BTw,STz), \\ d(ATz,TTw)d(BTw,STz)d(BTw,TTw) \end{matrix}\right\} - \emptyset\{m(ATz,BTw)\}$$

$$\text{where} \quad m(ATz, BTw) = \max\left\{\begin{array}{c} d^2(ATz, BTw), \\ d(ATz, STz)d(BTw, TTw), \\ d(ATz, TTw)d(BTw, STz), \\ \frac{1}{2}\left[\begin{array}{c} d(ATz, STz)d(ATz, TTw) \\ +d(BTw, STz)d(BTw, TTw) \end{array}\right] \end{array}\right\}$$

$$d^3(Tz, Tw) \leq pmax\left\{\begin{array}{c} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{array}\right\} - \emptyset\{d^2(Tz, Tw)\}$$

This gives $Tz = Tw$. Thus $Tz = Tw$ is unique common fixed point of the four self mappings S, T, A and B. This completes the proof.

Theorem 3.5 Theorem 3.1 remains true if a "weakly compatible property " is replaced by any one (Retaining the rest of the hypotheses) of the following:

(I) $\mathfrak{R}$-weakly commuting mappings,

(II) $\mathfrak{R}$-weakly commuting mappings of type (A_f),

(III) $\mathfrak{R}$-weakly commuting mappings of type (A_g),

(IV) $\mathfrak{R}$-weakly commuting mappings of type (P),

(V) Weakly commuting mappings.

Proof. Since all the conditions of Theorem 3.1 are satisfied, then both the pairs (A, S) and (B, T) have coincidence points. From the Theorem 3.1, we obtained w and v are the coincidence point of the pairs (A, S) and (B, T) respectively.

(I) we are given that the pairs (A, S) and (B, T) are $\mathfrak{R}$-weakly commuting mappings, then

$$d(ASw, SAw) \leq \mathfrak{R}\, d(Aw, Sw) = 0 \text{ and } d(BTv, TBv) \leq \mathfrak{R}\, d(Bv, Tv) = 0,$$

which amounts to say that $ASw = SAw$ and $BTv = TBv$. Thus the pairs (A, S) and (B, T)

are weakly compatible. Now from the proof of the Theorem 3.1, we have that z is a common fixed point theorems for the four self mappings S, T, A and B.

(II) In the case when the pairs (A, S) and (B, T) are $\mathfrak{R}$-weakly commuting mappings of type (A_f), then $d(ASw, SSw) \leq \mathfrak{R}\, d(Aw, Sw,) = 0$ implies that $ASw = SSw$.

Now $d(ASw, SAw) \leq d(ASw, SSw) + d(SSw, SAw) = 0 + 0$, gives $ASw = SAw$.

Similarly, $d(BTv, TTv) \leq \mathfrak{R}\, d(Bv, Tv) = 0$ implies that $BTv = TTv$.

Now $d(BTv, TBv) \leq d(BTv, TTv) + d(TTv, TBv) = 0 + 0$, gives $BTv = TTv$. Thus the pairs (A, S) and (B, T) are weakly compatible. Again from the Theorem 3.1, we have that the four self mappings S, T, A and B have a common fixed point in X.

(III) When the pairs (A, S) and (B, T) are $\mathfrak{R}$-weakly commuting mappings of type (A_g), then $d(SAw, AAw) \leq \mathfrak{R}\, d(Aw, Sw,) = 0$ implies that $SAw = AAw$.

Now $d(ASw, SAw) \leq d(ASw, AAw) + d(AAw, SAw) = 0 + 0$, gives $ASw = SAw$.

Similarly, $d(TBv, BBv) \leq \mathfrak{R}\, d(Bv, Tv) = 0$ implies that $TBv = BBv$.

Now $d(BTv, TBv) \leq d(BTv, BBv) + d(BBv, TBv) = 0 + 0$, gives $BTv = TTv$. Thus the pairs (A, S) and (B, T) are weakly compatible. Again from the Theorem 3.1, we have that the four self mappings S, T, A and B have a common fixed point in X.

(IV) When the pairs (A, S) and (B, T) are $\mathfrak{R}$-weakly commuting mappings of type (P), then $d(SSw, AAw) \leq \mathfrak{R}\, d(Sw, Aw,) = 0$ implies that $SSw = AAw$. Using triangular inequality we have $ASw = SAw$ also we have $BTv = TTv$. Thus the pairs (A, S) and (B, T) are weakly compatible. Again from the Theorem 3.1, we have that the four self mappings S, T, A and B have a common fixed point in X.

(V) Similarly in the case when (A, S) and (B, T) are weakly commuting mappings we can also prove that S, T, A and B have a common fixed point in X.

As an application of Theorem 3.1, we prove a common fixed point theorem for four finite families of mappings which runs as follow:

Theorem 3.6 Let $\{S_1, S_2, S_3, \dots . S_m\}$, $\{T_1, T_2, T_3, \dots . T_n\}$, $\{A_1, A_2, A_3, \dots . A_r\}$, $\{B_1, B_2, B_3, \dots . B_t\}$ are four finite families of self mappings of a metric space (X, d) such that $S = \{S_1, S_2, S_3, \dots . S_m\}$, $T = \{T_1, T_2, T_3, \dots . T_n\}$, $A = \{A_1, A_2, A_3, \dots . A_r\}$ and $B = \{B_1, B_2, B_3, \dots . B_t\}$ satisfy the condition (C1), (C2),

(C3) If One of subspace AX or BX or SX or TX is complete subspace of X

(i) A and S have a point of coincidence,

(ii) B and T have a point of coincidence.

Moreover, if $S_iS_j = S_jS_i$, $T_pT_q = T_qT_p$, $A_kA_l = A_lA_k$ and $B_uB_v = B_vB_u$,
for all $i, j \in I_1 = \{1,2,3, \dots \dots , m\}$, $p, q \in I_2 = \{1,2,3, \dots , n\}$, $k, l \in I_3 = \{1,2,3, \dots , r\}$ and $u, v \in I_4$=$\{1,2,3, \dots , t\}$, then for all ($i \in I_1, p \in I_2, k \in I_3$ and $u \in I_4$) S_i ,T_p , A_k and B_u have a common fixed point.

Proof. Since all the conditions of Theorem 3.1 are satisfied, then both the pairs $(A, S)$and (B, T) have coincidence points. From the Theorem 3.1, we obtained w and v are the coincidence point of the pairs $(A, S)$and (B, T) respectively. Now applying to component wise commutativity of various pairs, one can immediately prove that $AS = SA$ and $BT = TB$, hence obviously both the pairs $(A, S)$and (B, T) are coincidently commuting. From all the conditions of Theorem 3.1 are satisfied ensuring that z is a unique common fixed point. Now one need to show that z remains the fixed point of all the component maps. For this consider $S(S_iz)$= $\big((S_1, S_2, S_3, \dots . S_m)S_i\big)z = (S_1, S_2, S_3, \dots . S_{m-1})((S_mS_i)z))$

$$= (S_1, S_2, S_3, \dots . S_{m-1})(S_mS_iz) = (S_1, S_2, S_3, \dots . S_{m-2})\big(S_{m-1}S_i(S_mz)\big)$$

$$= (S_1, S_2, S_3, \dots . S_{m-2})\big(S_iS_{m-1}(S_mz)\big)$$

$$= \cdots\ S_1S_i(S_2, S_3, \dots . S_mz) = S_iS_1(S_2, S_3, \dots . S_mz) = S_i(Sz) = S_iz.$$

Similarly, we can prove that

$$S(A_kz) = A_k(Sz) = A_kz, A\big(A_k(z)\big) = A_k(Az) = A_kz$$

and $\qquad A(S_i\mathrm{z}) = S_i(Az) = S_iz.$

which shows that (for all i and k) S_iz and $A_k(z)$ are others fixed point of the pair (A, S). In the same manner we can prove that T_pz and $B_u(z)$ are others fixed point of the pair (B, T). Now applying the uniqueness of common fixed points of the pairs $(A, S)$and (B, T) we get

$z = S_i z = A_k(z) = T_p z = B_u(z)$. Hence z is a common fixed point of S_i ,T_p , A_k and $B_u(i \in I_1, p \in I_2, k \in I_3$ and $u \in I_4)$.

By setting $S = \{S_1, S_2, S_3, \dots . S_m\}$, $T = \{T_1, T_2, T_3, \dots . T_n\}$, $A = \{A_1, A_2, A_3, \dots . A_r\}$ and $B = \{B_1, B_2, B_3, \dots . B_t\}$ we deduce the following

Corollary 3.3 Let S, T, A and B are the four self mappings of a metric spaces (X, d) such that S_m ,T_n , A_r and B_t satisfies the conditions (C_1), (C_2) and (C_3). If one of the $S_m(X)$,T_n (X), $A_r(X)$ or $B_t(X)$ is a complete subspace of X, then S, T, A and B have a unique common fixed point provided $(A, S)$and (B, T) commutes.

4. (E.A.) PROPERTY AND (CLR) PROPERTY

In 2002, Aamri and EI Moutawakil [1] introduced the notion of E.A. property follows:

Definition 4.1[1] Let f and g be two self mappings of a metric space (X, d). We say that f and g satisfy (E.A) property if there exists a sequence $\{x_n\}$ in X such that

$$lim_n f x_n = lim_n g x_n = t \text{ for some } t \text{ in} X.$$

Remark 4.1 [1] It is to be noted that weak compatibility and E.A. property are independent to each other.

In 2011, Sintunavarat and Kumam [17] coined the idea of common limit range property (called CLR) which relaxes the requirement of completeness.

Definition 4.2[17] Two self mappings f and g on a metric space (X, d) are are said to satisfy the common limit in the range of g property if

$$lim_n f x_n = lim_n g x_n = gt \text{ for some } t \text{ in } X.$$

In what follows, the common limit in the range of g property will be denoted by CLRg property.

Theorem 4.1 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying the following conditions:

(C1) $S(X) \subset B(X), T(X) \subset A(X)$;

(C2) $$d^3(Sx, Ty) \leq \; pmax \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax, Sx)d(By, Ty) \\ +d(Ax, Sx)d^2(By, Ty) \end{bmatrix}, \\ d(Ax, Sx)d(Ax, Ty)d(By, Sx), \\ d(Ax, Ty)d(By, Sx)d(By, Ty) \end{matrix} \right\} - \emptyset\{m(Ax, By)\}$$

where $$m(Ax,By) = \max\left\{\begin{array}{c} d^2(Ax,By), d(Ax,Sx)d(By,Ty), \\ d(Ax,Ty)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Ty) \\ +d(By,Sx)d(By,Ty) \end{bmatrix} \end{array}\right\},$$

for all $x, y \in X, p \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\,(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.

(C3) one of subspace AX or BX or SX or TX is closed subset of X,

(C4) The pairs (A, S) and (B, T) are weakly compatible,

(C5) The pairs (A, S) and (B, T) satisfies E.A. property.

Then S, T, A and B have a unique common fixed point.

Proof: Suppose that the pairs A, S satisfies E.A. property then there exists a sequence $\{x_n\}$ in x such that $lim_n Ax_n = lim_n Sx_n = z$ for some z in X. Since $S(X) \subset B(X)$, there exists a sequence $\{y_n\}$ in X such that $By_n = Sx_n$. Hence $lim_n By_n = z$. Also $T(X) \subset A(X)$ so there exists a sequence $\{w_n\}$ in X such that $Tw_n = Ax_n$. Hence $lim_n Tw_n = z$.

Now suppose that BX is closed subset of X, then there exists $u \in X$ such that $z = Bu$. Subsequently, we have

$lim_n Ax_n = lim_n Sx_n = lim_n Tw_n = lim_n By_n = z = Bu$, for some $u \in X$.

First we claim that $Tu = z$.

Now putting $x = x_n$ and $y = u$ in (C2)

$$d^3(Sx_n, Tu) \leq\ pmax\left\{\begin{array}{c} \frac{1}{2}\begin{bmatrix} d^2(Ax_n, Sx_n)d(Bu,Tu) \\ +d(Ax_n, Sx_n)d^2(Bu,Tu) \end{bmatrix}, \\ d(Ax_n, Sx_n)d(Ax_n, Tu)d(Bz, Sx_n), \\ d(Ax_n, Tu)d(Bu, Sx_n)d(Bu,Tu) \end{array}\right\} - \emptyset\{m(Ax_n, Bu)\}$$

where $$m(Ax_n, Bu) = \max\left\{\begin{array}{c} d^2(Ax_n, Bu), \\ d(Ax_n, Sx_n)d(Bu,Tu), \\ d(Ax_n, Tu)d(Bu, Sx_n), \\ \frac{1}{2}\begin{bmatrix} d(Ax_n, Sx_n)d(Ax_n, Tu) \\ +d(Bu, Sx_n)d(Bu,Tu) \end{bmatrix} \end{array}\right\} = 0$$

Therefore, we get

$$d^3(z,Tu) \le pmax\begin{Bmatrix}\frac{1}{2}[0+0],\\ 0,\\ 0\end{Bmatrix} - \emptyset(0)$$

This gives $z = Tu$ and hence $z = Tu = Bu$. Since $T(X) \subset A(X)$ therefore there exists $v \in X$ such that $Tu = z = Av$.

Next we claim that $Sv = z$. On setting $x = v$ and $y = u$ in (C2) we get

$$d^3(Sv,Tu) \le pmax\begin{Bmatrix}\frac{1}{2}\begin{bmatrix}d^2(Av,Sv)d(Bu,Tu)\\ +d(Av,Sv)d^2(Bu,Tu)\end{bmatrix},\\ d(Av,Sv)d(Av,Tu)d(Bu,Sv),\\ d(Av,Tu)d(Bu,Sv)d(Bu,Tu)\end{Bmatrix} - \emptyset\{m(Av,Bu)\}$$

where $\quad m(Av,Bu) = \max\begin{Bmatrix}d^2(Av,Bu), d(Av,Sv)d(Bu,Tu),\\ d(Av,Tu)d(Bu,Sv),\\ \frac{1}{2}\begin{bmatrix}d(Av,Sv)d(Av,Tu)\\ +d(Bu,Sv)d(Bu,Tu)\end{bmatrix}\end{Bmatrix}$

$$m(Av,Bu) = \max\begin{Bmatrix}d^2(z,z), d(z,Sv)d(z,z), d(z,z)d(z,Sv),\\ \frac{1}{2}[d(z,Sv)d(z,z) + d(z,Sv)d(z,z)]\end{Bmatrix} = 0$$

Therefore, we get

$$d^3(Sv,z) \le \ pmax\begin{Bmatrix}\frac{1}{2}[d^2(z,Sv)d(z,z) + d(z,Sv)d^2(z,z)],\\ d(z,Sv)d(z,z)d(z,Sv),\\ d(z,z)d(z,Sv)d(z,z)\end{Bmatrix} - \emptyset(0)$$

This implies that $Sv = z$ and hence $Sv = Av = z$ so $Av = Sv = Tu = Bu = z$. Since the pairs A, S and B, T are weakly compatible and v and u are their coincidence point respectively, so we have $Az = A(Sv) = S(Av) = Sz, \quad Bz = B(Tu) = T(Bu) = Tz.$

Now we prove that z is a common fixed point of A, B, S and T. For this we prove that $Sv = Tz$. On setting $x = v$ and $y = z$ in (C2) we get

$$d^3(Sv,Tz) \le pmax\begin{Bmatrix}\frac{1}{2}\begin{bmatrix}d^2(Av,Sv)d(Bz,Tz)\\ +d(Av,Sv)d^2(Bz,Tz)\end{bmatrix},\\ d(Av,Sv)d(Av,Tz)d(Bz,Sv),\\ d(Av,Tz)d(Bz,Sv)d(Bz,Tz)\end{Bmatrix} - \emptyset\{m(Av,Bz)\}$$

where $$m(Av,Bz) = \max\left\{\begin{matrix} d^2(Av,Bz), d(Av,Sv)d(Bz,Tz), \\ d(Av,Tz)d(Bz,Sv), \\ \frac{1}{2}\begin{bmatrix} d(Av,Sv)d(Av,Tz) \\ +d(Bz,Sv)d(Bz,Tz) \end{bmatrix} \end{matrix}\right\}$$

$$m(Av,Bz) = \max\left\{\begin{matrix} d^2(Sv,Tz), d(z,z)d(Bz,Bz), \\ d(Sv,Tz)d(Tz,Sv), \\ \frac{1}{2}\begin{bmatrix} d(z,z)d(Sv,Tz) \\ +d(Tz,Sv)d(Bz,Tz) \end{bmatrix} \end{matrix}\right\} = d^2(Sv,Tz)$$

Therefore, we get

$$d^3(Sv,Tz) \le pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(z,z)d(Bz,Tz) \\ +d(z,z)d^2(Bz,Tz) \end{bmatrix}, \\ d(z,z)d(Sv,Tz)d(Tz,Sv), \\ d(Sv,Tz)d(Tz,Sv)d(Bz,Tz) \end{matrix}\right\} - \emptyset\{d^2(Sv,Tz)\}$$

This implies that $Sv = Tz$. and hence $z = Sv = Tz$ and $z = Tz = Bz$ So z is a common fixed point of B and T. Also we can prove that $Sv = z$ is also a common fixed point of A and S. Similarly we can complete the proof for cases in which AX or SX or TX is closed subset of X. The uniqueness follows easily. This completes the proof.

Now we prove the following theorem as an application of Theorem 4.1.

Theorem 4.2 Let S,T, A and B be four mappings of a complete metric space (X,d) into itself satisfying $(C_1), (C_3), (C_4), (C_5)$ and the following condition:

$$\int_0^{d^3(Sx,Ty)} \varphi(t)\,dt \le \int_0^{\mathcal{M}(x,y)} \varphi(t)\,dt$$

$$\mathcal{M}(x,y) = pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax,Sx)d(By,Ty) \\ +d(Ax,Sx)d^2(By,Ty) \end{bmatrix}, \\ d(Ax,Sx)d(Ax,Ty)d(By,Sx), \\ d(Ax,Ty)d(By,Sx)d(By,Ty) \end{matrix}\right\} - \emptyset\{m(Ax,By)\}$$

where $$m(Ax,By) = \max\left\{\begin{matrix} d^2(Ax,By), \\ d(Ax,Sx)d(By,Ty), \\ d(Ax,Ty)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Ty) \\ +d(By,Sx)d(By,Ty) \end{bmatrix} \end{matrix}\right\},$$

for all $x, y \in X$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\,(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Further, where $\varphi : \mathrm{R}^+ \to \mathrm{R}^+$ is a Lebesgue - integrable over R^+function which is summable on each compact subset of R^+, non-negative, and such that for each $\in > 0$, $\int_0^{\in} \varphi(t)\mathrm{d}t > 0$. Then S, T, A and B have a unique common fixed point.

Proof. The proof of the theorem follows on the same lines of the proof of the Theorem 4.1. on setting φ (t) = 1.

Next we prove a theorem for (CLR) property along with weakly compatible and closeness of one the subspaces.

Theorem 4.3 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying the following conditions:

(C1) $S(X) \subset B(X), T(X) \subset A(X)$;

$$\text{(C2)}\quad d^3(Sx, Ty) \leq\ pmax\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} d^2(Ax, Sx)d(By, Ty) \\ +d(Ax, Sx)d^2(By, Ty)\end{bmatrix}, \\ d(Ax, Sx)d(Ax, Ty)d(By, Sx), \\ d(Ax, Ty)d(By, Sx)d(By, Ty)\end{matrix}\right\} - \emptyset\{m(Ax, By)\}$$

$$\text{where}\quad m(Ax, By) = \max\left\{\begin{matrix} d^2(Ax, By), d(Ax, Sx)d(By, Ty), \\ d(Ax, Ty)d(By, Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax, Sx)d(Ax, Ty) \\ +d(By, Sx)d(By, Ty)\end{bmatrix}\end{matrix}\right\}$$

for all $x, y \in X$, $p \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\,(t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.

(C3) one of subspace AX or BX or SX or TX is closed subset of X,

(C4) The pairs (A, S) and (B, T) are weakly compatible,

(C5) The pairs (A, S) satisfiies CLR_A property or the pair (B, T) satisfies CLR_B property.

Then S, T, A and B have a unique common fixed point.

Proof: If the pair B, T satisfies CLR_B property so there exists a sequence $\{x_n\}$ in X such that $lim_n Bx_n = lim_n Tx_n = z \in BX$. Sinch $T(X) \subset A(X)$ so for each $\{x_n\}$ in X there corresponds a sequence $\{y_n\}$ in X such that $Tx_n = Ay_n$. Therefore, $lim_n Ay_n = lim_n Tx_n = z \in BX$. Thus we have $lim_n Ay_n = lim_n Bx_n = lim_n Tx_n = z$.

Now suppose that BX is a closed subset of X, there exists a point $u \in X$ such that $Bu = z$.

Now we show that $lim_n Sy_n = z$. Putting $x = y_n$ and $y = x_n$. We have

$$d^3(Sy_n, Tx_n) \leq \ pmax\begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Ay_n, Sy_n)d(Bx_n, Tx_n) \\ +d(Ay_n, Sy_n)d^2(Bx_n, Tx_n) \end{bmatrix}, \\ d(Ay_n, Sy_n)d(Ay_n, Tx_n)d(Bx_n, Sy_n), \\ d(Ay_n, Tx_n)d(Bx_n, Sy_n)d(Bx_n, Tx_n) \end{Bmatrix} - \ \emptyset\{m(Ay_n, Bx_n)\}$$

$$m(Ay_n, Bx_n) = \max\begin{Bmatrix} d^2(Ay_n, Bx_n), d(Ay_n, Sy_n)d(Bx_n, Tx_n), \\ d(Ay_n, Tx_n)d(Bx_n, Sy_n), \\ \frac{1}{2}\begin{bmatrix} d(Ay_n, Sy_n)d(Ay_n, Tx_n) \\ +d(By, Sy_n)d(Bx_n, Tx_n) \end{bmatrix} \end{Bmatrix}$$

$$d^3(Sy_n, z) \leq \ pmax\begin{Bmatrix} \frac{1}{2}[d^2(z, Sy_n)d(z,z) + d(z, Sy_n)d^2(z,z)], \\ d(z, Sy_n)d(z,z)d(z, Sy_n), \\ d(z,z)d(z, Sy_n)d(z,z) \end{Bmatrix} - \emptyset\{m(z,z)\}$$

$$m(z,z) = \max\begin{Bmatrix} d^2(z,z), d(z, Sy_n)d(z,z), \\ d(z,z)d(z, Sy_n), \\ \frac{1}{2}\begin{bmatrix} d(z, Sy_n)d(z,z) \\ +d(z, Sy_n)d(z,z) \end{bmatrix} \end{Bmatrix} = 0$$

$$d^3(Sy_n, z) \leq \ \ pmax\begin{Bmatrix} \frac{1}{2}[d^2(z, Sy_n)d(z,z) + d(z, Sy_n)d^2(z,z)], \\ d(z, Sy_n)d(z,z)d(z, Sy_n), \\ d(z,z)d(z, Sy_n)d(z,z) \end{Bmatrix} - \emptyset\{m(z,z)\}$$

which implies that $lim_n d(Sy_n, z) = 0$. Hence $lim_n Ay_n = lim_n Bx_n = lim_n Tx_n = lim_n Sy_n = z = Bu$ for some u in X. From the proof of theorem 2.6 we can easily prove that z is a common fixed point of $A, B, S\ and\ T$. Also one can easily prove that the pair A, S satisfies $\mathrm{CLR_A}$ property. Similarly we can complete the proof for cases in which AX or TX or SX is a closed subset of X. This completes the proof.

Now we prove the following theorem as an application of Theorem 4.3.

Theorem 4.4 Let S,T, A and B be four mappings of a complete metric space (X, d) into itself satisfying $(C_1), (C_3), (C_4), (C_5)$ and the following condition:

$$\int_0^{d^3(Sx,Ty)} \varphi(t)\ dt \ \leq \int_0^{\mathcal{M}(x,y)} \varphi(t)\ dt$$

$$\mathcal{M}(x,y) = pmax \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} d^2(Ax,Sx)d(By,Ty) \\ +d(Ax,Sx)d^2(By,Ty) \end{bmatrix}, \\ d(Ax,Sx)d(Ax,Ty)d(By,Sx), \\ d(Ax,Ty)d(By,Sx)d(By,Ty) \end{Bmatrix} - \emptyset\{m(Ax,By)\}$$

where $$m(Ax,By) = \max \begin{Bmatrix} d^2(Ax,By), \\ d(Ax,Sx)d(By,Ty), \\ d(Ax,Ty)d(By,Sx), \\ \frac{1}{2}\begin{bmatrix} d(Ax,Sx)d(Ax,Ty) \\ +d(By,Sx)d(By,Ty) \end{bmatrix} \end{Bmatrix},$$

for all $x, y \in X$ and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset\ (t) = 0 \Leftrightarrow t = 0$ and $\emptyset(t) > 0$ for each $t > 0$. Further, where $\varphi : \mathrm{R}^+ \to \mathrm{R}^+$ is a Lebesgue - integrable over R^+function which is summable on each compact subset of R^+, non-negative, and such that for each $\in > 0$, $\int_0^{\in} \varphi(t)\mathrm{d}t > 0$. Then S, T, A and B have a unique common fixed point.

Proof. The proof of the theorem follows on the same lines of the proof of the Theorem 4.3. on setting φ (t) = 1.

CONCLUSION

In this paper, we prove a common fixed point theorem for six self mapping using weakly compatible mapping in a metric space. At the last we give corollaries and example in support of our theorem.

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AUTHORS CONTRIBUTIONS

All authors contributed equally to the writing of this manuscript. All authors read and approved the final version.

CONFLICT OF INTEREST

All the authors declare that they have no competing interests regarding this manuscript.

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