Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 6, 7523-7534
https://doi.org/10.28919/jmcs/5900
ISSN: 1927-5307

NUMERICAL SIMULATION OF DIFFERENTIAL –DIFFERENCE EQUATIONS WITH SMALL DELAY IN CONVECTIVE TERM AND REACTION TERM

KUMAR RAGULA[1], G.B.S.L. SOUJANYA[2,*]

[1]Department of Mathematics, Rajiv Gandhi University of Knowledge Technologies, Basar, India

[2]Department of Mathematics, University Arts & Science College, Kakatiya University, Warangal, India

Abstract: In this paper, we suggested a numerical scheme for solving singularly perturbed differential-difference equation with small shift. First, Taylor series used to replace the given problem as singularly perturbed boundary value problem and then subsequently a fourth order finite difference scheme is employed to solve this problem. Convergence of the method is evaluated. By considering numerical experiments, the effect of small shift on the boundary layer solution of the problem is demonstrated.

Keywords: singularly perturbed differential-difference equation; delay; tridiagonal system; truncation error.

2010 AMS Subject Classification: 65L10, 65L11, 65L12.

1. INTRODUCTION

In the area of differential equation having delay, calculating its solution was an immense task, and was of considerable significance due to versatility of these equations in the mathematical modeling of processes in various application fields [2, 17]. For the detailed theory of delay differential equations, also known as functional differential equations, one may refer to [4, 6]. The numerical

*Corresponding author
E-mail address: gbslsoujanya@gmail.com
Received April 21, 2021

solution of singular perturbation problems is very well described in [3 5, 7, 12, 15]. In [1], the author derived a numerical scheme using finite differences for the solution of functional differential equations of second order. The authors in [8] presented a numerical method for solving boundary layer problems having delay, which works well, when delay argument is bigger one as well as smaller one. Kumara Swamy et al. [9] suggested a numerical integration scheme for solving delay differential equations with twin layers or oscillatory behavior. Lange and Miura [10] analyzed on the problems that display layer behavior at one or both boundaries using Laplace transforms. In [11], the same authors studied the problems having solutions which have turning point behavior. Phaneendra et al. [13] suggested a higher order compact numerical scheme for the solution of boundary layer problem with delay term. The same authors in [14] used Trapezoidal rule of integration to solve the delay differential equations having dual layers or oscillatory structure. Soujanya and Reddy [16] employed Simpson's rule of integration for the problems of delay differential equations with layer composition.

2. PROBLEM DESCRIPTION

Consider a singularly perturbed linear two - point boundary value problem having small delay of the form:

$$\varepsilon\omega''(\theta)+p(\theta)\omega'(\theta-\delta)+q(\theta)\omega(\theta-\delta)+r(\theta)\omega(\theta)=f(\theta) \quad (1)$$

on (0, 1), under the boundary

$$\omega(\theta)=\varphi(\theta) \quad \text{on } -\delta\le\theta\le 0,\ \ \omega(1)=\gamma, \quad (2)$$

where the functions $p(\theta), q(\theta), r(\theta), f(\theta)$ and $\varphi(\theta)$) are smooth, ε is small perturbation parameter, $0<\varepsilon<<1$ and the delay parameter $\delta\ (0<\delta<1)$ is of $o(\varepsilon)$ satisfying the condition $(\varepsilon-\delta p(\theta)+\frac{\delta^2}{2}q(\theta))<0$ and $q(\theta)+r(\theta)\le 0, \forall\theta\in(0,1)$.

Expanding the terms $\omega(\theta-\delta)$ and $\omega'(\theta-\delta)$ by Taylor series, as the solution $\omega(\theta)$ of the problem Eq. (1) with Eq. (2) is sufficiently differentiable, we have

$$\omega'(\theta-\delta)\approx\omega'(\theta)-\delta\omega''(\theta) \quad (3a)$$

$$\omega(\theta-\delta)\approx\omega'(\theta)-\delta\omega'(\theta)+\frac{\delta^2}{2}\ \omega''(\theta) \quad (3b)$$

Using (3) in (1), we get an equivalent problem as

$$\left(\varepsilon - \delta p(\theta) + \frac{\delta^2}{2} q(\theta)\right)\omega''(\theta) + a(\theta)\omega'(\theta) + b(\theta)\omega(\theta) = f(\theta) \tag{4}$$

Eq. (4) is a second order singular perturbation problem.

Here, $$\tilde{\varepsilon} = (\varepsilon - \delta p(\theta) + \frac{\delta^2}{2} q(\theta)) \tag{5a}$$

$$a(\theta) = p(\theta) - \delta q(\theta) \tag{5b}$$

$$b(\theta) = q(\theta) + r(\theta) \tag{5c}$$

3. NUMERICAL METHOD

Discretize the region [0, 1] into N subregions of mesh size $h = 1/N$ so that $\theta_i = ih$, for i =0, 1, 2, …, N are the nodes.

At $\theta = \theta_i$ the Eq. (4) becomes

$$\tilde{\varepsilon}\omega_i'' + a_i\omega_i' + b_i\omega_i = f_i \tag{6}$$

Using the central difference formulae for ω'_i and ω_i'' in new form as

$$\omega_i'' \cong D^+D^-\omega_i - \frac{h^2}{12}\omega_i^{(4)} + R_1 \tag{7}$$

$$\omega_i' \cong D^{\pm}\omega_i - \frac{h^2}{6}\omega_i''' + R_2 \tag{8}$$

where $D^+D^-\omega_i = \frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}$, $D^{\pm}\omega_i = \frac{\omega_{i+1} - \omega_{i-1}}{2h}$, $R_1 = -\frac{2h^4\omega^{(6)}(\xi)}{6!}$

$R_2 = -\frac{h^4\omega^{(5)}(\eta)}{5!}$ for $\xi, \eta \in [\theta_{i-1}, \theta_{i+1}]$.

From the differential Eq. (6), we obtain ω'''_i, $\omega_i^{(4)}$ as

$$\omega_i''' = \left[-\frac{a_i}{\tilde{\varepsilon}}\omega''_i - \frac{(a_i' + b_i)}{\tilde{\varepsilon}}\omega'_i - \frac{b_i'}{\tilde{\varepsilon}}\omega + \frac{f'}{\tilde{\varepsilon}}\right]$$

$$\omega_i^{(4)} = \left[\frac{a_i{}^2}{\tilde{\varepsilon}^2} - \frac{(2a_i' + b_i)}{\tilde{\varepsilon}}\right]\omega''_i + \left[\frac{a_i(a_i' + b_i)}{\tilde{\varepsilon}^2} - \frac{(a_i'' + 2b_i')}{\tilde{\varepsilon}}\right]\omega'_i + \left[\frac{ab_i'}{\tilde{\varepsilon}^2} - \frac{b_i''}{\tilde{\varepsilon}}\right]\omega_i + \frac{1}{\tilde{\varepsilon}}f_i''$$

Using these expressions in Eq. (7), Eq. (8) and then from Eq. (6), we get

$$\tilde{\varepsilon}\left\{\begin{array}{l}\left[1 - \frac{h^2 a_i{}^2}{12\tilde{\varepsilon}^2} + \frac{h^2(2a_i' + b_i)}{12\tilde{\varepsilon}}\right]\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left[\frac{h^2(a_i'' + 2b_i')}{12\tilde{\varepsilon}} - \frac{h^2 a_i(a_i' + b_i)}{12\tilde{\varepsilon}}\right]\frac{(\omega_{i+1} - \omega_{i-1})}{2h} \\ -\left[\frac{h^2 b_i''}{12\tilde{\varepsilon}} - \frac{a_i b_i' h^2}{12\tilde{\varepsilon}^2}\right]\omega_i - \frac{h^2}{12\tilde{\varepsilon}}f_i''\end{array}\right\}$$

$$+a_i\left[\frac{a_i h^2}{6\tilde{\varepsilon}}\left(\frac{\omega_{i-1} - 2\omega_i + \omega_{i+1}}{h^2}\right) + \left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right)\frac{(\omega_{i+1} - \omega_{i-1})}{2h} + \frac{h^2}{6\tilde{\varepsilon}}b'\omega_i - \frac{h^2 f_i'}{6\tilde{\varepsilon}}\right] + b_i\omega_i = f_i \tag{9}$$

Eq. (9) can be written as $E_i\omega_{i-1} - F_i\omega_i + G_i\omega_{i+1} = H_i$, for $i = 1, 2, \ldots, N\text{-}1$ (10)

where

$$E_i = \frac{\tilde{\varepsilon}}{h^2} - \frac{a_i^2}{12\tilde{\varepsilon}} + \frac{(2a_i' + b_i)}{12} + \frac{a_i^2}{6\tilde{\varepsilon}} - \frac{h}{24}(a_i'' + 2b_i') + \frac{ha_i(a_i' + b_i)}{24\tilde{\varepsilon}} - \frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right)$$

$$F_i = \frac{2a_i^2}{12\tilde{\varepsilon}} - \frac{2\tilde{\varepsilon}}{h^2} - \frac{2(2a_i' + b_i)}{12} - \frac{2a_i^2}{6} + \frac{h^2 b_i''}{12} - \frac{h^2 a_i b_i'}{12\tilde{\varepsilon}} + \frac{h^2 a_i^2 b_i'}{6\tilde{\varepsilon}} + b_i$$

$$G_i = \frac{\tilde{\varepsilon}}{h^2} - \frac{a_i^2}{12\tilde{\varepsilon}} + \frac{(2a_i' + b_i)}{12} + \frac{a_i^2}{6\tilde{\varepsilon}} + \frac{h}{24}(a_i'' + 2b_i')_- \frac{ha_i(a_i' + b_i)}{24\tilde{\varepsilon}} + \frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right)$$

$$H_i = \frac{\tilde{\varepsilon}h^2}{12\tilde{\varepsilon}} f_i'' + \frac{a_i h^2}{6\tilde{\varepsilon}} f_i' + f_i$$

We solve the tridiagonal system Eq. (10) by using Thomas algorithm.

4. CONVERGENCE ANALYSIS

The system of Eq. (10) in matrix-vector form is given by

$$AY = C\,. \tag{11}$$

Here $A = (m_{i\,j})$, $1 \le i, j \le N\text{-}1$ is a tridiagonal matrix of order N-1, with

$$m_{i\,i+1} = \frac{\tilde{\varepsilon}}{h^2} - \frac{a_i^2}{12\tilde{\varepsilon}} + \frac{(2a_i' + b_i)}{12} + \frac{a_i^2}{6\tilde{\varepsilon}} + \frac{h}{24}(a_i'' + 2b_i')_{\frac{ha_i(a_i' + b_i)}{24\tilde{\varepsilon}}} + \frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right)$$

$$m_{i\,i} = \frac{2a_i^2}{12\tilde{\varepsilon}} - \frac{2\tilde{\varepsilon}}{h^2} - \frac{2(2a_i' + b_i)}{12} - \frac{2a_i^2}{6} + \frac{h^2 b_i''}{12} - \frac{h^2 a_i b_i'}{12\tilde{\varepsilon}} + \frac{h^2 a_i^2 b_i'}{6\tilde{\varepsilon}} + b_i$$

$$m_{i\,i-1} = \frac{\tilde{\varepsilon}}{h^2} - \frac{a_i^2}{12\tilde{\varepsilon}} + \frac{(2a_i' + b_i)}{12} + \frac{a_i^2}{6\tilde{\varepsilon}} - \frac{h}{24}(a_i'' + 2b_i') + \frac{ha_i(a_i' + b_i)}{24\tilde{\varepsilon}} - \frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right)$$

and $C = (d_i)$ is a column vector with $d_i = \frac{\tilde{\varepsilon}h^2}{12\tilde{\varepsilon}} f_i'' + \frac{a_i h^2}{6\tilde{\varepsilon}} f_i' + f_i$ for $i = 1, 2, \ldots, N\text{-}1$

with local truncation error

$$|\tau_i| \le \max_{x_{i-1} \le x \le x_{i+1}} \left\{\frac{h^4 a(\theta)}{5!}\left|\omega^{(5)}(\theta)\right|\right\} + \max_{x_{i-1} \le x \le x_{i+1}} \left\{\frac{2h^4\tilde{\varepsilon}}{6!}\left|\omega^{(6)}(\theta)\right|\right\}$$

i.e.,
$$|\tau_i| \le o(h^4) \tag{12}$$

and $Y = (\omega_0, \omega_1, \omega_2, \ldots, \omega_N)^t$.

Let $\overline{Y} = (\overline{\omega}_0, \overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_N)^t$ denotes the actual solution and the local truncation error be $T(h) = (T_0(h), T_1(h), \ldots, T_N(h))^t$, then we have

$$A\overline{Y} - T(h) = C \tag{13}$$

Using Eq. (11) and Eq. (13), we get

$$A\left(\overline{Y} - Y\right) = T(h) \tag{14}$$

Hence, the error equation is $AE = T(h)$ (15)

where $E = \overline{Y} - Y = (e_0, e_1, e_2, \ldots, e_N)^t$.

Clearly, we have

$$S_i = \sum_{j=1}^{N-1} m_{i\,j} = -\frac{\tilde{\varepsilon}}{h^2} + \frac{a_i{}^2}{12\tilde{\varepsilon}} - \frac{(2a_i' + b_i)}{12} - \frac{a_i{}^2}{6\tilde{\varepsilon}} + \frac{a_i h^2 b_i''}{12} - \frac{h^2 a_i b_i{}^1}{12\tilde{\varepsilon}} + \frac{a_i{}^2 h^2 b_i{}^1}{6\tilde{\varepsilon}} + bi + \frac{h a_i''}{24\tilde{\varepsilon}}$$
$$+ \frac{h b_i'}{12} - \frac{h a_i a_i'}{24\tilde{\varepsilon}} - \frac{h a_i b_i}{24\tilde{\varepsilon}} + \frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right), \text{ for } i = 1$$

$$S_i = \sum_{j=1}^{N-1} m_{i\,j} = b - \frac{h^2 b_i''}{12} - \frac{a_i b_i' h^2}{12\tilde{\varepsilon}} + \frac{a_i h^2 b_i'}{6\tilde{\varepsilon}} = b_i + o\left(h^2\right) = B_{i_0}, \text{ for } i = 2,3,\ldots,N-2$$

$$S_i = \sum_{j=1}^{N-1} m_{i\,j} = -\frac{\tilde{\varepsilon}}{h^2} + \frac{a_i{}^2}{12\tilde{\varepsilon}} - \frac{(2a_i' + b_i)}{12} - \frac{a_i{}^2}{\tilde{\varepsilon}} - \frac{h(a_i'' + 2b_i')}{24} + \frac{h a_i(a_i' + b_i)}{24\tilde{\varepsilon}}$$
$$-\frac{a_i}{2h}\left(1 + \frac{h^2}{6\tilde{\varepsilon}}(a_i' + b_i)\right) + \frac{a_i h^2 b_i''}{12} - \frac{h^2 a_i b_i{}^1}{12\tilde{\varepsilon}} + \frac{a_i{}^2 h^2 b_i{}^1}{6\tilde{\varepsilon}} + b_i, \text{ for } i = N-1$$

By choosing sufficiently small h, we get irreducible and monotone matrix A. It gives the existence of A^{-1} and its elements are non-negative.

Hence from Eq. (15), we get $E = A^{-1}T(h)$ (16)

Also, using the matrix theory [18], we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i}\, S_i = 1\ ,\ k = 1\ (1)\ N\text{-}1 \tag{17}$$

where $\overline{m}_{k,i}$ is (k, i) element of the matrix A^{-1}.

Therefore, for some i_0 between 1 and N-1, we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{B_{i_0}} \le \frac{1}{|B_{i_0}|} \tag{18}$$

From Eq. (16), Eq. (18) and Eq. (12), we get

$$e_j = \sum_{i=1}^{N-1} \bar{m}_{k,i} T_i(h), \quad j = 1,2,\ldots, N\text{-}1$$

which implies $$e_j \le \frac{o(h^4)}{|B_{i_0}|}, \qquad j = 1,2,\ldots, N\text{-}1 \tag{19}$$

where $B_{i_0} = b_i$.

Therefore, $$\|E\| = 0(h^4).$$

Hence, the proposed method has a fourth order convergent on uniform mesh.

5. Numerical Examples

We consider four examples to demonstrate the proposed method computationally. The maximum point-wise errors at all the mesh points are calculated using the double mesh principle $E_\varepsilon^N = \max_{0\le j\le N} \left|(\omega_\varepsilon^N)_j - (\omega_\varepsilon^{2N})_j\right|$ when exact solution is not available for the problems.

Example 1: Our first problem is the following differential equation with variable coefficients $-\varepsilon\omega''(\theta) + (1+\theta)\omega'(\theta-\delta) - e^{-2\theta}\omega(\theta-\delta) + e^{-\theta}\omega(x) = 0$ with $\omega(\theta) = 1, \quad -\delta \le \theta \le 0$ and $\omega(1) = -1$.

Table 1 shows the maximum absolute error values obtained by the present scheme for various values of δ and N with $\varepsilon = 10^{-2}$. The effect of the small parameter on the boundary layer solutions is shown in Figure. 1.

Example 2: Secondly, we consider the inhomogeneous equation

$-\varepsilon\omega''(\theta) + (1+\theta)\omega'(\theta-\delta) - e^{-2\theta}\omega(\theta-\delta) + e^{-\theta}\omega(x) = 1$ under the conditions $\omega(\theta) = 1, \quad -\delta \le \theta \le 0$ and $\omega(1) = -1$

The maximum absolute error values are given in Table 2 for $\varepsilon = 10^{-2}$ and different values of the delay parameter and N. Figure 2 shows the influence of the small parameter on the solutions for the boundary layer.

Example 3: $-\varepsilon\omega''(\theta) + (1+\theta)\omega'(\theta-\delta) + e^{-\theta}\omega(\theta) = 1$ with the conditions $\omega(\theta) = 1$, $-\delta \le \theta \le 0$ and $\omega(1) = -1$

Table 3 shows the maximum absolute error values for $\varepsilon = 10^{-2}$ with different values of the delay parameter. Figures 3 demonstrate the influence of small parameter on the solutions of boundary layers.

Example 4: $-\varepsilon\omega''(\theta) + (1+\theta)\omega'(\theta-\delta) + e^{-\theta}\omega(\theta) = 0$ with $\omega(\theta) = 1, \quad -\delta \le \theta \le 0$ and $\omega(1) = -1$

The maximum absolute error values for $\varepsilon = 10^{-2}$ are presented in Table 4 with different values of the delay parameter. Figures 4 display the influence of small parameter on the solutions of the boundary layer.

6. DISCUSSIONS AND CONCLUSION

For the solution of singularly perturbed differential equations with delay parameter, a numerical method has been developed which uses higher orders of finite differences. Four examples were solved for different values of delay and perturbation parameter in order to illustrate the applicability of the method. The maximum absolute error values are presented in Tables 1-4. It is observed that, the present method approximates the exact solution very well for $h > \varepsilon$ and $h \le \varepsilon$. It is also noticed that the error decreases as the number of subintervals N increases. The influence of the delay parameter on solutions was analyzed and shown in the graphs. Figures 1-4 demonstrate that the thickness of a boundary layer decreases as the value of the delay increases. In addition, the proposed approach is simple and efficient technique for addressing singularly perturbed differential – difference problems.

Table 1. The maximum absolute error values in the solution of Example 1

$\delta \downarrow \quad N \rightarrow$	32	64	128	256	512	1024
$\delta = 0.3\varepsilon$	1.0198e-01	1.2790e-02	9.4382e-04	7.8752e-05	1.6921e-05	4.2306e-06
$\delta = 0.5\varepsilon$	5.6795e-02	5.8630e-03	4.0715e-04	5.7102e-05	1.4280e-05	3.5704e-06
$\delta = 0.7\varepsilon$	3.3507e-02	2.9926e-03	2.2552e-04	4.9301e-05	1.2329e-05	3.0824e-06
$\delta = 0.9\varepsilon$	2.0769e-02	1.6566e-03	1.7301e-04	4.3298e-05	1.0827e-05	2.7070e-06

Table 2. The maximum absolute error values in the solution of Example 2

$\delta \downarrow \quad N \rightarrow$	32	64	128	256	512	1024
$\delta = 0.3\varepsilon$	6.8088e-02	8.6165e-03	6.5347e-04	6.1724e-05	1.3880e-05	3.4705e-06
$\delta = 0.5\varepsilon$	3.8149e-02	4.0039e-03	3.0046e-04	4.6857e-05	1.1719e-05	2.9300e-06
$\delta = 0.7\varepsilon$	2.2671e-02	2.0800e-03	1.7802e-04	4.0450e-05	1.0116e-05	2.5292e-06
$\delta = 0.9\varepsilon$	1.4173e-02	1.1768e-03	1.4185e-04	3.5507e-05	8.8797e-06	2.2201e-06

Table 3. The maximum absolute error values in the solution of Example 3

$\delta \downarrow \quad N \rightarrow$	32	64	128	256	512	1024
$\delta = 0.3\varepsilon$	1.0581e-01	1.1282e-02	6.9964e-04	1.7784e-04	4.6704e-05	1.2049e-05
$\delta = 0.5\varepsilon$	5.4409e-02	3.9143e-03	5.7951e-04	1.5082e-04	3.9894e-05	1.0153e-05
$\delta = 0.7\varepsilon$	2.8632e-02	1.9410e-03	4.9681e-04	1.3312e-04	3.4603e-05	8.7429e-06
$\delta = 0.9\varepsilon$	1.5050e-02	1.6882e-03	4.4015e-04	1.1863e-04	3.0442e-05	7.6628e-06

Table 4. The maximum absolute error values in the solution of Example 4.

$\delta \downarrow \quad N \rightarrow$	32	64	128	256	512	1024
$\delta = 0.3\varepsilon$	7.9279e-02	8.5562e-03	4.7217e-04	1.1931e-04	3.1452e-05	8.1579e-06
$\delta = 0.5\varepsilon$	4.1023e-02	3.0446e-03	3.9288e-04	1.0175e-04	2.7153e-05	6.9294e-06
$\delta = 0.7\varepsilon$	2.1789e-02	1.3287e-03	3.3763e-04	9.0673e-05	2.3712e-05	5.9990e-06
$\delta = 0.9\varepsilon$	1.1621e-03	1.1607e-03	2.9945e-04	8.1410e-05	2.0957e-05	5.2789e-06

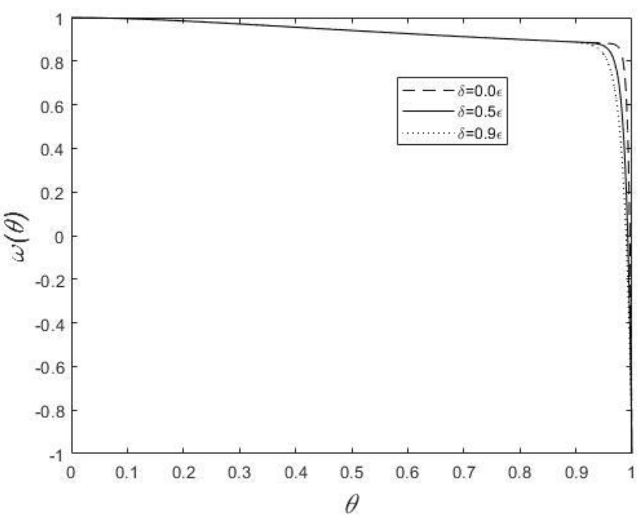

Fig. 1. Solution profile in Example 1 for different values of δ with $\varepsilon = 10^{-2}$

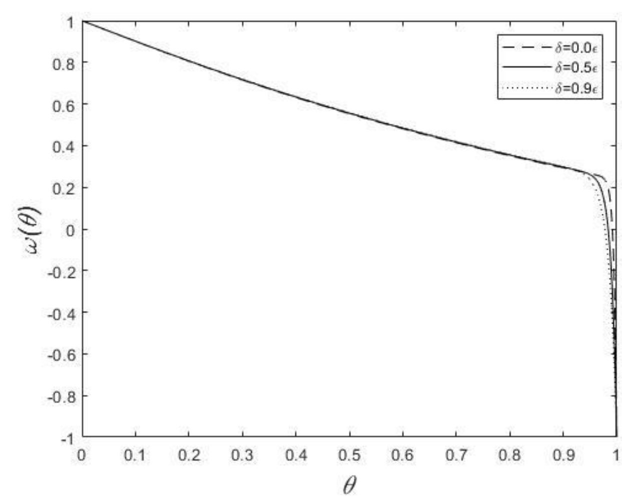

Fig. 2. Numerical solution of Example 2 for different values of δ with $\varepsilon = 10^{-2}$

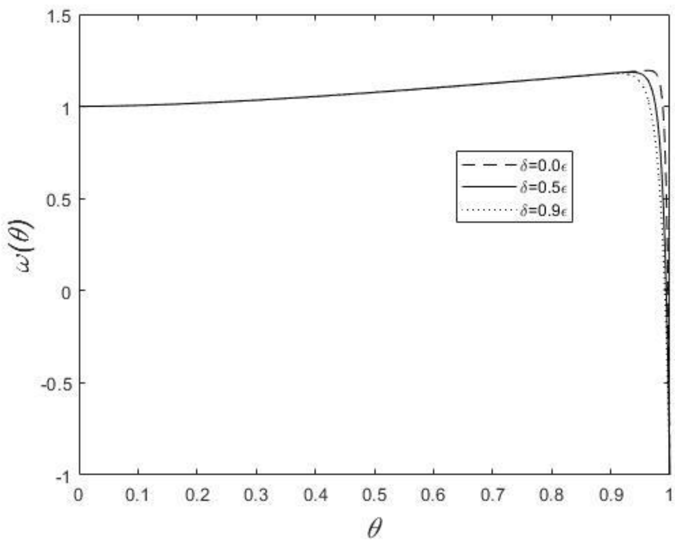

Fig. 3. Layer behavior in the solution of Example 3 for different values of δ with $\varepsilon = 10^{-2}$

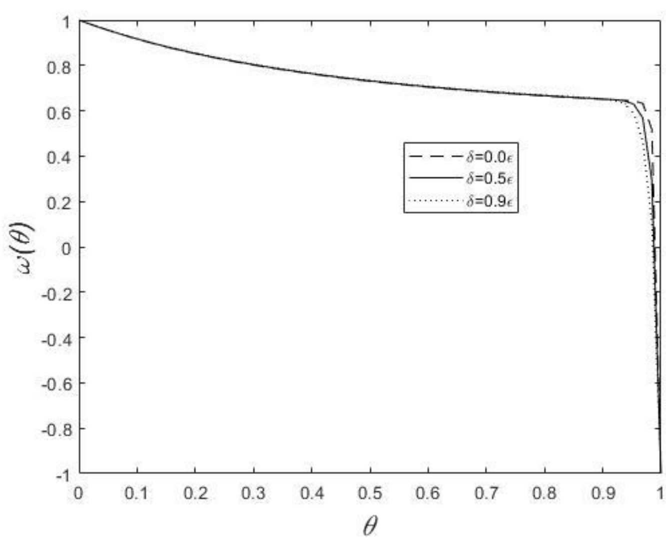

Fig. 4. Numerical solution of Example 4 for different values of δ with $\varepsilon = 10^{-2}$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] C.W. Cryer, The numerical solution of boundary value problems for second order functional differential equations by finite differences, Numer. Math. 20 (1973), 288-289.

[2] M.W. Derstine, F.A.H.H.M. Gibbs, D.L. Kaplan, Bifurcation gap in a hybrid optical system, Phys. Rev. A, 26 (1982), 3720–3722.

[3] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, Uniform numerical methods for problems with initial and boundary layers, Doole Press, Dublin, 1980.

[4] M.A. Feldstein, Discretization methods for retarded ordinary differential equations, Dissertation, University of California, Los Angeles, 1964.

[5] E.C. Gartland, Uniform higher-order difference schemes for singularly perturbed two point boundary value problems, Math. Comput. 48 (1987), 551 -564.

[6] J.K. Hale, Functional Differential Equations, Springer, New York, 1971.

[7] P.W. Hemker, A Numerical study to Stiff two-point boundary problems, Mathematisch Centrum, Amsterdam, 1977.

[8] M.K. Kadalbajoo, K.K. Sharma, A numerical method on finite difference for boundary value problems for singularly perturbed delay differential equations, Appl. Math. Comput. 197 (2008), 692-707.

[9] D. Kumara Swamy, K. Phaneendra, A. Benerji Babu, Y.N. Reddy, Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behavior, Ain Shams Eng. J. 6 (1) (2015), 391-398.

[10] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential – difference equations, SIAM J Appl. Math. 42 (3) (1982), 502-531.

[11] C.G. Lange, R.M. Miura, Singular perturbation analysis of boundary value problems for differential – difference equations V. Small shifts with layer behavior, SIAM J. Appl. Math. 54 (1) (1994), 249-272.

[12] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions, World Scientific, Singapore, 1996.

[13] K. Phaneendra, Y.N. Reddy, GBSL. Soujanya, A seventh order numerical method for singular perturbed differential-difference equations with negative shift, Nonlinear Anal., Model. Control, 16 (2) (2011), 206–219.

[14] K. Phaneendra, D. Kumaraswamy, Y.N. Reddy, Computational method for singularly perturbed delay differential equations with twin layers or oscillatory behaviour, Khayyam J. Math. 4 (2018), 110–122.

[15] H.-G. Roos, M. Stynes, L. Tobiska, Numerical Methods for Singularly Perturbed Differential Equations, Springer Berlin Heidelberg, Berlin, Heidelberg, 1996.

[16] GBSL. Soujanya, Y. N. Reddy, Computational method for singularly perturbed delay differential equations with layer or oscillatory behaviour, Appl. Math. Inform. Sci. 10 (2016), 527-536.

[17] R.B. Stein, Some models of neuronal variability, Biophys. J. 7 (1) (1967), 37–68.

[18] R.S. Varga, Matrix iterative analysis, Prentice-Hall, Englewood Cliffs, 1962.