Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4640-4649
https://doi.org/10.28919/jmcs/5901
ISSN: 1927-5307

QUALITATIVE RESULTS FOR FINITE SYSTEM OF R-L FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

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Abstract. Qualitative results such as existence and uniqueness of finite system of Riemann-Liouville (R-L) fractional differential equations with initial time difference are obtained. Monotone technique coupled with method of lower and upper solutions is developed to obtain existence and uniqueness of solutions of finite system of R-L fractional differential equations with initial time difference.

Keywords: existence and uniqueness; initial time difference; fractional differential equations; monotone method.

2010 AMS Subject Classification: 34A12, 34C60, 34A45.

1. Introduction

Due to wide applications of fractional calculus in sciences, engineering, nature and social sciences numerous methods of solving fractional differential equations are developed [1, 8, 9]. V.Lakshmikantham et.al [6] obtained local and global existence results for solutions of Riemann-Liouville fractional differential equations.The Caputo fractional differential equation with periodic boundary conditions have been studied in [3, 4] and developed monotone method. Existence and uniqueness of solution of Riemann-Liouville fractional differential equation with

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Received April 21, 2021

integral boundary conditions is proved in [11, 12]. Monotone method for Riemann-Liouville fractional differential equations with initial conditions is established by McRae [7]. Vasundhara Devi considered [2] the general monotone method for periodic boundary value problem of Caputo fractional differential equation. Recently, initial value problems involving Riemann-Liouville fractional derivative was studied by authors [5, 13]. Yaker et.al. proved existence and uniqueness of solutions of fractional differential equations with initial time difference for locally Holder continuous functions [14]. Authors have generalized these results for the class of continuous functions [10] and extended for system. In this paper, we consider the finite system of Riemann-Liouville fractional differential equations with initial time difference when the function on the right hand side is mixed quasi-monotone and construct two monotone convergent sequences to obtain existence and uniqueness of solution for the finite system.
The paper is organized as follows: In section 2, basic definitions and results are given. Section 3 is devoted to obtain main results .

2. Basic Results

Some basic definitions and results used for the development of monotone technique for the problem are given in this section.
The Riemann-Liouville fractional derivative of order $q(0<q<1)$ [**?**] is defined as

$$D_a^q u(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-q-1}u(\tau)d\tau, \quad \text{for } a \leq t \leq b.$$

Lemma 2.1. [2] *Let $m \in C_p(J,\mathbb{R})$ and for any $t_1 \in (t_0,T]$ we have $m(t_1)=0$ and $m(t)<0$ for $t_0 \leq t \leq t_1$. Then it follows that $D^q m(t_1) \geq 0$.*

Theorem 2.1. [11] *Let $v,w \in C_p([t_0,T],\mathbb{R}), f \in C([t_0,T]\times\mathbb{R},\mathbb{R})$ and*

(i) $D^q v(t) \leq f(t,v(t))$

and

(ii) $D^q w(t) \geq f(t,w(t))$,

$t_0 < t \leq T$. Assume $f(t,u)$ satisfy one sided Lipschitz condition

$$f(t,u)-f(t,v) \leq L(u-v), \quad u \geq v, L>0.$$

Then $v^0 < w^0$, *where* $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ *and* $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$, *implies* $v(t) \leq w(t), t \in [t_0, T]$.

Corollary 2.1. *The function* $f(t,u) = \sigma(t)u$, *where* $\sigma(t) \leq L$, *is admissible in Theorem 2.1 to yield* $u(t) \leq 0$ *on* $t_0 \leq t \leq T$.

The results proved by Yakar et.al. for the following problem

$$D^q u(t) = f(t,u),\ u(t)(t-t_0)^{1-q}|_{t=t_0} = u^0 \tag{2.1}$$

where $0 < q < 1$, $f \in C[R^+ \times \mathbb{R}, \mathbb{R}]$, are generalized by authors [10] for class of continuous functions $u(t)$.

The corresponding Volterra fractional integral is given by

$$u(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s,u(s)) ds \tag{2.2}$$

where

$$u^0(t) = \frac{u(t)(t-t_0)^{1-q}}{\Gamma(q)}$$

and that every solution of (2.2) is a solution of (2.1).

In this paper, we develop monotone technique coupled with lower and upper solutions for the class of continuous functions for the following finite system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution for the problem.

$$D^q u_i(t) = f_i(t, u_1(t), u_2(t), \ldots, u_N(t)), \quad u_i(t)(t-t_0)^{1-q}|_{t=t_0} = u_i{}^0 \tag{2.3}$$

where $i = 1, 2, \ldots, N, t \in J = [t_0, T] \quad f_i$ in $C(J \times \mathbb{R}^n, \mathbb{R}), 0 < q < 1$.

Definition 2.1. *A pair of functions* $v = (v_1, v_2, \ldots, v_N)$ *and* $w = (w_1, w_2 \ldots, w_N)$ *in* $C_p(J, \mathbb{R}), p = 1-q$ *are said to be ordered lower and upper solutions* $(v_1, v_2, \ldots, v_N) \leq (w_1, w_2, \ldots, w_N)$ *of the problem* (2.3) *if*

$$D^q v_i(t) \leq f_i(t, v_1(t), v_2(t), \ldots, v_N), \qquad v_i(t)(t-t_0)^{1-q}|_{t=t_0} = v_i{}^0$$

and

$$D^q w_i(t) \geq f_i(t, w_1(t), w_2(t), \ldots, w_N(t)), \qquad w_i(t)(t-t_0)^{1-q}|_{t=t_0} = w_i{}^0.$$

Definition 2.2. *A function $f_i \in C([0,T] \times R^N, R^N)$ is said to satisfy mixed quasimonotone property if for each $i, f_i(t, u_i, [u]_{r_i}, [u]_{s_i})$ is monotone nondecreasing in $[u]_{r_i}$ and monotone nonincreasing in $[u]_{s_i}$.*

When either r_i or s_i is equal to zero a special case of the mixed quasimonotone property is defined as follows:

Definition 2.3. *A function $f_i \in C([0,T] \times R^N, R^N)$ is said to be quasimonotone nondecreasing (nonincreasing) if for each $i, u_i \leq v_i$ and $u_j = v_j, i \neq j$, then $f_i(t, u_1, u_2, ...,$ $u_N) \leq f_i(t, v_1, v_2, ..., v_N)$ $\left(f_i(t, u_1, ..., u_N) \geq f_i(t, v_1, ..., v_N)\right)$.*

3. Qualitative Results

This section is devoted to develop monotone method for system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution of the problem (2.3).

Theorem 3.1. *Assume that*

(E_1) *$v = (v_1, v_2, \ldots, v_N) \in C_p[t_0, t_0+T], \mathbb{R}], t_0, T > 0, w = (w_1, w_2, \ldots, w_N) \in C_p^*[\tau_0, \tau_0 + T], \mathbb{R}]$ is continuous and $p = 1-q$ where*
$C_p(J,R) = \{u(t) \in C(J,R)$ and $u(t)(t-t_0)^p \in C(J,R)\}, J = [t_0, t_0+T]$,
$C_p^*(J^*,R) = \{u(t) \in C(J^*,R)$ and $u(t)(t-t_0)^p \in C(J^*,R)\}, J^* = [\tau_0, \tau_0+T]$,
$f \in C[[t_0, \tau_0+T] \times \mathbb{R}, \mathbb{R}]$ and

$$D^q v_i(t) = f_i(t, v_1(t), v_2(t), \ldots, v_N(t)), \quad t_0 \leq t \leq t_0+T,$$

$$D^q w_i(t) = f_i(t, w_1(t), w_2(t), \ldots, w_N(t)), \quad \tau_0 \leq t \leq \tau_0+T,$$

$$v_i^0 \leq u_i^0 \leq w_i^0, \text{ where } v_i^0 = v_i(t)(t-t_0)^{1-q}\}_{t=t_0}, w_i^0 = w_i(t)(t-\tau_0)^{1-q}\}_{t=\tau_0}$$

(E_2) *$f_i(t, u_1, u_2, \ldots, u_N)$ is mixed quasimonotone in t for each u_i and $v_i(t) \leq w_i(t+\eta), t_0 \leq t \leq t_0+T$, where $\eta = \tau_0 - t_0$*

(E_3) *f_i satisfies one-sided Lipschitz condition,*

$$f_i(t,u_1,u_2,\dots,u_N) - f_i(t,\overline{u}_1,\overline{u}_2,\dots,\overline{u}_N) \geq -M_i[u_i - \overline{u}_i], for\ \overline{u}_i \leq u_i, M_i \geq 0.$$

Then there exists monotone sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ such that

$$\{v^n(t)\} \to v(t) = (v_1,v_2,\dots,v_n) \quad and \quad \{w^n(t)\} \to w(t) = (w_1,w_2,\dots,w_n)\ as\ n \to \infty$$

where $v(t)$ and $w(t)$ are minimal and maximal solutions of the problem (2.3) *respectively.*

Proof. Let $w_{i0}(t) = w_i(t+\eta)$ and $v_{i0}(t) = v_i(t)$ $i = 1,2$ for $t_0 \leq t \leq t_0 + T$, where $\eta = \tau_0 - t_0$. Since $f(t,u_1,u_2,\dots,u_N)$ is quasimonotone nondecreasing in t for each u_i we have $D^q w_0(t) = D^q w_i(t+\eta) \geq f(t+\eta, w_1(t+\eta), w_2(t+\eta),\dots,u_N(t+\eta)) \geq f(t,w_1(t),w_2(t),\dots,w_N(t))$
and

$$w_0^0 = w_{i0}(t)(t-t_0)^{1-q}\}_{t=t_0} = w+i(t+\eta)(t-t_0)^{1-q}\}_{t=t_0} = w_i(t)(t-t_0)^{1-q}\}_{t=t_0} = w^0$$

Also,

$$D^q v_{i0}(t) = D^q v_i(t) \leq f(t,v_{10}(t),v_{20}(t),\dots,v_{N0}(t))$$

and

$$v_{i0}^0 = v_{i0}(t)(t-t_0)^{1-q}\}_{t=t_0} = v_i(t)(t-t_0)^{1-q}\}_{t=t_0} = v_i^0, v_i^0 \leq u_i^0 \leq w_i^0$$

which proves that v_{i0} and w_{i0} are lower and upper solutions of IVP (2.3) respectively.
For any $\theta(t) = (\theta_1,\theta_2,\dots,\theta_N)$ in $C_p(J,\mathbb{R})$ such that for $\alpha_{i0} \leq \theta_i \leq \beta_{i0}, \alpha_{i0} \leq \theta_i \leq \beta_{i0}$ on J, consider the following linear system of fractional differential equations

$$\begin{aligned} D^q u_i(t) &= f_i(t,\theta_1(t),\dots,\theta_N(t)) - M_i[u_i(t) - \theta_i(t)], \\ u_i(0) &= u_i(t)(t-t_0)^{1-q}\}_{t=t_0} \end{aligned} \tag{3.1}$$

Since the right hand side of IVP (3.1) satisfies Lipschitz condition, unique solution of IVP (3.1) exists on J.
For each $\eta(t)$ and $\mu(t)$ in $C_p(J,\mathbb{R})$ such that $v_i^0(0) \leq \eta_i(t), w_i^0(0) \leq \mu_i(t)$, define a mapping A by $A[\eta,\mu] = u(t)$ where $u(t)$ is the unique solution of the problem (3.1). Firstly, we prove that

(A_1) $v_i^0 \leq A[v_i^0, w_i^0], \quad w_i^0 \geq A[w_i^0, v_i^0]$

(A_2) A possesses the monotone property on the segment

$$[v^0,w^0] = \left\{(u_1,u_2,\dots,u_N) \in C(J,\mathbb{R}) : v_i^0 \le u_i \le w_i^0\right\}.$$

Set $A[v^0,w^0] = v^1(t)$, where $v^1(t) = (v_1^1, v_2^1, \dots, v_n^1)$ is the unique solution of system (3.1) with $\eta_i = v_i^0(0)$.

Setting $p_i(t) = v_i^0(t) - v_i^1(t)$ we see that

$$\begin{aligned} D^q p_i(t) &\le f_i(t, v_1^0(t), v_2^0(t), \dots, v_N^0(t)) - f_i(t, v_1^1(t), v_2^1(t), \dots v_N^0(t)) \\ &\le -M_i p_i(t) \\ \text{and} \quad & p_i(t) \le 0. \end{aligned}$$

Applying Corollary 2.1, we get $p_i(t) \le 0 \quad \text{on} \quad 0 \le t \le T$ and hence $v_i^0(t) - v_i^1(t) \le 0$ which implies $v_i^0 \le A[v^0,w^0]$. Set $A[v^0,w^0] = w^1(t)$, where $w^1(t) = (w_1^1, w_2^1)$ is the unique solution of the problem (3.1) with $\mu_i = w_i^0(t)$. Setting $p_i(t) = w_i^0(t) - w_i^1(t)$, similarly by Corollary 2.1, we have $w_i^0 \ge w_i^1$. Hence $w^0 \ge A[w^0,v^0]$. Let $\eta, \beta, \mu \in [v^0,w^0]$ with $\eta \le \beta$. Suppose that $A[\eta,\mu] = u(t), A[\beta,\mu] = v(t)$. Then setting $p_i(t) = u_i(t) - v_i(t)$ we find that

$$\begin{aligned} D^q p_i(t) &= f_i(t, \eta_1, \dots, \eta_N) - f_i(t, \beta_1, \dots, \beta_N) - M_i[u_i(t) - \eta_i(t)] \\ &\qquad + M_i[v_i(t) - \beta_i(t)] \\ &\le -M_i p_i(t) \\ \text{and} \quad & p_i(t) \le 0. \end{aligned}$$

As before in (A_1), we have $A[\eta,\mu] \le A[\beta,\mu]$.

Similarly, if $\eta(t), \gamma(t), \mu(t) \in [v^0,w^0]$ be such that $\gamma(t) \le \mu(t)$. Suppose that $A[\eta,\gamma] = u(t), A[\eta,\mu] = v(t)$ we can prove that $A[\eta,\gamma] \ge A[\eta,\mu]$. Thus it follows that the mapping A possesses monotone property on the segment $[v^0,w^0]$.

Define the sequences

$$v_i^n(t) = A[v_i^{n-1}, w_i^{n-1}], \quad w_i^n(t) = A[w_i^{n-1}, v_i^{n-1}]$$

on the segment $[v^0, w^0]$ by

$$D^q v_i^n(t) = f_i(t, v_1^{n-1}, \ldots, v_N^{n-1}) - M_i[v_i^n - v_i^{n-1}], \quad v_i^n(t)(t-t_0)^{1-q}|_{t=t_0} = v_i^{n0}$$
$$D^q w_i^n(t) = f_i(t, w_1^{n-1}, \ldots, w_N^{n-1}) - M_i[w_i^n - w_i^{n-1}], \quad w_i^n(t)(t-t_0)^{1-q}|_{t=t_0} = w_i^{n0}$$

From (A_1), we have $v_i^0 \leq v_i^1, \quad w_i^0 \geq w_i^1$. Assume that $v_i^{k-1} \leq v_i^k, \quad w_i^{k-1} \geq w_i^k$. To prove $v_i^k \leq v_i^{k+1}, \quad w_i^k \geq w_i^{k+1}$ and $v_i^k \geq w_i^k$, define $p_i(t) = v_i^k(t) - v_i^{k+1}(t)$. Thus

$$\begin{aligned}
D^q p_i(t) &= f_i(t, v_1^{k-1}, \ldots, v_N^{k-1}) - M_i[v_i^k - v_i^{k-1}] \\
&\qquad - \{f_i(t, v_1^k(t), \ldots, v_N^k(t)) - M_i[v_i^{k+1}(t) - v_i^k(t)]\} \\
&\leq -M_i[v_i^{k-1} - v_i^k] - M_i[v_i^k - v_i^{k-1}] + M_i[v_i^{k+1}(t) - v_i^k(t)] \\
&\leq -M_i[v_i^k(t) - v_i^{k+1}(t)] \\
&\leq -M_i p_i(t)
\end{aligned}$$

$$\text{and} \quad p_i(t) \leq 0.$$

It follows from Corollary 2.1 that $p_i(t) \leq 0$, which gives $v_i^k(t) \leq v_i^{k+1}(t)$. Similarly we can prove $w_i^k(t) \geq w_i^{k+1}(t)$ and $v_i^k(t) \geq w_i^k(t)$. By induction, it follows that

$$v_i^0(t) \leq v_i^1(t) \leq v_i^2(t) \leq \ldots \leq v_i^n(t) \leq w_i^n(t) \leq w_i^{n-1}(t) \leq \ldots \leq w_i^1(t) \leq w_i^0(t).$$

Thus the sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ are bounded from below and bounded from above respectively and monotonically nondecreasing and monotonically non-increasing on J. Hence point-wise limit exist and are given by

$$\lim_{n\to\infty} v_i^n(t) = v_i(t), \quad lim_{n\to\infty} w_i^n(t) = w_i(t) \text{ on } J$$

.

Using corresponding Volterra fractional integral equations

$$v_i^n(t) = v_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, v_1^n(s), \ldots, v_N^n(s)) - M_i[v_i^n - v_i^{n-1}] \right\} ds$$
$$w_i^n(t) = w_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \left\{ f_i(s, w_1^n(s), \ldots, w_N^n(s)) - M_i[w_1^n - w_1^{n-1}] \right\} ds,$$

as $n \to \infty$, we get

$$v_i(t) = \frac{v_i^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}f(s, v_1^n(s), \ldots, v_N^n(s))ds$$
$$w_i(t) = \frac{w_i^0(t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}f(s, w_1^n(s), \ldots, w_N^n(s))ds$$

where $v_i^0 = v_i(t)(t-t_0)^{1-q}|_{t=t_0}$. It follows that $v(t)$ and $w(t)$ are solutions of system (2.3). Lastly, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solution of the problem (2.3). Let $u(t) = (u_1, \ldots, u_N)$ be any solution of (2.3) other than $v(t)$ and $w(t)$, so that there exists k such that $v_i^k(t) \le u_i(t) \le w_i^k(t)$ on $[t_0, T]$ and set $p_i(t) = v_i^{k+1}(t) - u_i(t)$ so that

$$\begin{aligned} D^q p_i(t) &= f_i(t, v_1^k, \ldots, v_N^k) - M_i[v_i^{k+1} - v_i^k] - f_i(t, u_1, \ldots, u_N) \\ &\ge -M_i p_i(t) \\ \text{and} \quad p_i(t) &\ge 0. \end{aligned}$$

Thus $v_i^{k+1}(t) \le u_i(t)$ on J. Since $v_i^0(t) \le u_i(t)$ on J, by induction it follows that $v_i^k(t) \le u_i(t)$ for all k. Similarly, we can prove $u_i \le w_i^k$ for all k on J. Hence $v_i^k(t) \le u_i(t) \le w_i^k(t)$ on J. Taking limit as $n \to \infty$, it follows that $v_i(t) \le u_i(t) \le w_i(t)$ on J. Now, we obtain the uniqueness of solution of the problem (2.3) in the following

Theorem 3.2. *Assume that*

(U_1) *Assumptions* (E_1) *and* (E_3) *of Theorem 3.1 holds.*

(U_2) $f_i = f_i(t, u_1, u_2, ..., u_N)$ *satisfies Lipschitz condition,*

$$|f_i(t, u_1, u_2, u_N) - f_i(t, \overline{u}_1, \overline{u}_2, \overline{u}_N)| \ge -M_i|u_i - \overline{u}_i|, M_i \ge 0$$

then there exists unique solution of the problem (2.3).

Proof. We know that $v(t) \leq w(t)$. It is sufficient to prove $v(t) \geq w(t)$. For this, if $p_i(t) = w_i(t) - v_i(t)$ then we have

$$\begin{aligned} D^q p_i(t) &\leq D^q w_i(t) - D^q v_i(t) \\ &\leq M_i(w_i(t) - v_i(t)) \\ &\leq -M_i p_i(t) \\ \text{and} \quad p_i(t) &= 0. \end{aligned}$$

Applying Corollary 2.1, we obtain $p_i(t) \leq 0$ implies $w_i(t) \leq v_i(t)$. Thus $v(t) = u(t) = w(t)$ is the unique solution of (2.3) on $[t_0, t_0 + T]$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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