Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 5063-5070
https://doi.org/10.28919/jmcs/5902
ISSN: 1927-5307

# TOTAL GRAPH ASSOCIATED TO $\Gamma-$SEMIGROUP OF $\mathbb{Z}_n$

PINKU CHANDRA DEY*, KUNTALA PATRA

Department of Mathematics, Gauhati University, Guwahati, India



**Abstract.** In this paper we have considered two sets $\mathbb{Z}_n$ (set of integers modulo n) and $\Gamma = U_n$ (set of unit elements of $\mathbb{Z}_n$) so that $\mathbb{Z}_n$ forms a $\Gamma-$semigroup with respect to $U_n$. Now a new graph structure associated to this $\Gamma-$semigroup can be introduced by taking all the elements of $\mathbb{Z}_n$ as the vertices of the graph and any two distinct vertices $x$ and $y$ of $\mathbb{Z}_n$ are adjacent if there exist an $\alpha$ in $U_n$ such that $x+\alpha+y \equiv 0(modn)$. This definition is a slight modification of the concept of total graph defined by [2]. We call this graph structure the total graph associated to $\Gamma-$semigroup $(\mathbb{Z}_n(\Gamma))$ and denote it by $G(\mathbb{Z}_n(\Gamma))$. This paper focuses on determination of some graphical parameters like the degree of the vertices, number of edges, Girth, Diameter, Planarity and Traversibility of $G(\mathbb{Z}_n(\Gamma))$.

**Keywords:** total graph; $\Gamma-$semigroup.

**2010 AMS Subject Classification:** 05C25, 05C31.

## 1. INTRODUCTION

In 1964 N. Nobusawa[7] introduced a new notion named as $\Gamma$ to study the algebraic structure in his paper "On generalization of ring theory" and named this as $\Gamma$ algebraic system or as $\Gamma-$ring. Motivated by this concept M.K. Sen[8] 1981 extended the definition of $\Gamma-$ring to $\Gamma-$semigroup and defined the structure by taking two non empty sets $S$ and $\Gamma$ and by taking two mappings $S\times\Gamma\times S \to S$ written as $(a,\alpha,b) \to a\alpha b$ and $\Gamma\times S\times\Gamma \to \Gamma$ written as $(\alpha,a,\beta) \to$

*Corresponding author

E-mail address: pikulmg@gmail.com

Received April 21, 2021

$\alpha a\beta$ which satisfies the identity $a\alpha(b\beta c) = a(\alpha b\beta)c = (a\alpha b)\beta c$, for all $a,b,c \in S$ and $\alpha,\beta \in \Gamma$.

M.K. Sen and N.K. Saha[9] in their paper redefined $\Gamma-$semigroup which make the previous definition little weak. They modified the relation slightly in their paper by considering two non empty sets $S$ and $\Gamma$ and the set $S$ is said to be a $\Gamma-$semigroup if for any $a,b,c \in S$ and for any $\alpha,\beta \in \Gamma$, the following relation holds:

1. $a\alpha b \in S$
2. $(a\alpha b)\beta c = a\alpha(b\beta c)$

Graph theory plays an important role to represent various algebraic structure with the help of graph. This establishment of relation of algebraic structure and Graph structure was first introduced by I. Beck[5]where he made a connection between ring theory and graph theory by defining zero divisor graph where any two distinct vertices of a commutative ring are made to adjacent if and only if their product is zero.In his paper he was interested in coloring of the vertices of the commutative ring. this definition was modified by various other mathematician to give new structure. In 2008 D.F. Anderson and A. Badwi[2] in their paper introduced a new graph structure and named it as total graph of a commutative ring where any two distinct vertices of a commutative ring$R$ are adjacent if their sum contains in zero divisor of $R$ i.e $Z(R)$. Motivated by these concepts we have introduced a graph structure ”total graph associated to $\Gamma-$semigroup of $\mathbb{Z}_n$ and denote it by $G(\mathbb{Z}_n(\Gamma))$ by considering the non empty sets $S = \mathbb{Z}_n$ and $\Gamma =$units elements of $\mathbb{Z}_n$ clearly $\mathbb{Z}_n$ forms a $\Gamma-$semigroup and any two distinct vertices $x$ and $y$ of $\mathbb{Z}_n$ are adjacent if and only if we can find an $\alpha$ such that $x+\alpha+y \equiv 0(mod n)$. This definition is a slight modification of the definition given in[2].

## 2. Preliminaries

We have referred to Harary[4], C. Godsil and Royle[1] for some basic standard terminology. A graph $G$ involving two sets known as vertex set $V(G)$ and edge set $E(G)$ where any two elements of $V(G)$ are connected by some relation and constitutes the edge set $E(G)$ and the ordered pair $G(V,E)$ together defines a graph structure. and we say that $a$ and $b$ of the vertex set $V(G)$ are adjacent if there exist a edge between this two. The degree of a vertex $a$ is defined as the minimum number of vertices adjacent to $a$ and is denoted by $deg(a)$.If degree of each vertices are same then the graph $G(V,E)$ is termed as regular graph and if each of the vertices are adjacent to all the remaining vertices except itself (simple graph)then the graph is a complete graph. The graph $G(V,E)$ is said to be a connected graph if any two distinct elements of the vertex set are adjacent.

A sequence of the type $v_0, e_1, v_1, e_2, v_2, .....e_n, v_n$ of the vertices and edges constitutes a walk. If degree of each vertices of a graph is two then then the graph is termed as cycle.

A graph is said to be Eulerian if it can be traversed by by crossing every edge of the graph exactly once and if the graph can be traversed by crossing each vertices exactly once then it is said to be Hamiltonian graph.

A graph is said to be a planar graph if it can be embedded in a plane where no two edges intersects except at the common vertices of the graph.

## 3. MAIN RESULTS

$\phi(n)$, known as Eulers $\phi$ function is defined as $\phi(n) = \{m \in \mathbb{Z} : (m,n) = 1 \text{ and } m < n\}$. This function is used more frequently as we proceed further.

**Theorem**: The degree of any vertex $p$ of $\mathbb{Z}_n$ of the total graph associated to $\Gamma-$semigroup of $\mathbb{Z}_n$ $(G(\mathbb{Z}_n(\Gamma))$ is $\phi(n)$ when $n$ is even.

**Proof:** For any $n$ we have $|U_n| = \phi(n)$.

Let us consider an element $\alpha$ of $u_n$

then for any $p \in \mathbb{Z}_n, \exists$ an $\beta = n - \alpha \in U_n$ such that

$\alpha - p + p + \beta \equiv 0(modn)$

therefore $p$ is adjacent to all the vertices of the type $\alpha - p$

now we just need to check the similarity of these two elements.

since $n$ is even the elements of $U_n$ are all odd.

hence $2p \not\equiv \alpha(modn)$

$\Rightarrow p \not\equiv \alpha - p(modn)$.

hence $p$ and $\alpha - p$ are distinct.

hence $deg(p) = \phi(n)$.

**Theorem**: The degree of any vertex $p$ of $\mathbb{Z}_n$ of the total graph associated to $\Gamma-$semigroup of $\mathbb{Z}_n$ $(G(\mathbb{Z}_n(\Gamma))$ is $\phi(n)$ when $n$ is odd and $(p,n) \neq 1$ and is $\phi(n) - 1$ whenever $n$ is odd and

$(p,n) = 1$.

**Proof:** Case 1: when $n$is odd $(p,n) \neq 1$

By previous theorem it is clear that any vertex $p$ is adjacent to all the other vertices of the type

$\alpha - p$

here $(p,n) \neq 1$ ,hence $p \notin U_n$

therefore $2p \not\equiv \alpha(modn)$

$\Rightarrow p \not\equiv \alpha - p(modn)$.

therefore in this case also these two vertices are distinct and hence $deg(p) = \phi(n)$.

Case 2: when $n$ is odd and $(p,n) = 1$

here $p \in U_n$

$\Rightarrow 2p \equiv \alpha(modn)$

$\Rightarrow p \equiv \alpha - p(modn)$

But p is not adjacent to itself by our definition .

Hence $deg(p) = \phi(n) - 1$.

**Corollary**: Number of edges in the total graph associated to $\Gamma-$ semigroup$G(\mathbb{Z}_n(\Gamma))$ is $\frac{1}{2}n\phi(n)$ when $n$ is even and $\frac{1}{2}(n-1)\phi(n)$ when $n$ is odd.

**Proof:** Case 1: When $n$ is even. A very well known results states that the sum of the degrees of the vertices equals the total number of edges.

$\Rightarrow n\phi(n) = 2\times$ number of edges

$\Rightarrow$ number of edges $= \frac{1}{2}n\phi(n)$.

Case 2: When $n$ is odd

By the same argument we have,

$\phi(n)\phi(n) - 1 + n - \phi(n)\phi(n) = 2\times$ number of edges

$\Rightarrow \phi(n)^2 - \phi(n) + n\phi(n) - \phi(n)^2 = 2\times$ number of edges

$\Rightarrow \phi(n)(n-1) = 2\times$ number of edges

$\Rightarrow$ number of edges $= \frac{1}{2}\phi(n)\{\phi(n) - 1\}$.

**Theorem**: The girth of total graph associated to $\Gamma-$semigroup is 3, for any odd number greater than 3 and 4 for any even number greater than 2 and $n \not\equiv 0(mod3)$.

**Proof:** For $n = 2$ and $n = 3$ it is very obvious to verify that the $G(\mathbb{Z}_n(\Gamma))$ is not a cycle.

for $n$ odd and $n > 3$,

We can always construct a cycle $C_1 = \{0, 2, n-1, 0\}$ of length 3 and hence girth is 3. Now for

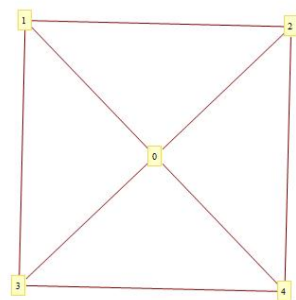



FIGURE 1. $G(\mathbb{Z}_5(\Gamma))$

$n$ even and $n \not\equiv 0(mod3)$.

We can again construct a cycle $C_2 = \{0, 1, 2, n-1, 0\}$ of length 4 and hence the girth is 4 in this case.

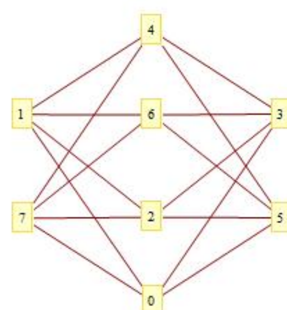



FIGURE 2. $G(\mathbb{Z}_8(\Gamma))$

**Theorem:** The total graph associated to a $\Gamma-$semigroup($G(\mathbb{Z}_n(\Gamma))$) is hamiltonian if

1. $n$ is a prime number greater than 3.

2. $n$ is even.

3. $n$ is odd and $n \not\equiv 0(mod3)$.

**Proof:** Let us suppose that $n$ is a prime number greater than 3. For each such $n$ we can always obtain a cycle of the form

$C_1 = \{n-2, n-4, n-6, .......3, 1, 0, 2, 4, ...n-3, n-1, n-2\}$

and this cycle contains all the edges and crosses each edge exactly once and thus contains an hailtonian cycle and hence the graph is hamiltonian for given $n$.

Now by considering $n$ an even number we can again consider a cycle of the form

$C_2 = \{0, n-1, 2, n-3, ....., n-4, 3, n-2, 1, 0\}$

This cycle also contains all the edges and traverses the graph by crossing all the edges exactly once. Hence the graph is hamitonian as it contains an hamiltonian cycle.

Lastly let us consider $n$ to be an odd number and $n \not\equiv 0(mod3)$ a cycle of the form

$C_3 = \{n-1, 0, n-2, 1, ......., \frac{n-3}{2}, \frac{n-1}{2}, n-1\}$

can be constructed where the cycle contains all the edges by traversing all the edges exactly once and therefore hamiltonian.

**Theorem:** The total graph associated to $\Gamma-$semigroup $G(\mathbb{Z}_n(\Gamma))$ is connected for all $n$.

**Proof:** By our previous theorem we have seen that $G(\mathbb{Z}_n(\Gamma))$ is hamiltonian for all $n$ except for $n$ which satisfy $n \not\equiv 0(mod3)$. Therefore we can conclude that $G(\mathbb{Z}_n(\Gamma))$ is connected except for $n$ satisfying $n \not\equiv 0(mod3)$.

For $n \not\equiv 0(mod3)$ we can find a cycle

$C = \{n-1, 0, n-2, ......., \frac{n-1}{2}\}$

Therfore $G(\mathbb{Z}_n(\Gamma))$ is connected for all $n$.

**Theorem:** The total graph associated to $\Gamma-$semigroup $G(\mathbb{Z}_n(\Gamma))$ is Eulerian for every even $n$ greater than 2.

**Proof:** The result which will be used in proving this theorem states that a graph is eulerian if and only if each of the vertices has even degree.

Now as we have proved that for every even $n$ the degree of the vertices is $\phi(n)$, and $\phi(n)$ is even for all even integer grater than 2 therefore we can easily conclude that $G(\mathbb{Z}_n(\Gamma))$ is Eulerian for all even $n$.

For all odd $n$ some of the vertices has degree $\phi(n)$ and some has degree $\phi(n)-1$.

But $\phi(n)-1$ is odd for every integer greater than 2. Therefore we can conclude that some of the vertices has odd degree for odd $n$ and thus not Eulerian.

Some of the important theorems which will be used are staded below:

**Theorem:** If a graph has a subgraph homeomorphic to $K_5$ and $K_3, 3$ then it cannot be planar.

**Theorem:** The total graph associate to $\Gamma-$semigroup $G(\mathbb{Z}_n(\Gamma))$ is planar for $n$ =1, 2, 3, 4, 5, 6.

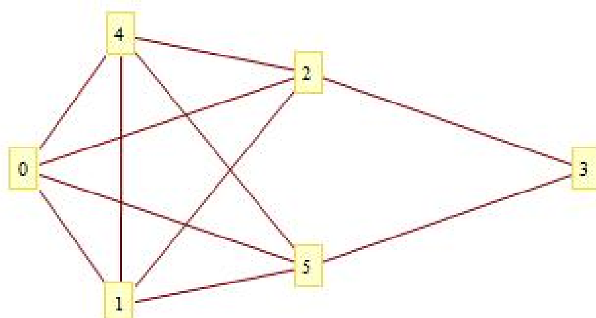
**Proof:** For $n$ =1, 2, 3, 4, 5, and 6 we can easily draw a planar representation of the graph structure $G(\mathbb{Z}_n(\Gamma))$.

Now we know that a simple graph is planar if it has a vertice of degree 6. But for $n \geq 15$ the degree of each vertices has degree greater than 6. Therefore cannot be Plannar.

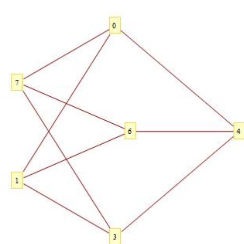
For $n = 8$, 10, 14 we can easily verify that the graph is not Plannar.

For $n = 11$, 13 the degree of each vertices is greater than 6 as$\phi(11) = 10$ and $\phi(13) = 12$ and thus not Plannar.

For $n = 7, 9, 12$ we have subgraphs homeomorphic to $K_5$ and $K_{3,3}$ making the graph non planar.




This is an example of subgraph of $G(\mathbb{Z}_7(\Gamma))$ homeomorphic to $K_5$.



This is the example of subgraph of $G(\mathbb{Z}_9(\Gamma))$ which is homeomorphic to $K_{3,3}$.

## Conflict of Interests

The authors declare that there is no conflict of interests.

## REFERENCES

[1] C. Godsil, G. Royle, Algebraic graph theory, Springer Graduate Texts in Mathematics No. 207, New York, 2001.

[2] D.F. Anderson, A. Badawi, The total graph of a commutative ring, J. Algebra. 320 (2008), 2706–2719.

[3] D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra. 217 (1999), 434–447.

[4] F. Harary, Graph theory, Addison-Wesley, Reading, MA, 1969.

[5] I. Beck, Coloring of commutative rings, J. Algebra. 116 (1988), 208–226.

[6] N. Biggs, Algebraic graph theory, Second Edition, Cambridge University Press, (1993)

[7] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math. 1 (1964), 81-89.

[8] M.K. Sen, On $\Gamma-$semigroup, in: Proceedings of International conference on Algebra and it's Application, Decker Publication, New York, (1981), 301-308.

[9] M.K. Sen, N.K. Saha, on $\Gamma-$ semigroup I, Bull. Cal. Math. Soc. 78 (1986), 181-186.

[10] N.K. Saha, On $\Gamma-$ semigroup II, Bull. Cal. Math. Soc. 79 (1987), 331-335.

[11] N.K. Saha, On $\Gamma-$ semigroup III, Bull. Cal. Math. Soc. 80 (1988), 1-12.