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SOME RESULTS ON SECURE DOMINATION IN ZERO-DIVISOR GRAPHS

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Abstract. Let $\Gamma(G) = (V(\Gamma(G)), E(\Gamma(G)))$ be a zero-divisor graph. A dominating set S of $V(\Gamma(G))$ is a secure dominating set of $\Gamma(G)$ if for every vertex $x \in V(\Gamma(G)) - S$, there exists $y \in N_{\Gamma(G)}(x) \cap S$ such that $(S - \{y\}) \cup \{x\}$ is a domination set. The minimum cardinality of a secure dominating set of $\Gamma(G)$ is called secure domination number. In this paper, the secure domination number of zero-divisor graphs is obtained and also studied the structure of this parameter in $\Gamma(Z_n)$.

Keywords: zero-divisor graph; secure domination.

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1. INTRODUCTION

In 1988, Istvan Beck [2] introduced zero-divisor graphs and studied the coloring properties of a graph, whose vertices are all the elements of the ring and two vertices are adjacent if their product is equal to 0. This definition was redefined and simplified in 1999 by Anderson and Livingston [1] to the zero-divisor graphs. The vertices of the zero-divisor graph are all non-zero zero-divisors and two distinct vertices x and y are adjacent if and only if $xy = 0$. On Compared with the Beck's zero-divisor graph, Anderson and Livingston [1] excluded zero element, thus the properties of the zero-divisors in the ring were clearly studied. In 1962, Berge [3] defined the

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concept of the domination number of a graph, calling this as "coefficient of external Stability" and Ore [8] used the name dominating set and domination number for the same concept. In 1977 Cockayne and Hedetniemi [4] made an interesting and extensive survey of the results know at that time about dominating sets in graphs. They have used the notation $\gamma(G)$ for the domination number of a graph, which has become very popular since then. The survey paper of Cockayne and Hedetniemi [4] has generated lot of interest in the study of domination in graphs. The study of domination parameters and related topics is one of most rich and fast developing area in graph theory. There are 2000 research papers were published in different journals. Collection of results and open problems on various dominating sets were published in [6, 7]. Some works on zero-divisor graphs can be found in [9, 10]. In this paper, secure dominating set of a zero-divisor graph is defined and also the secure domination number of $\Gamma(Z_n)$ are obtained under various constraints. Finally some results related to this parameter are stated and proved.

2. Preliminaries

Definition 2.1. Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be its set of zero-divisors. The zero-divisor graph of R, denoted by $\Gamma(R) = (V(\Gamma(R)), E(\Gamma(R)))$ is the (undirected) graph with vertices $V(\Gamma(R)) = Z(R)^* = Z(R) - \{0\}$, the non-zero zero-divisors of R, and for distinct $x, y \in V(\Gamma(R))$, the vertices x and y are adjacent if and only if $xy = 0$. $\Gamma(R)$ is the null graph if and only if R is an integral domain.

Note: Here after, we consider a commutative ring R by Z_n and the zero-divisor graph $\Gamma(R)$ by $\Gamma(Z_n)$.

Definition 2.2. Let $\Gamma(Z_n)$ be a zero-divisor graph. A subset S of $V(\Gamma(Z_n))$ is a dominating set if every vertex in $V(\Gamma(Z_n)) - S$ is adjacent to atleast one vertex in S.

The minimum of the cardinality of a dominating set is the domination number of a graph $\Gamma(Z_n)$ and its denoted by $\gamma(\Gamma(Z_n))$.

Definition 2.3. Let $\Gamma(Z_n)$ be a simple graph with the vertex set $V(\Gamma(Z_n))$. The neighborhood of x is the set $N_{\Gamma(Z_n)}(x) = \{y \in V(\Gamma(Z_n)) \mid y \odot_n x = 0\}$.

Definition 2.4. A vertex $x \in V(\Gamma(Z_n))$ is said to be a pendant vertex if and only if it has degree 1.

Definition 2.5. Let $\Gamma(Z_n)$ be a zero-divisor graph. The degree of a vertex y is denoted by $d(y)$ and defined as number of edges incident on it.

The maximum degree of $\Gamma(Z_n)$ is denoted by $\Delta(\Gamma(Z_n)) = \max\{d(y) : y \in V(\Gamma(Z_n))\}$.

3. Secure Domination Number of $\Gamma(Z_n)$

Definition 3.1. A dominating set S of $\Gamma(Z_n)$ is said to be a secure dominating set if for each vertex $x \in V(\Gamma(Z_n)) - S$, there exist $y \in N_{\Gamma(Z_n)}(x) \cap S$, such that $(S - \{y\}) \cup \{x\}$ is the dominating set.

The minimum cardinality of a minimal secure dominating set is called a secure domination number. It is denoted by the symbol $\gamma^{sd}(\Gamma(Z_n))$, the corresponding minimum secure dominating set is denoted by γ^{sd}-set.

Theorem 3.2. *Let $\Gamma(Z_{2p})$ be a zero-divisor graph with prime number $p \geq 3$.*
Then $\gamma^{sd}(\Gamma(Z_{2p})) = p - 1$.

Proof. Let $\Gamma(Z_{2p})$ be a zero-divisor graph and p be any prime number with $p \geq 3$.
Then $V(\Gamma(Z_{2p})) = \{2, 4, 6, .., 2(p-1), p\}$
Let $x = 2(p-1)$ and $y = p$ be two vertices, then $x \odot_{2p} y = 2(p-1) \odot_{2p} p = 0$ (since $x.y$ is a multiple of $2p$).
Hence $xy \in E(\Gamma(Z_{2p}))$.
Similarly, let $x \in V(\Gamma(Z_{2p})) - \{y\}$ and $y = p$ then $y \odot_{2p} x = 0$
Consider $S = V(\Gamma(Z_{2p})) - \{y\}$ is a dominating set and it has $p - 1$ elements.
Take $x \in S$, then for every $y \in V(\Gamma(Z_{2p})) - S$, there exists $x \in N_{\Gamma(Z_{2p})}(y) \cap S$ such that $(S - \{x\}) \cup \{y\})$ is also a dominating set.
Thus S is a $\gamma^{sd} - set$.
Hence $\gamma^{sd}(\Gamma(Z_{2p})) = p - 1$. □

Theorem 3.3. *Let $\Gamma(Z_{3p})$ be a zero-divisor graph with prime number $p > 3$.*
Then $\gamma^{sd}(\Gamma(Z_{3p})) = 2$

Proof. Let $\Gamma(Z_{3p})$ be a zero-divisor graph and $p > 3$ be a prime number.

The vertex set $V(\Gamma(Z_{3p})) = \{3, 6, 9, , , , 3(p-1), p, 2p\}$.

Let $y \in \Gamma(Z_{3p})$ with $d(y) = \Delta(\Gamma(Z_{3p}))$

Let z be a another vertex with $d(z) = \Delta$ in $\Gamma(Z_{3p})$

either $z = p, y = 2p$ (or) $z = 2p, y = p$

Then $z \odot_{3p} y = 2p \odot_{3p} p \neq 0$ (since $z.y$ is not divisible by $3p$)

$zy \notin E(\Gamma(Z_{3p}))$

Let x be any vertex in $V(\Gamma(Z_{3p})) - \{z, y\}$ such that $z. \odot_{3p} x = y \odot_{3p} x = 0$

Then $zx, yx \in E(\Gamma(Z_{3p}))$

Let $S = \{z, y\} \subseteq V(\Gamma(Z_{3p}))$ is a domonating set.

For every $x \in V(\Gamma(Z_{3p})) - S$,there exists $z \in N_{\Gamma(Z_{3p})}(x) \cap S$ such that $(S - \{z\}) \cup \{x\} = \{y, x\}$ is also a dominating set. Thus S is a $\gamma^{sd} - set$.

Hence $\gamma^{sd}(\Gamma(Z_{3p})) = 2$. □

Theorem 3.4. *Let $\Gamma(Z_{4p})$ be a zero-divisor graph with prime number $p \geq 5$. Then $\gamma^{sd}(\Gamma(Z_{4p})) = 3$.*

Proof. Let $\Gamma(Z_{4p})$ be a zero-divisor graph and $p \geq 5$ be a prime number.

The vertex set of $V(\Gamma(Z_{4p}) = \{2, 4, 6, ..., 2(2p-1), p, 2p, 3p\}$

Let $x = 2p$ and t is any even number from 2 to $2(2p-1)$.

Clearly $x \odot_{4p} t = 2p \odot_{4p} 2(2p-1) = 0$ (since $x.t$ is a multiple of $4p$)

Hence $xt \in E(\Gamma(Z_{4p}))$

Also, let $x = 2p, y = p$ and $z = 3p$ are in $V(\Gamma(Z_{4p})$.

Then $x \odot_{4p} y = 2p \odot_{4p} p \neq 0$, (since $x.y$ is not divisible by $4p$).

$y \odot_{4p} z = p \odot_{4p} 3p \neq 0$ and, (since $y.z$ is not divisible by $4p$).

$x \odot_{4p} z = 2p \odot_{4p} 3p \neq 0$ (since $x.z$ is not divisible by $4p$).

$xy, yz, zx \notin E(\Gamma(Z_{4p}))$

Let $u_n = 4n \in V(\Gamma(Z_{4p}))$ where $n = 1, 2, 3, \ldots, p-1$.

Then $u_n \odot_{4p} y = u_n \odot_{4p} z = 0$ where $n = 1, 2, 3, \ldots, p-1$.

Hence $u_n y, u_n z \in E(\Gamma(Z_{4p}))$.

Choose $u_n^* \in \{u_n\}$ and let $S = \{x, y, u_n^*\} \subseteq V(\Gamma(Z_{4p}))$ be a dominating set.

For every $u_n^{**} \in V(\Gamma(Z_{4p})) - S$, there exists $y \in N_{\Gamma(Z_{4p})}(u_n^{**}) \cap S$ such that $(S - y) \cup \{u_n^{**}\} = \{x, u_n^{*}, u_n^{**}\}$ is also a dominating set, for some $u_n^{*}, u_n^{**} \in \{u_n\}$ and $u_n^{*} \neq u_n^{**}$.

Thus S is a $\gamma^{sd} - set$.

Hence $\gamma^{sd}(\Gamma(Z_{4p})) = 3$. □

Theorem 3.5. *Let $\Gamma(Z_{5p})$ be a zero-divisor graph with prime number $p > 5$. Then $\gamma^{sd}(\Gamma(Z_{5p})) = 4$.*

Proof. Let $\Gamma(Z_{5p})$ be a zero-divisor graph with $p > 5$.

The vertex set of $V(\Gamma(Z_{5p})) = \{5, 10, 15, ..., 5(p-1), p, 2p, 3p, 4p\}$.

Now, the vertex set $V(\Gamma(Z_{5p}))$ can be partition in to two parts V_1 and V_2 with $V_1 \cup V_2 = V(\Gamma(Z_{5p}))$

Consider $V_1 = \{5, 10, 15, ..., 5(p-1)\}$ and $V_2 = \{p, 2p, 3p, 4p\}$.

Let $x, y \in V_1$ Then $x \odot_{5p} y \neq 0$, (since $x.y$ is not divisible by $5p$).

Similarly,

Let $z, w \in V_2$. Then $z \odot_{5p} w \neq 0$, (since $z.w$ is not divisible by $5p$).

Therefore $xy, zw \notin E(\Gamma(Z_{5p}))$

Hence no vertices of V_1 is adjacent to any vertices of V_1.

Similarly, no vertices of V_2 is adjacent to any vertices of V_2.

let $x \in V_1$ and $y \in V_2$.

Then $x \odot_{5p} y = 0$. (since $x.z$ is a multiple of $5p$)

Hence $xy \in E(\Gamma(Z_{5p}))$.

Therefore, every vertex in V_1 is adjacent to any vertices in V_2.

Consider $S = V_2 \subseteq V(\Gamma(Z_{5p}))$ is a dominating set.

For every $y \in V(\Gamma(Z_{5p})) - S$, there exists $p \in N_{Z_{5p}}(y) \cap S$ such that $(S - \{p\}) \cup \{y\} = \{y, 2p, 3p, 4p\}$ is also a dominating set. Thus S is a $\gamma^{sd} - set$.

Hence $\gamma^{sd}(\Gamma(Z_{5p})) = 4$. □

Theorem 3.6. *Let $\Gamma(Z_{7p})$ be a zero-divisor graph with prime number $p > 7$. Then $\gamma^{sd}(\Gamma(Z_{7p})) = 6$.*

Proof. Let $\gamma^{sd}(\Gamma(Z_{7p}))$ be a zero-divisor graph and $p > 7$ be a prime number.

The vertex set of $\gamma^{sd}(\Gamma(Z_{7p})) = \{7, 14, 21, .., 7(p-1), p, 2p, 3p, 4p, 5p, 6p\}$.

Clearly , the vertex V can be partition in to two parts V_1 and V_2.

$V_1 = \{7, 14, 21, ..., 7(p-1)\}$ and $V_2 = \{p, 2p, 3p, 4p, 5p, 6p\}$.

Let $x \in V_1, z \in V_2$

Then $x \odot_{7p} z = 0$ (since $x.z$ is a multiple of $7p$)

Hence $xz \in E(\Gamma(Z_{7p}))$.

Let $x, y \in V_1$

Then $x \odot_{7p} y \neq 0$ (since $x.y$ is not divisible by $7p$)

Similarly,

Let $z, w \in V_2$

Then $z \odot_{7p} w \neq 0$ (since $z.w$ is not divisible by $7p$)

$$xy, zw \notin E((\Gamma(Z_{7p})))$$

Therefore, every vertex in V_1 is adjacent to any vertex in V_2 and vice versa.

Let $S = V_2 \subseteq V(\Gamma(Z_{7p}))$ be a dominating set.

For every $w \in V(\Gamma(Z_{7p})) - S$, there exist $p \in N_{Z_{7p}}(w) \cap S$ such that $(S - \{p\}) \cup \{w\} = \{w, 2p, 3p, 4p, 5p, 6p\}$ is a dominating set.

Thus S is a $\gamma^{sd} - set$.

Hence $\gamma^{sd}(\Gamma(Z_{7p})) = 6$. □

Theorem 3.7. *Let $\Gamma(Z_{2p})$ be a zero-divisor graph where $p \geq 3$, with p vertices and maximum vertex degree $\Delta(\Gamma(Z_{2p}))$ then $\gamma^{sd}(\Gamma(Z_{2p})) = 2p - 2 - \Delta(\Gamma(Z_{2p}))$ if and only if $\Gamma(Z_{2p})$ is a star graph.*

Proof. Let y be a vertex with maximum degree $\Delta(\Gamma(Z_{2p}))$

If $\Gamma(Z_{2p})$ is a star, with y is the root, then $\Gamma(Z_{2p})$ has exactly $\Delta(\Gamma(Z_{2p}))$ branches from y.

Then the number of leaves (edges) in $\Gamma(Z_{2p})$ is exactly $\Delta(\Gamma(Z_{2p}))$

Since $\gamma^{sd}(\Gamma(Z_{2p})) = p - 1$.

That is, $p - 1 = 2p - 2 - \Delta(\Gamma(Z_{2p})) = p - 1$.

Therefore $\gamma^{sd}(\Gamma(Z_{2p})) = 2p - \Delta(\Gamma(Z_{2p})) - 2$.

Conversely, if $\Gamma(Z_{2p})$ is not a star, then there exist a vertex other than y with degree not less than 3 in $\Gamma(Z_{2p})$

This shows that $\Gamma(Z_{2p})$ has more than $\Delta(\Gamma(Z_{2p}))$ edges, which is contradiction.

Hence the theorem. □

Theorem 3.8. *Let $\Gamma(Z_{p^n q})$ be a zero-divisor graph with p and q are distinct prime and $q > p, n > 1$, then $\gamma^{sd}(\Gamma(Z_{p^n q})) = 3$.*

Proof. Let $\Gamma(Z_{p^n q})$ be a zero-divisor graph with p and q are distinct prime and $n > 1$

Let $x = p^{n-1}q \in V(\Gamma(Z_{p^n q})), y = p^n \in V(\Gamma(Z_{p^n q}))$

Clearly $x \odot_{p^n q} y = p^{n-1}q \odot_{p^n q} p^n = 0$ (since $x.y$ is a multiple of $p^n q$)

Hence $xy \in E(\Gamma(Z_{p^n q}))$

Let $z = 2p^n$, $w = q \in V(\Gamma(Z_{p^n q}))$

Clearly $z \odot_{p^n q} w = 2p^n \odot_{p^n q} q = 0$ (since $z.w$ is a multiple of $p^n q$)

Hence $zw \in E(\Gamma(Z_{p^n q}))$

Let $S = \{p^{n-1}q, p^n, 2p^n\}$ is a dominating set. For every $q \in V(\Gamma(Z_{p^n q})) - S$, there exists $2p^n \in N_{Z_{p^n q}}(q) \cap S$ such that $(S - \{2p^n\}) \cup \{q\} = \{p^{n-1}q, p^n, q\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(\Gamma(Z_{p^n q})) = 3$ □

Theorem 3.9. *Let $\Gamma(Z_{p^2})$ be a zero-divisor graph with prime number p. Then $\gamma^{sd}(\Gamma(Z_{p^2}) = 1$.*

Proof. Let $\Gamma(Z_{p^2})$ be a zero-divisor graph with prime number p.

The vertex set of $V(\Gamma(Z_{p^2})) = \{p, 2p, 3p, .., (p-1)p\}$.

Clearly p is adjacent to all the vertices in $\Gamma(Z_{p^2})$ also each vertices adjacent to remaining all the vertices in $\Gamma(Z_{p^2})$.

Let $S = \{p\}$ is a dominating set. For every $2p \in V(\Gamma(Z_{p^2})) - S$, there exists $p \in N_{Z_{p^2}}(2p) \cap S$ such that $(S - \{p\}) \cup 2p = \{2p\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(\Gamma(Z_{p^2}) = 1$. □

Theorem 3.10. *Let $\Gamma(Z_{2^n})$ be a zero-divisor graph with $n > 3$ is a positive integer. Then $\gamma^{sd}(\Gamma(Z_{2^n})) = 2$.*

Proof. The vertex set of $V(\Gamma(Z_{2^n})) = \{2,4,6,...,2^{n-1},2^{n-1}+2,...,2(2^{n-1}-1)\}$.

Let $x \in \Gamma(Z_{p^2})$ has a maximum degree. $d(x) = 2^{n-1}-2$.

Let $x \in 2^{n-1}$ and y be any other vertex in $\Gamma(Z_{2^n})$

Let take $y = 2^n + 4$, then $x \odot_{2^n} y = 2^{n-1} \odot_{2^n} (2^n+4) = 0$ (since $x.y$ is a multiple of 2^n)

Hence $xy \in E(\Gamma(Z_{2^n})$

Let $S = \{2^{n-1}, \frac{2^{n-1}}{2}\}$ is a dominating set. For every $\frac{3}{2}(2^{n-1}) \in V(\Gamma(Z_{2^n})) - S$, there exists $\frac{2^{n-1}}{2} \in N_{Z_{2^n}}(\frac{3}{2}(2^{n-1})) \cap S$ such that $(S - \{\frac{2^{n-1}}{2}\}) \cup \{\frac{3}{2}(2^{n-1})\} = \{2^{n-1}, \frac{3}{2}(2^{n-1})\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(\Gamma(Z_{2^n})) = 2$. □

Theorem 3.11. *Let $\Gamma(Z_{3^n})$ be a zero-divisor graph with a positive integer $n > 2$. Then $\gamma^{sd}(\Gamma(Z_{3^n})) = 1$.*

Proof. Let $\Gamma(Z_{3^n})$ be a zero-divisor graph with $n \geq 3$ is a positive integer.

The vertex set $V(\Gamma(Z_{3^n})) = \{3,6,9,...,3^{n-1},...,2(3^{n-1}),...,3(3^{n-1}-1\}$.

Since, $\Gamma(Z_{3^n})$ has no pendent vertex.

Therefore, there exists $x,y \in \Gamma(Z_{3^n})$ is adjacent to all the vertices in $\Gamma(Z_{3^n})$.

Let $z \in \Gamma(Z_{3^n})$, then $z \odot_{3^n} x = z \odot_{3^n} y = 0$ and

let $x = 3^{n-1}$ and $y = 2(3^{n-1})$ Then $x \odot_{3^n} y = 3^{n-1} \odot_{3^n} 2(3^{n-1}) = 0$ (since $x.y$ is a multiple of 3^n)

Hence $xy \in E(\Gamma(Z_{3^n}))$.

Let $S = \{3^{n-1}\}$ is a dominating set.

For every $2(3^{n-1}) \in V(\Gamma(Z_{3^n})) - S$, there exist $3^{n-1} \in N_{Z_{3^n}}(2(3^{n-1})) \cap S$ such that $(S - \{3^{n-1}\}) \cup \{2(3^{n-1})\} = \{2(3^{n-1})\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(\Gamma(Z_{3^n})) = 1$.. □

Theorem 3.12. *Let $\Gamma(Z_{p^n})$ be a zero-divisor graph with $p > 2$ is prime and $n > 2$. Then $\gamma^{sd}(\Gamma(Z_{p^n})) = 1$.*

Proof. From theorem 3.9 and 3.11, there exist $x,y \in \Gamma(Z_{p^n})$ is adjacent to all the vertices in $\Gamma(Z_{p^n})$.

The vertex set $V(\Gamma(Z_{p^n}))$ is $\{p, 2p, 3p, ..., p^{n-1}, ..., 2p^{n-1}, ...3p^{n-1}, ..., p(p^{n-1}-1)\}$

Clearly $\Gamma(Z_{p^n})$ has no pendent vertex.

Let $x = p^{n-1}, y = 2p^{n-1}$ then $x \odot_{p^n} y = 0$ (since $x.y$ is a multiple of p^n)

Hence $xy \in E(\Gamma(Z_{p^n}))$

Let $S = \{p^{n-1}\}$ is a dominating set.

For every $2(p^{n-1}) \in V(\Gamma(Z_{p^n})) - S$, there exists $p^{n-1} \in N_{Z_{p^n}}(2(p^{n-1})) \cap S$ such that $(S - \{p^{n-1}\}) \cup \{2(p^{n-1})\} = \{2(p^{n-1})\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(\Gamma(Z_{p^n})) = 1$. □

Theorem 3.13. *Let $\Gamma(Z_n), n = p^s q^t$ be a zero-divisor graph with p and q are prime numbers $p < q$ and s, t are positive integers with $s > 1, t > 1$. Then $\gamma^{sd}(Z_n) = 2$.*

Proof. Let $\Gamma(Z_n), n = p^s q^t$ be a zero-divisor graph.

The vertex set of $V(\Gamma(Z_n)) = V(\Gamma(Z_{p^s q^t})) = \{p, 2p, 3p, ..., n-p, q, 2q, 3q, ..., n-q\}$.

Let $u, v \in V(\Gamma(Z_n))$.

Let $p^s q^t | u.v$. Clearly $u \odot_{p^s q^t} v = 0$

Hence $uv \in E(\Gamma(Z_n))$.

Since $(u, v) \neq 1$ and there exist any vertex $w \in \Gamma(Z_n)$ either $n|u.w$ or $n|v.w$.

Then $u \odot_{p^s q^t} w = 0$ or $v \odot_{p^s q^t} w = 0$. Thus, every vertex in $\Gamma(Z_n)$ is adjacent to either u or v.

Let $S = \{p^{s-1}q^t, p^s q^{t-1}\}$ is a dominating set.

For every $2p^s q^{t-1} \in V(\Gamma(Z_n)) - S$, there exists $p^s q^{t-1} \in N_{Z_{p^s q^t}}(2p^s q^{t-1}) \cap S$ such that $(S - \{p^s q^{t-1}\}) \cup \{2p^s q^{t-1}\} = \{p^{s-1}q^t, 2p^s q^{t-1}\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(Z_{p^s q^t}) = 2$ □

Theorem 3.14. *Let $\Gamma(Z_n), n = p^s q^t r^k$ be a zero-divisor graph with p, q, r are prime numbers and s, t, k are positive integers with $s, t, k > 1$. Then $\gamma^{sd}(Z_n) = 3$.*

Proof. Let $\Gamma(Z_n), n = p^s q^t r^k$ be a zero-divisor graph.

Then $V(\Gamma(Z_{p^s q^t r^k})) = \{p, 2p, 3p, ..., n-p, q, 2q, 3q, ..., n-q, r, 2r, 3r, ..., n-r\}$

Let $u = pq, v = qr$ and $w = pr$ in $V(\Gamma(Z_{p^s q^t r^k}))$.

Then $u.v = pq^2r, v.w = pqr^2$ and $u.w = p^2qr$ implies that $n|uv, n|vw$ and $n|uw$.

Therefore the vertices u,v,w are adjacent and the graph $\Gamma(Z_n)$ has a K_3 subgraph.

Clearly $(u,v,w) = 1$ which implies u,v,w are relatively prime and $(u,w) \neq 1$ and $(v,w) \neq 1$.

Let x be any other vertex in $\Gamma(Z_n)$ then $ux = 0$ or $wx = 0$, which implies that x is adjacent to any one of the vertex from $\{u,v,w\}$.

Let $S = \{p^{s-1}q^tr^k, p^sq^{t-1}r^k, p^sq^tr^{k-1}\}$ is a dominating set.

Case (i):

Suppose, for every $2p^sq^{t-1}r^k \in V(\Gamma(Z_n)) - S$, there exist $p^sq^{t-1}r^k \in N_{Z_{p^sq^tr^k}}(2p^sq^{t-1}r^k) \cap S$ such that $(S - \{p^sq^{t-1}r^k\}) \cup \{2p^sq^{t-1}r^k\} = \{p^{s-1}q^tr^k, 2p^sq^{t-1}r^k, p^sq^tr^{k-1}\}$ is a dominating set.

Case (ii):

For every $2p^sq^tr^{k-1} \in V(\Gamma(Z_n)) - S$, there exist $p^sq^tr^{k-1} \in N_{Z_{p^sq^tr^k}}(2p^sq^tr^{k-1}) \cap S$ such that $(S - \{p^sq^tr^{k-1}\}) \cup \{2p^sq^tr^{k-1}\} = \{p^{s-1}q^tr^k, p^sq^{t-1}r^k, 2p^sq^tr^{k-1}\}$ is a dominating set.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(Z_(p^sq^tr^k)) = 3$. □

Theorem 3.15. *Let $\Gamma(Z_n), n = p_1^{e_1}p_2^{e_2}......p_k^{e_k}$ be a zero-divisor graph with $p_1, p_2, ..., p_k$ are distinct primes and the $e_1, e_2, ..., e_k$ are positive integers greater than* 1. *Then $\gamma^{sd}(Z_n) = k$.*

Proof. Using theorem (3.13), we get $\gamma_s(\Gamma(Z_(p^sq^t))) = 2$ and using theorem (3.14), we get $\gamma_s(Z_(p^sq^tr^k)) = 3$.

Similarly,

Let $u = p_1p_2, v = p_2p_3, ..., w = p_{k-1}p_k$ in $\Gamma(Z_n)$. Then $u.v = p_1p_2^2p_3, v.w = p_2p_3^2p_4, \ldots$ implies that $n|uv, n|vw$.

Therefore the vertices $u, v, ..., w$ are adjacent , proceeding the same way, we get $\Gamma(Z_n)$ has a subgraph of K_k.

Clearly $(u, v, ...w) = 1$. Let x be any vertex in $\Gamma(Z_n)$ the any one of the following is true. $(a)ux = 0 (b)vx = 0....(k)wx = 0$ .ie., the remaining vertices in $\Gamma(Z_n)$ is adjacent to any one of vertex in $K_k = \{u, v, ...w\}$ also,

we find the secure dominating set $S = \{p_1^{e_1-1}p_2^{e_2}......p_k^{e_k}, p_1^{e_1}p_2^{e_2-1}......p_k^{e_k}, p_1^{e_1}p_2^{e_2}......p_k^{e_k-1}\}$. Therefore the secure domination number is k.

Thus S is a γ^{sd}-set.

Hence $\gamma^{sd}(Z_n) = k$. $\square$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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