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FINSLER INFINITY SUPERHARMONIC FUNCTIONS

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Abstract. We investigate a simple proof on properties of a non-negative Finsler infinity superharmonic function such as positivity, Harnack inequality, Liouville property and Lipschitz continuity using Finsler distance function. We also present Hopf boundary point lemma for a Finsler infinity subharmonic function.

Keywords: Viscosity solution; Finsler Minkowski norm; Finsler distance function.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be open and connected set. In this paper we have presented properties of non-negative Finsler infinity superharmonic function in Ω; that is properties of a non-negative viscosity supersolution of

$$-\Delta^N_{F;\infty} u = 0. \tag{1.1}$$

The normalized Finsler infinity Laplacian operator $\Delta^N_{F;\infty}$ is a nonlinear, singular and degenerate elliptic. It is defined by

$$\Delta^N_{F;\infty} u(x) = \langle D^2 u DF(Du(x)), DF(Du(x)) \rangle, \tag{1.2}$$

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where F is a Finsler minkowski norm in $\mathbb{R}^n$. The Finsler minkowski norm F in $\mathbb{R}^n$ is defined as follows: Let $F : \mathbb{R}^n \to \mathbb{R}_0^+$ be a function satisfying the following properties.

(1) (**Regularity**) $F \in C^2(\mathbb{R}^n \setminus \{o\})$.

(2) (**Positive homogeneity**) F is positively homogeneous of degree 1; that is

$$F(t\xi) = tF(\xi) \ \ \forall \xi \in \mathbb{R}^n, \text{and } \forall t > 0.$$

(3) (**Strong Convexity**) $D^2(F^2)(\xi) > \mathbf{0}$ on $\mathbb{R}^n \setminus \{o\}$.

A function $F : \mathbb{R}^n \to \mathbb{R}_0^+$ that satisfies regularity, positive homogeneity and strong convexity is called a Finsler-Minkowski norm on $\mathbb{R}^n$. We can see that $F(o) = 0, \langle DF(\xi), \xi \rangle = F(\xi) \ \forall \xi \in \mathbb{R}^n \setminus \{o\}, D^2F(\xi)\xi = o$ on $\mathbb{R}^n \setminus \{o\}$ and $F(\xi) > 0 \ \forall \xi \in \mathbb{R}^n \setminus \{o\}$. (cf. [2, 11]). The proof of the following Lemma can be found in [2].

Lemma 1.1. *Let F be a Finsler-Minkowski norm. The following properties hold.*

(1) F satisfies the triangle inequality. That is

$$F(\xi + \varepsilon) \leq F(\xi) + F(\varepsilon) \quad \forall \xi, \varepsilon \in \mathbb{R}^n.$$

Equality holds iff $\varepsilon = \kappa\xi$ for some $\kappa \geq 0$.

(2) If $w \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n \setminus \{o\}$, then

$$\langle w, DF(\xi) \rangle \leq F(w).$$

Equality holds if and only if $w = \kappa\xi$ for some $\kappa \geq 0$.

We define $F^* : \mathbb{R}^n \to \mathbb{R}_0^+$ by

$$F^*(p) = \sup_{F(\eta)=1} \langle p, \eta \rangle = \sup_{\xi \neq o} \frac{\langle p, \xi \rangle}{F(\xi)}.$$

Let $\alpha = \inf_{|\xi|=1} \dfrac{1}{F(\xi)}$ and $\beta = \sup_{|\xi|=1} \dfrac{1}{F(\xi)}$. We have $0 < \alpha \leq \beta$ and

$$\alpha |x_0 - x| \leq F^*(x_0 - x) \leq \beta |x_0 - x|. \tag{1.3}$$

We may write (1.3) as

$$\alpha \leq F^*(\xi) \leq \beta$$

on the set $\{\xi : |\xi| = 1\}$. We have also $\dfrac{\alpha}{\beta} F^*(x - x_0) \leq F^*(x_0 - x) \leq \dfrac{\beta}{\alpha} F^*(x - x_0)$.

Remark 1.2. *F^* satisfies all properties that F satisfies. (See* [2]*).*

Lemma 1.3. *Let F be a Finsler-Minkowski norm. Then*

(1) $F^*(DF(\xi)) = 1 \quad \forall\, \xi \in \mathbb{R}^n \setminus \{o\}$.

(2) $F(DF^*(p)) = 1 \quad \forall\, p \in \mathbb{R}^n \setminus \{o\}$.

(3) The map $FDF : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and

$$(FDF)^{-1} = F^*DF^*.$$

(4) $DF(DF^*(p)) = \dfrac{1}{F^*(p)} p$.

For the proof of Lemma (1.3) we refer the reader to [5, 12].

Lemma (1.3) (4) and properties of F^* gives the following remark.

Remark 1.4. *For any $x \in \mathbb{R}^n \setminus \{x_0\}$,*

$$\Big\langle [DF^*(x_0-x)DF^*(x_0-x)^t]\, DF(DF^*(x_0-x)), DF(DF^*(x_0-x)) \Big\rangle = 1.$$

Problems involving the operator (1.2) have been extensively studied in [4, 10, 11, 12, 15].

Many authors have studied Harnack inequality for equation (1.1) where $F(\xi) = |\xi|$, see for instance [1, 3, 8, 9, 14] and references therein. In the recent paper [12] the Harnack inequality was introduced for inhomogeneous equation involving the operator (1.2). The paper [12] relies on comparison with quadratic F^* cones to prove Harnack inequality.

The following notations have been used.

$B(x,r)$ = Euclidean ball center at x and radius $r > 0$

$\langle \cdot, \cdot \rangle$ = The usual inner product

$\mathbb{R}_0^+$ = $[0, \infty)$

$C^2(\Omega)$ = Twice continuously differentiable on Ω

$Du(x)$ = The gradient of u at x

$D^2u(x)$ = The Hessian matrix of u at x

$u \succ_{x_0} \varphi$ = $u - \varphi$ has local minimum at x_0

$B_F(x_0,r)$:= $\{x : F^*(x_0 - x) < r\}$,

$\partial B_F(x_0,r)$:= $\{x : F^*(x_0 - x) = r\}$ and

$dist_F(x_0, \partial\Omega)$ = $\inf\{F^*(x_0 - x) : x \in \partial\Omega\}$.

We organized this paper as follows. In section two we state main results of the paper. Section three is devoted the definition of viscosity solution and proof of Lemma (2.1). In section four we give the proofs of Theorems (2.2),(2.3),(2.4),(2.5) and (2.6), respectively.

2. STATEMENT OF MAIN RESULTS

Let $p \in \Omega$, with $0 < r \leq dist_F(p, \partial\Omega)$. We define the Finsler distance function $d(x) = r - F^*(p-x)\ \forall x \in \overline{B}_F(p,r)$.

Lemma 2.1. *Let u be non-negative Finsler infinity superharmonic function in Ω. If $u(p) > 0$, then $u(x) \geq u(p)\dfrac{d(x)}{d(p)}\ \forall x \in B_F(p,r)$.*

Theorem 2.2. *(Positivity) Let u be non-negative Finsler infinity superharmonic function in Ω. If u is positive somewhere in Ω, then u is positive everywhere in Ω.*

Theorem 2.3. *(Harnack inequality) Let u be non-negative Finsler infinity superharmonic function in Ω; let $p \in \Omega, 0 < r < dist_F(p, \partial\Omega)$. Then*

$$\inf_{B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)} u \geq \frac{1}{4}\left[1 - \frac{\alpha}{\beta\kappa}\right] \sup_{B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)} u(x), 1 < \kappa < \infty. \tag{2.1}$$

Remark 2.4. *(Liouville property) If u is a non-negative Finsler infinity harmonic function in $\mathbb{R}^n$, then u is a constant function in $\mathbb{R}^n$.*

Lemma 2.5. *(Hopf) Suppose Ω satisfies the interior sphere condition at some $y \in \partial\Omega$, i.e. there exists $B_F(x_0, r) \subset \Omega$ such that $y \in \partial B_F(x_0, r) \cap \partial\Omega$. Let u be a Finsler infinity harmonic function in Ω such that $u(y) = \inf_\Omega u$ and $u(x_0) > u(y)$. Then u satisfies*

$$\liminf_{x \to y} \frac{u(x) - u(y)}{d(x)} > 0,$$

where $d(x) = r - F^*(x_0 - x)$.

Theorem 2.6. *(Lipschitz Continuity) Let $p \in \Omega$ and $dis(p, \partial\Omega) = r$. If u is a non-negative superharmonic function in Ω, then*

$$|u(x) - u(y)| \leq 2M\frac{\beta^2}{\alpha r}|x-y|, \forall x, y \in B_F\left(p, \frac{\alpha r}{\beta}\right),$$

where $M = \sup_\Omega u$.

3. VISCOSITY SOLUTION

In this section we give the definition of viscosity solution to problem (1.1) [11]. The lower and upper Finsler infinity Laplacian of a twice differentiable function φ at $x_0 \in \Omega$ are respectively denoted by $\Delta^-_{F;\infty}\varphi(x_0)$ and $\Delta^+_{F;\infty}\varphi(x_0)$. Which are defined by

$$\Delta^-_{F;\infty}\varphi(x_0) = \begin{cases} \langle D^2\varphi(x_0)DF(D\varphi(x_0)), DF(D\varphi(x_0))\rangle & \text{if } D\varphi(x_0) \neq o \\ \\ \min\{\langle D^2\varphi(x_0)e, e\rangle : F^*(e) = 1\} & \text{if } D\varphi(x_0) = o. \end{cases} \tag{3.1}$$

and

$$\Delta^+_{F;\infty}\varphi(x_0) = \begin{cases} \langle D^2\varphi(x_0)DF(D\varphi(x_0)), DF(D\varphi(x_0))\rangle & \text{if } D\varphi(x_0) \neq o \\ \\ \max\{\langle D^2\varphi(x_0)e, e\rangle : F^*(e) = 1\} & \text{if } D\varphi(x_0) = o, \end{cases} \tag{3.2}$$

Definition 3.1. *(1) A function $u \in USC(\Omega, \mathbb{R})$ is called a viscosity subsolution of* (1.1) *if for every function $\varphi \in C^2(\Omega, \mathbb{R})$ and point $x_0 \in \Omega$ such that $u \prec_{x_0} \varphi$ we have*

$$-\Delta^+_{F;\infty}\varphi(x_0) \leq 0.$$

In this case we write $-\Delta^N_{F;\infty}\varphi(x_0) \leq 0$.

(2) A function $u \in USC(\Omega, \mathbb{R})$ is called a viscosity supersolution of (1.1) *if for every function $\varphi \in C^2(\Omega, \mathbb{R})$ and point $x_0 \in \Omega$ such that $u \succ_0 \varphi$ we have*

$$-\Delta^-_{F;\infty}\varphi(x_0) \geq 0.$$

In this case we write $-\Delta^N_{F;\infty}u(x_0) \geq 0$.

(3) A function $u \in C(\Omega, \mathbb{R})$ is called a viscosity solution of (1.1) *if u is both a viscosity subsolution and supersolution of* (1.1).

A viscosity subsolution of (1.1) is called Finsler infinity subharmonic where as a viscosity supersolution of (1.1) is called Finsler infinity superharmonic.

Lemma 3.2. *Let $d(x) = r - F^*(x_0 - x), \forall x \in B_F(x_0, r)$. Then for $x \neq x_0$ we have*

$$\Delta^N_{F;\infty}d^\alpha(x) = \alpha(\alpha-1)d^{\alpha-2}(x), \alpha > 1.$$

Proof. For $x \neq x_0$, we observe that

$$D(d^{\alpha}(x)) = \alpha d^{\alpha-1}(x) DF^{*}(x_0 - x)$$

and

$$D^2 d^{\alpha}(x) = \alpha(\alpha-1) d^{\alpha-2}(x) DF^{*}(x_0-x) DF^{*}(x_0-x) - \alpha d^{\alpha-1}(x) D^2 F^{*}(x_0-x).$$

We note that $D(d^{\alpha}(x)) \neq o$ for $x \neq x_0$, and $DF(DF^{*}(x_0-x)) = \dfrac{x_0-x}{F^{*}(x_0-x)}$ for any $x \in \mathbb{R}^n \setminus \{x_0\}$. We know that $\langle D^2F^{*}(x)\, x, x\rangle = 0$ for any $x \in \mathbb{R}^n$ and hence by Remark (1.4) we obtain

$$\Delta^{N}_{F;\infty} d^{\alpha}(x) = \alpha(\alpha-1) d^{\alpha-2}(x).$$

□

Proof of Lemma (2.1). Since $u(p) > 0$, there exist $k > 0$ such that $u(p) = \frac{r}{k}$. Let $u_c(x) = \frac{c}{r} u(x)$ and $v(x) = \frac{d(x)}{r}, 0 < c < k$. Then

$$u_c(p) = \frac{c}{r} u(p) = \frac{c}{r}\frac{d(p)}{k} = \frac{c}{k} < 1.$$

For $x \in \partial B_F(p,r), v(x) = \dfrac{d(x)}{r} = 0$. We have also $d(p) = r - F^{*}(0) = r$ and thus $v(p) = 1$. Let $w = u_c - v$, for a fixed c. Then

$$w(p) = u_c(p) - v(p) = \frac{c}{k} - 1 < 0,$$

and

$$w(x) = \frac{c}{r} u(x) - \frac{d(x)}{r} = \frac{c}{r} u(x) \geq 0, \text{ on } \partial B_F(p,r).$$

Thus w has a negative minimum in $B_F(p,r)$. This minimum value occurs at p. We show this by contradiction. Suppose there is a point $x_c \neq p$ such that

$$w(x_c) < w(p) < 0.$$

Now

$$v^{\alpha}(x) = \left(\frac{d(x)}{r}\right)^{\alpha}, \alpha > 1$$

and

$$w_{\alpha}(x) = u_c(x) - v^{\alpha}(x).$$

Thus $w_\alpha(p) = u_c(p) - 1 < 0$ and on $\partial B_F(p,r), w_\alpha(x) \geq 0$. We can choose α sufficiently close to 1 such that the point of minimum of w_α, denoted by $x_{c,\alpha} \neq p$ and $w_\alpha(x_{c,\alpha}) < w_\alpha(p) = u_c(p) - 1 < 0$. This indicates $x_{c,\alpha} \notin \partial B_F(p,r)$.

Again now

$$\frac{r}{c} w_\alpha(x) = u(x) - \frac{r}{c} v^\alpha(x) = u(x) - \frac{d^\alpha(x)}{cr^{\alpha-1}}$$

has a negative minimum at $x_{c,\alpha} \neq p$. We notice that $v^\alpha(x)$ is C^2 near $x_{c,\alpha}$ and as u is Finsler infinity superharmonic, we have

$$-\Delta^N_{F;\infty} \left(\frac{d^\alpha(x_{c,\alpha})}{cr^{\alpha-1}} \right) \geq 0.$$

By Lemma (3.2) we have

$$\Delta^N_{F;\infty} \left(\frac{d^\alpha(x_{c,\alpha})}{cr^{\alpha-1}} \right) = \frac{\alpha(\alpha-1) d^{\alpha-2}(x_{c,\alpha})}{cr^{\alpha-1}} > 0.$$

Which is a contradiction. Hence the minimum of w occurs at p.

Therefore, $u_c(x) - v(x) \geq u_c(p) - 1$. Which implies

$$\frac{c}{r} u(x) - \frac{d(x)}{r} \geq \frac{c}{r} u(p) - 1 \;\forall x \in B_F(p,r) \text{and for all } c < k.$$

As $c \to k$ we have $ku(x) - d(x) \geq ku(p) - d(p) = 0$. This implies $ku(x) \geq d(x)$. Since $k = \dfrac{d(p)}{u(p)}$, we obtain

$$u(x) \geq u(p) \frac{d(x)}{d(p)}.$$

□

4. PROOFS

Proof of Theorem (2.2). Let $x_0 \in \Omega$ such that $u(x_0) > 0$. Consider the set

$$S = \{x \in \Omega : u(x) > 0\}.$$

Since $x_0 \in S, S \neq \emptyset$. For each $x \in S$, there is an open set V containing x such that $u(y) > 0$ for all $y \in V$. Hence S is open. Let $\{x_n\}$ be a sequence of points in S converges to $x \in \Omega$. Since Ω is open, there is a ball $B(x,\delta)$ contained in Ω for some $\delta > 0$. We observe that $|x-y| \leq \frac{1}{\alpha} F^*(x-y) < \delta$ for $y \in B_F(x,\alpha\delta)$. So, $B_F\left(x, \frac{\alpha\delta}{4}\right) \subset B(x,\delta)$. The Finsler ball $B_F(x,\alpha\delta)$ contains point

z of the sequence $\{x_n\}$. Here we have $u(z) > 0$ and $B_F(x, \alpha\delta) \subset B_F(z, (\alpha+\beta)\delta)$. If $z = x$, nothing is done. So we assume $z \neq x$. In this case, by Lemma (2.1) we have

$$u(x) \geq u(z) \frac{(\alpha+\beta)\delta - F^*(z-x)}{(\alpha+\beta)\delta} > 0.$$

Thus $x \in S$, that is S is closed. We see that S is both open and closed. It follows that $S = \Omega$. Therefore; u is positive in Ω. □

Proof of Theorem (2.3). If $u = 0$ in Ω, then (2.1) holds true. So we assume $u(x) > 0$ for some $x \in \Omega$. By Theorem (2.2), $u > 0$ in Ω. If $x \in B_F(p, r)$, by Lemma(2.1)

$$u(x) \geq u(p) \left[\frac{d(x)}{d(p)}\right] = u(p) \left[\frac{r - F^*(p-x)}{r}\right] \tag{4.1}$$

For all $x \in B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)$, Equation (4.1) becomes

$$u(x) \geq u(p) \left[1 - \frac{\alpha}{\beta\kappa}\right]. \tag{4.2}$$

Taking infinimum of (4.2) over $B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)$ we get

$$\inf_{B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)} u \geq u(p) \left[1 - \frac{\alpha}{\beta\kappa}\right] \tag{4.3}$$

Take $x \in B_F\left(p, \frac{\alpha r}{\beta\kappa}\right)$, $p \in B_F(x, \frac{r}{\kappa})$. Let R be a mid point point of the segment joining the points p and x. Let $F^*(x-P) = l$. In $B_F(x, l)$,

$$u(R) \geq u(x) \left[\frac{l - F^*(x-R)}{l}\right] \geq \frac{u(x)}{2} \tag{4.4}$$

In $B_F\left(R, \dfrac{\alpha r}{\beta\kappa}\right)$,

$$\begin{aligned} u(p) \geq u(R) \left[\frac{d(p)}{d(x)}\right] &= u(x) \left[\frac{\frac{\alpha r}{\beta\kappa} - F^*(R-p)}{\frac{\alpha r}{\beta\kappa}}\right] \\ &\geq u(x) \left[\frac{\frac{\alpha r}{\beta\kappa} - \frac{\alpha r}{2\beta\kappa}}{\frac{\alpha r}{\beta\kappa}}\right] = \frac{u(x)}{2} \qquad (4.5) \end{aligned}$$

From (4.3), (4.4) and (4.5) we obtain

$$\begin{aligned} \inf_{B_F\left(p,\frac{\alpha r}{\beta\kappa}\right)} u &\geq u(p)\left[1-\frac{\alpha}{\beta\kappa}\right] \\ &\geq \left[1-\frac{\alpha}{\beta\kappa}\right]\frac{u(x)}{4}, \forall x \in B_F\left(p,\frac{\alpha r}{\beta\kappa}\right). \end{aligned} \tag{4.6}$$

Taking the supremum of (4.6) over $B_F\left(p,\frac{\alpha r}{\beta\kappa}\right)$ we get

$$\inf_{B_F\left(p,\frac{\alpha r}{\beta\kappa}\right)} u \geq \frac{1}{4}\left[1-\frac{\alpha}{\beta\kappa}\right] \sup_{B_F\left(p,\frac{\alpha r}{\beta\kappa}\right)} u(x).$$

□

Proof of Remark (2.4). Take two distinct points x and z in $\mathbb{R}^n$. Consider the ball $B_F(z,r)$ with $r > F^*(z-x)$. By Lemma (2.1),

$$u(z) \leq u(x)\frac{d(z)}{d(x)},$$

and $d(z) = d(x) + F^*(z-x) = r$. Letting $r \to \infty$ we get $u(z) \leq u(x)$. Interchanging the roles of x and z we get the reverse inequality. Therefore; u is constant in $\mathbb{R}^n$. □

Proof of Lemma (2.5). Let $w(x) = u(x) - u(y)$. Then w is a non negative Finsler infinity supperharmonic function in $B_F(x_0,r)$ and hence from Lemma (2.1) we have

$$\frac{u(x)-u(y)}{d(x)} \geq \frac{u(x_0)-u(y)}{r}.$$

We conclude that

$$\liminf_{x\to y}\frac{u(x)-u(y)}{d(x)} > 0.$$

□

Proof of Theorem (2.6). For all $x \in B_F\left(p,\frac{\alpha r}{\beta}\right)$, we have

$$u(x) \geq u(p)\left[\frac{\frac{\alpha r}{\beta} - F^*(p-x)}{\frac{\alpha r}{\beta}}\right].$$

So,

$$u(x) - u(p) \geq -u(p)\frac{\beta^2}{\alpha r}|p-x|. \tag{4.7}$$

If $x \in B_F\left(p, \dfrac{\alpha r}{\beta}\right)$, then $p \in B_F(x,r)$ and thus

$$u(p) - u(x) \geq -u(x)\frac{\beta^2}{\beta r}|p-x|. \tag{4.8}$$

Combining (4.7) and (4.8), we get $|u(x)-u(p)| \leq M\dfrac{\beta^2}{\alpha r}|p-x|$, $\forall x \in B_F(p, \frac{\alpha r}{\beta})$, where $M = \sup_{\Omega} u$. Consequently,

$$|u(x)-u(y)| \leq |u(x)-u(p)| + |u(y)-u(p)| \leq 2M\frac{\beta^2}{\alpha r}|x-y|, \ \forall x, y \in B_F\left(p, \frac{\alpha r}{\beta}\right).$$

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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