Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 4, 4695-4710
https://doi.org/10.28919/jmcs/5918
ISSN: 1927-5307

LATTICE VALUED NEUTROSOPHIC SETS

JOSE JAMES, SUNIL C. MATHEW*

Department of Mathematics, St. Thomas College, Palai, Kottayam, Kerala, 686574, India

Abstract. In this paper, the authors introduce lattice valued neutrosophic sets and study their properties. A decomposition theorem for lattice valued neutrosophic sets is obtained. Lattice valued neutrosophic mappings are defined and verified that its properties are consistent with their crisp counterparts. Finally, a topology of lattice valued neutrosophic sets is introduced.

Keywords: neutrosophic set; neutrosophic mapping; lattice; neutrosophic topology.

2010 AMS Subject Classification: 03E72.

1. Introduction

Zadeh's[24](1965) fuzzy set theory and fuzzy logic brought wide applications in the domain of uncertainities. Fuzzy sets successfully handled cases where an element partially belongs to a set. But fuzzy sets could not handle those cases where uncertainities arose due to incomplete data. Such a scenario inspired the introduction of intuitionistic fuzzy sets by Atanassov[1]. Though intuitionistic fuzzy set theory was efficient in modeling incomplete information, it could not handle indeterminate and inconsistent data.

*Corresponding author

E-mail address: sunilcmathew@gmail.com

Received April 23, 2021

In 1998, Smarandache[19] proposed the concept of neutrosophic sets, which generalizes fuzzy sets, interval valued fuzzy sets, intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets. In addition to the truth-membership and the falsity-membership, an independent indeterminancy membership function defines a neutrosophic set. Several extensions of neutrosophic sets such as interval valued neutrosophic sets[21], bipolar neutrosophic sets[4], neutrosophic soft sets[15] etc, were studied by researchers to deal with a variety of problems. Neutrosophic set theory proved to have wide applications in decision making problems[16] and medical image processing[12]. Single valued neutrosophic sets were introduced by Wang et al.[20]. It became a hot area of research due to its applicability to practical problems[9, 2]. Single valued neutrosophic relations were studied by Kim et al[10]. In 2005, Smarandache defined various notions of neutrosophic topologies[18]. Kim, J. et al[11] studied ordinary single valued neutrosophic topologies. M. EL-Gayyar[5] introduced smooth neutrosophic topological spaces.

Chang's[3] introduction of fuzzy set theory into topology initiated extensive research in fuzzy set theory. Goguen[7] replaced the unit interval in a fuzzy set by a lattice to define L-fuzzy sets and subsequently introduced L-fuzzy topology[8](known as the Chang-Goguen L-fuzzy topology). Later, several authors looked into the interaction between lattice theory and topology in different directions.

In this paper, we introduce lattice valued neutrosophic sets and evaluate its basic properties. We show that a lattice valued neutrosophic set can be decomposed into level subsets. Lattice valued neutrosophic mappings and inverse lattice valued neutrosophic mappings are defined to connect different lattice valued neutrosophic sets. In the final section, a topology of lattice valued neutrosophic sets is introduced.

2. PRELIMINARIES

In this section, a brief overview of neutrosophic set theory is provided. Essential concepts and results from lattice theory are also discussed.

2.1. Neutrosophic Set Theory.

Since the introduction of neutrosophic sets by Smarandache[19], various authors introduced different types of operations on neutrosophic sets and studied their properties.

In this paper, we choose only the most naturally defined types of operations, which agrees with human intuition.

Definition 2.1. [19]*(Neutrosophic set) Let X be a set, with a generic element in X denoted by x. A neutrosophic set A in X is characterized by three membership functions: a truth membership function T_A, an indeterminancy membership function I_A and a falsity membership function F_A, where $\forall x \in X$, $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of* $]0^-, 1^+[$. *There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$.*

Definition 2.2. [20]*(Single Valued Neutrosophic set) A neutrosophic set A is said to be a single valued neutrosophic set(SVN Set) if $\forall x \in X$, $T_A(x), I_A(x), F_A(x) \in [0,1]$.*

Definition 2.3. [23]*(Inclusion) A single valued neutrosophic set A is contained in another single valued neutrosophic set B, or say $A \subseteq B$ if and only if $\forall x \in X$*

$$T_A(x) \leq T_B(x),\ I_A(x) \geq I_B(x),\ F_A(x) \geq F_B(x)$$

Two SVN Sets are said to be equal if and only if $A \subseteq B$ and $A \supseteq B$.

Definition 2.4. [20]*(Complement) Let A be a SVN set in X. The complement of A is denoted A^c, where $\forall x \in X$*

$$T_{A^C}(x) = F_A(x),\ I_{A^C}(x) = 1 - I_A(x),\ F_{A^C}(x) = T_A(x)$$

Definition 2.5. [17] *Let A and B be two SVN sets in X.*

(1) The union of A and B is a SVN set C, denoted $C = A \cup B$, where $\forall x \in X$

$$T_C(x) = max\{T_A(x), T_B(x)\},\ I_C(x) = min\{I_A(x), I_B(x)\},\ F_C(x) = min\{F_A(x), F_B(x)\}$$

(2) The intersection of A and B is a SVN set D, denoted $D = A \cap B$, where $\forall x \in X$

$$T_D(x) = min\{T_A(x), T_B(x)\},\ I_D(x) = max\{I_A(x), I_B(x)\},\ F_D(x) = max\{F_A(x), F_B(x)\}$$

2.2. Lattice Theory. Here we recall a few of the fundamental definitions and results from lattice theory.

Definition 2.6. [13]*(Lattice) Let L be a poset. L is called a lattice if any two of its elements a and b have a greatest lower bound ("meet") denoted by* $a \wedge b$ *and a least upper bound ("join") denoted by* $a \vee b$. *A lattice L is said to be complete when each of its subsets has an l.u.b and g.l.b in L. In particular, the smallest element* 0_L *and the greatest element* 1_L *will exist in L as the join of the empty set and the meet of the empty set respectively. Therefore, all lattices we consider in this paper are assumed to contain atleast* 0_L *and* 1_L.

Definition 2.7. [13] *A lattice L is said to be distributive if* $\forall x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.8. [13] *Let L be a complete lattice. L is said to be infinitely distributive if L satisfies the following conditions:*

(1) $\forall a \in L$ *and* $\forall\, B \subseteq L$, $a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$*(First infinite distributive law)*
(2) $\forall a \in L$ *and* $\forall\, B \subseteq L$, $a \vee \bigwedge B = \bigwedge_{b \in B} (a \vee b)$*(Second infinite distributive law)*

Theorem 2.9. [13] *Let L be a complete lattice and A,* $B \subseteq L$.

(1) L satisfies the first infinite distributive law iff $\bigvee A \wedge \bigvee B = \bigvee_{a \in A\ b \in B} (a \wedge b)$
(2) L satisfies the second infinite distributive law iff $\bigwedge A \vee \bigwedge B = \bigwedge_{a \in A\ b \in B} (a \vee b)$

Definition 2.10. [13] *Let L be a complete Lattice. L is called completely distributive, if* $\forall \{\{a_{i,j} : j \in J_i\} : i \in I\} \subseteq \mathscr{P}(L) \setminus \{\phi\}$ *the following equalities hold:*

(1) $\bigwedge_{i \in I} (\bigvee_{j \in J_i} a_{i,j}) = \bigvee_{\phi \in \prod_{i \in I} J_i} (\bigwedge_{i \in I} a_{i,\phi(i)})$
(2) $\bigvee_{i \in I} (\bigwedge_{j \in J_i} a_{i,j}) = \bigwedge_{\phi \in \prod_{i \in I} J_i} (\bigvee_{i \in I} a_{i,\phi(i)})$.

The following notions are from [6]. An element $l \in L \setminus \{0_L\}$ is called a co-prime element if, for any finite subset $K \subset L$ satisfying $l \leq \bigvee K$, $\exists\, k \in K$ such that $l \leq k$. The set of all co-prime elements of L will be denoted by $c(L)$. An element $l \in L \setminus \{1_L\}$ is called a prime element if it's a co-prime element of L^{op}. The set of all prime elements of L will be denoted by $p(L)$. There exists a stronger form of inequality in a lattice, known as 'way below relation'. A point

$a \in L$ is said to be 'way below' $b \in L$, denoted $a \ll b$, if for every directed set $D \subset L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$. The relation $\ll$ is stronger than $\leq$ in the sense- $\forall a,b \in L$ $a \ll b \implies a \leq b$.

Theorem 2.11. [6] *If L is completely distributive, then c(L) is a join-generating set of L and p(L) is a meet-generating set of L. i.e., every element in L is the supremum of all the co-primes way below it and the dual statement also holds.*

Definition 2.12. [13] *A mapping* $' : L \to L$ *is called order-reversing involution, if* $\forall a,b \in L$, $(a')' = a$ *and* $a \leq b \implies a' \geq b'$.

3. Lattice Valued Neutrosophic sets

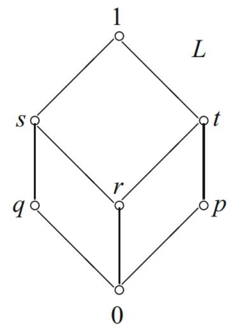
In this section we introduce the concept of Lattice Valued Neutrosophic (LVN) set. Basic operations on LVN sets are defined and their respective properties are investigated. The introduced operations behave analogous to their crisp counterparts. All lattices considered in the following sections are assumed to be complete and infinitely distributive.

Definition 3.1. *Let X be a set and (L, $\leq$) a nontrivial complete and distributive lattice. A lattice valued neutrosophic set A is characterized by three membership functions: a truth membership function* T_A, *an indeterminancy membership function* I_A *and a falsity membership function* F_A, *where* $\forall x \in X$, $T_A(x)$, $I_A(x)$ *and* $F_A(x) \in L$.

LVNS(X) will denote the set of all lattice valued neutrosophic sets on X. Let $A \in LVNS(X)$, then A may be represented for convenience as $\{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$.

Remark 3.2. *In the case* $L = ([0,1], \leq)$, *the given definition reduces to the definition of a SVN set. It's an obvious observation that the crisp sets, fuzzy sets, L-Fuzzy sets, intuitionistic fuzzy sets and intuitionistic L-Fuzzy sets are all special cases of an LVN set.*

Example 3.3. *Take* $X = \{a,b,c\}$ *and the lattice L as in the following figure.*
$A = \{< a,q,r,t >, < b,1,0,t >, < c,1,0,s >\}$ *is an LVN set on X.*



Definition 3.4. *(Inclusion) A lattice valued neutrosophic set A is contained in another lattice valued neutrosophic set B, or say $A \subseteq B$ if and only if $\forall x \in X$,*

$$T_A(x) \leq T_B(x),\ I_A(x) \geq I_B(x),\ F_A(x) \geq F_B(x).$$

Two LVN sets are said to be equal if and only if $A \subseteq B$ and $A \supseteq B$.

The assumption of an order reversing involution enables us, first of all, to give a reasonable definition for closedness and some related notions.

Definition 3.5. *Let A be an LVN set in X. The complement of A is denoted by A^c, where $\forall x \in X$,*

$$T_{A^C}(x) = F_A(x),\ I_{A^C}(x) = (I_A(x))',\ F_{A^C}(x) = T_A(x).$$

Definition 3.6. *Let A and B be two LVN sets in X.*

(1) The union of A and B is a LVN set C, denoted $C = A \cup B$, where $\forall x \in X$,

$$T_C(x) = T_A(x) \vee T_B(x),\ I_C(x) = I_A(x) \wedge I_B(x),\ F_C(x) = F_A(x) \wedge F_B(x).$$

(2) The inersection of A and B is a LVN set D, denoted $D = A \cap B$, where $\forall x \in X$,

$$T_D(x) = T_A(x) \wedge T_B(x),\ I_D(x) = I_A(x) \vee I_B(x),\ F_D(x) = F_A(x) \vee F_B(x).$$

The lattice L being infinitely distributive, arbitrary union and arbitrary intersection can be defined in the obvious way.

Theorem 3.7. *The following properties hold when $A, B, C \in LVNS(X)$.*

(1) $A \cup B = B \cup A$ (Commutativity)

(2) $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$ (Associativity)

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*(Distributivity)*

(4) $A \cup A = A$ *and* $A \cap A = A$*(Idempotency)*

(5) $A \cup (A \cap B) = A$ *and* $A \cap (A \cup B) = A$ *(Absorption)*

Definition 3.8. *The empty LVN set and full LVN set are defined respectively as*

$(0_L, 1_L, 1_L) = \{< x, 0_L, 1_L, 1_L > | x \in X\}$ *and* $(1_L, 0_L, 0_L) = \{< x, 1_L, 0_L, 0_L > | x \in X\}$.

Theorem 3.9. *The neutrosophic empty set and full set satisfies the following equalities:*

(1) $(0_L, 1_L, 1_L) \cup (1_L, 0_L, 0_L) = (1_L, 0_L, 0_L)$, $(0_L, 1_L, 1_L) \cap (1_L, 0_L, 0_L) = (0_L, 1_L, 1_L)$, $(1_L, 0_L, 0_L) \cup (1_L, 0_L, 0_L) = (1_L, 0_L, 0_L)$ *and* $(1_L, 0_L, 0_L) \cap (1_L, 0_L, 0_L) = (1_L, 0_L, 0_L)$.

(2) $A \cap (0_L, 1_L, 1_L) = (0_L, 1_L, 1_L)$, $A \cup (0_L, 1_L, 1_L) = A$, $A \cap (1_L, 0_L, 0_L) = A$ *and* $A \cup (1_L, 0_L, 0_L) = (1_L, 0_L, 0_L)$.

Remark 3.10. *The equalities* $A \cup A^C = (1_L, 0_L, 0_L)$ *and* $A \cap A^C = (0_L, 1_L, 1_L)$ *need not hold in LVNS(X).*

Theorem 3.11. *Let A and B be two LVN sets in X, the following results hold:*

(1) $A, B \subseteq A \cup B$

(2) $A \cap B \subseteq A, B$

(3) $(A^c)^c = A$

(4) $(A \cup B)^c = A^c \cap B^c$ *and* $(A \cap B)^c = A^c \cup B^c$ *(Demorgan Laws)*

Some of the lattice theoretic properties of L carries over to LVNS(X), as stated in the following theorem:

Theorem 3.12. *Consider LVNS(X) and define* $\forall A, B \in LVNS(X)$ $A \leq B$ *iff* $A \subseteq B$, $A \vee B = A \cup B$ *and* $A \wedge B = A \cap B$ *then,* $(LVNS(X), \leq)$ *is a complete lattice. Moreover,*

(1) *L is infinitely distributive iff LVNS(X) is infinitely distributive.*

(2) *L is completely distributive iff LVNS(X) is completely distributive.*

Definition 3.13. *Let* $A \in LVNS(X)$ *and* $d, e, f \in L$. *Define a LVN set* $(d, e, f)A$ *by*

$$(d, e, f)A = \{< x, d \wedge T_A(x), e \vee I_A(x), f \vee F_A(x) > | x \in X\}.$$

The set $(d, e, f)A$ *will be called the (d,e,f)-layer of A.*

Definition 3.14. *Let $A \in LVNS(X)$ and $d,e,f \in L$. Define the (a,b,c)-level of A as the crisp set*

$$A_{a,b,c} = \{x \in X | T_A(x) \geq a, I_A(x) \leq b, F_A(x) \leq c\}.$$

Definition 3.15. *Suppose we are given a crisp set A, we can convert A into a LVN set $N(A)$ as follows:*

$$T_{N(A)}(x) = \begin{cases} 1_L & \text{if } x \in A \\ 0_L & \text{otherwise} \end{cases}, \; I_{N(A)}(x) = \begin{cases} 0_L & \text{if } x \in A \\ 1_L & \text{otherwise} \end{cases}, \; F_{N(A)}(x) = \begin{cases} 0_L & \text{if } x \in A \\ 1_L & \text{otherwise} \end{cases}$$

In particular, for a given (a,b,c)-level $A_{a,b,c}$ of a LVN set A, $N(A_{a,b,c})$ is as follows:

$$T_{N(A_{a,b,c})}(x) = \begin{cases} 1_L & \text{if } x \in A_{a,b,c} \\ 0_L & \text{otherwise} \end{cases}, \; I_{N(A_{a,b,c})}(x) = \begin{cases} 0_L & \text{if } x \in A_{a,b,c} \\ 1_L & \text{otherwise} \end{cases},$$

$$F_{N(A_{a,b,c})}(x) = \begin{cases} 0_L & \text{if } x \in A_{a,b,c} \\ 1_L & \text{otherwise} \end{cases}$$

Thus, the LVN set $(a,b,c)N(A_{a,b,c})$ will have the following truth, indeterminancy and falsity values:

$$T_{(a,b,c)N(A_{a,b,c})}(x) = \begin{cases} a & \text{if } x \in A_{a,b,c} \\ 0_L & \text{otherwise} \end{cases}, \; I_{(a,b,c)N(A_{a,b,c})}(x) = \begin{cases} b & \text{if } x \in A_{a,b,c} \\ 1_L & \text{otherwise} \end{cases},$$

$$F_{(a,b,c)N(A_{a,b,c})}(x) = \begin{cases} c & \text{if } x \in A_{a,b,c} \\ 1_L & \text{otherwise} \end{cases}$$

Theorem 3.16. *(Decomposition theorem for LVN sets) Let A be a LVN set on X, then*

$$\underset{a,b,c \in L}{\cup} (a,b,c)N(A_{a,b,c}) = A$$

Proof:

For convenience sake, denote $\underset{a,b,c \in L}{\cup} (a,b,c)N(A_{a,b,c})$ by $dec(A)$.
Let $x \in X$ and $(T_A(x), I_A(x), F_A(x)) = (\alpha_1, \alpha_2, \alpha_3)$.
We will prove $(T_{dec(A)}(x), I_{dec(A)}(x), F_{(dec(A)}(x)) = (\alpha_1, \alpha_2, \alpha_3)$.

$$T_{dec(A)}(x) = \bigvee_{a,b,c \in L} T_{(a,b,c)N(A_{a,b,c})}(x) = \bigvee_{a \leq \alpha_1, b \geq \alpha_2, c \geq \alpha_3} a = \alpha_1$$

$$I_{dec(A)}(x) = \bigwedge_{a,b,c\in L} I_{(a,b,c)N(A_{a,b,c})}(x) = \bigwedge_{a\leq\alpha_1,b\geq\alpha_2,c\geq\alpha_3} b = \alpha_2$$

$$F_{dec(A)}(x) = \bigwedge_{a,b,c\in L} F_{(a,b,c)N(A_{a,b,c})}(x) = \bigwedge_{a\leq\alpha_1,b\geq\alpha_2,c\geq\alpha_3} c = \alpha_3$$

We can do better if we assume that L is completely distributive, in that case, since $c(L)$ is a join generating set and $p(L)$ is a meet generating set, the following statement holds:

Theorem 3.17. *Let L be a completely distributive lattice and for $a \in c(L)$ and $b,c \in p(L)$ take $\tilde{A}_{(a,b,c)} = \{x \in X | T_A(x) \gg a, I_A(x) \ll b, F_A(x) \ll c\}$. Then,*

$$\bigcup_{a\in c(L),\ b,c\in p(L)} (a,b,c)N(\tilde{A}_{a,b,c}) = A$$

4. LATTICE VALUED NEUTROSOPHIC MAPPINGS

In this section, we define the concept of lattice valued neutrosophic mappings(LVN mappings) and inverse lattice valued neutrosophic mappings. Some of their properties are also investigated.

Definition 4.1. *Let $A \in LVNS(X)$, $B \in LVNS(Y)$ and f be a mapping from X to Y.*

(1) Define $f^{\rightarrow}$ as a LVN mapping from LVNS(X) to LVNS(Y) induced by f as

$$f^{\rightarrow}(A) = \{< y, T_{f^{\rightarrow}(A)}(y), I_{f^{\rightarrow}(A)}(y), F_{f^{\rightarrow}(A)}(y) > | y \in Y\}, \text{ where}$$

$$T_{f^{\rightarrow}(A)}(y) = \bigvee_{x\in f^{-1}(y)} \{T_A(x)\},\ \ I_{f^{\rightarrow}(A)}(y) = \bigwedge_{x\in f^{-1}(y)} \{I_A(x)\},\ \ F_{f^{\rightarrow}(A)}(y) = \bigwedge_{x\in f^{-1}(y)} \{F_A(x)\}$$

(2) Define $f^{\leftarrow}$ as a inverse LVN mapping from LVNS(Y) to LVNS(X) induced by f as

$$f^{\leftarrow}(B) = \{< x, T_{f^{\leftarrow}(B)}(x), I_{f^{\leftarrow}(B)}(x), F_{f^{\leftarrow}(B)}(x) > | x \in X\}$$

$$= \{< x, T_B(f(x)), I_B(f(x)), F_B(f(x)) > | x \in X\}$$

It's easy to observe that the notion of LVN mappings is a generalization of SVNR mappings introduced by Yang[22]

Theorem 4.2. *Let $A_1, A_2 \in LVNS(X)$, $B_1, B_2 \in LVNS(Y)$ and f be a function from X to Y. The following properties are true:*

(1) $f^{\rightarrow}(A_1 \cup A_2) = f^{\rightarrow}(A_1) \cup f^{\rightarrow}(A_2)$

(2) $f^{\leftarrow}(B_1 \cup B_2) = f^{\leftarrow}(B_1) \cup f^{\leftarrow}(B_2)$ and $f^{\leftarrow}(B_1 \cap B_2) = f^{\leftarrow}(B_1) \cap f^{\leftarrow}(B_2)$

(3) $B_1 \subseteq B_2 \implies f^{\leftarrow}(B_1) \subseteq f^{\leftarrow}(B_2)$ *and* $A_1 \subseteq A_2 \implies f^{\rightarrow}(A_1) \subseteq f^{\rightarrow}(A_2)$

Theorem 4.3. *Let $A \in LVNS(X)$, $B \in LVNS(Y)$ and f be a function from X to Y. The following hold:*

(1) $f^{\rightarrow}(f^{\leftarrow}(B)) \subseteq B$*, equality holds if f is a surjection.*
(2) $f^{\leftarrow}(f^{\rightarrow}(A)) \supseteq A$*, equality holds if f is an injection.*

Proof:

(1) Take $B = \{< y, T_B(y), I_B(y), F_B(y) > | y \in Y\}$

$$f^{\rightarrow}(f^{\leftarrow}(B)) = f^{\rightarrow}(\{< x, T_B(f(x)), I_B(f(x)), F_B(f(x)) > | x \in X\})$$

$$= \{< y, \bigvee_{x\in f^{-1}(y)} \{T_B(f(x))\}, \bigwedge_{x\in f^{-1}(y)} \{I_B(f(x))\}, \bigwedge_{x\in f^{-1}(y)} \{F_B(f(x))\} > | y \in Y\}$$

Thus, $f^{\rightarrow}(f^{\leftarrow}(B)) \subseteq B$.

(2) Take $A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\}$

$$f^{\leftarrow}(f^{\rightarrow}(A)) = f^{\leftarrow}(\{< y, \bigvee_{z\in f^{-1}(y)} \{T_A(z)\}, \bigwedge_{z\in f^{-1}(y)} \{I_A(z)\}, \bigwedge_{z\in f^{-1}(y)} \{F_A(z)\} > | y \in Y\})$$

$$= \{< x, \bigvee_{z\in f^{-1}(f(x))} \{T_A(z)\}, \bigwedge_{z\in f^{-1}(f(x))} \{I_A(z)\}, \bigwedge_{z\in f^{-1}(f(x))} \{F_A(z)\} > | x \in X\}$$

Thus, $f^{\leftarrow}(f^{\rightarrow}(A)) \supseteq A$.

Theorem 4.4. *If L is infinitely distributive, then* $f^{\rightarrow}(A_1 \cap A_2) \subseteq f^{\rightarrow}(A_1) \cap f^{\rightarrow}(A_2)$*. Equality holds if f is injective.*

Proof:

For $x \in X$ and $y \in Y$,

$$A_1 \cap A_2 = \{< x, T_{A_1}(x) \wedge T_{A_2}(x), I_{A_1}(x) \vee I_{A_2}(x), F_{A_1}(x) \vee F_{A_2}(x) > | x \in X\}$$

$$f^{\rightarrow}(A_1 \cap A_2)(y) =< y, \bigvee_{x\in f^{-1}(y)} \{T_{A_1}(x) \wedge T_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{I_{A_1}(x) \vee I_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{F_{A_1}(x) \vee F_{A_2}(x)\} >$$

$$f^{\rightarrow}(A_1)(y) =< y, \bigvee_{x\in f^{-1}(y)} \{T_{A_1}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{I_{A_1}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{F_{A_1}(x)\} >$$

$$f^{\rightarrow}(A_2)(y) =< y, \bigvee_{x\in f^{-1}(y)} \{T_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{I_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y)} \{F_{A_2}(x)\} >$$

$$f^{\rightarrow}(A_1)\cap f^{\rightarrow}(A_2)(y) =< y, \bigvee_{x\in f^{-1}(y)}\{T_{A_1}(x)\}\wedge \bigvee_{x\in f^{-1}(y)}\{T_{A_2}(x)\},$$

$$\bigwedge_{x\in f^{-1}(y)}\{I_{A_1}(x)\}\vee \bigwedge_{x\in f^{-1}(y)}\{I_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y)}\{F_{A_1}(x)\}\vee \bigwedge_{x\in f^{-1}(y)}\{F_{A_2}(x)\}>$$

$$=< y, \bigvee_{x\in f^{-1}(y), x\in f^{-1}(y)}\{T_{A_1}(x)\wedge T_{A_2}(x)\},$$

$$\bigwedge_{x\in f^{-1}(y), x\in f^{-1}(y)}\{I_{A_1}(x)\vee I_{A_2}(x)\}, \bigwedge_{x\in f^{-1}(y), x\in f^{-1}(y)}\{F_{A_1}(x)\vee F_{A_2}(x)\}>$$

$$\implies f^{\rightarrow}(A_1\cap A_2)\subseteq f^{\rightarrow}(A_1)\cap f^{\rightarrow}(A_2).$$

Theorem 4.5. *Let $f: X\rightarrow Y$ be a function and $f^{\rightarrow}: LVNS(X)\rightarrow LVNS(Y)$ be an LVN mapping. The following results hold:*

(1) $f^{\rightarrow}$ is injective iff f is injective

(2) $f^{\rightarrow}$ is surjective iff f is surjective

(3) $f^{\rightarrow}$ is bijective iff f is bijective

(4) $f^{\rightarrow}$ is injective iff $f^{\leftarrow}\circ f^{\rightarrow} = I$, the identity function on LVNS(X).

Theorem 4.6. *Let $f: X\rightarrow Y$ and $g: Y\rightarrow Z$ be ordinary functions. Then*

(1) $g^{\rightarrow}\circ f^{\rightarrow} = (gf)^{\rightarrow}$

(2) $f^{\leftarrow}\circ g^{\leftarrow} = (gf)^{\leftarrow}$

Definition 4.7. *An LVN Point on X is an LVN set $x_{(a,b,c)}\in LVNS(X)$ defined by*

$$T_{x_{(a,b,c)}}(z) = \begin{cases} a & \text{if } z = x \\ 0_L & \text{otherwise} \end{cases}, \; I_{x_{(a,b,c)}}(z) = \begin{cases} b & \text{if } z = x \\ 1_L & \text{otherwise} \end{cases}, \; F_{x_{(a,b,c)}}(z) = \begin{cases} c & \text{if } z = x \\ 1_L & \text{otherwise} \end{cases}$$

The tuple (a,b,c) is called the height of the LVN point $x_{(a,b,c)}$.

One can easily observe that any given LVN set can be written as the union of LVN points contained in it.

Theorem 4.8. *An LVN mapping preserves LVN points with height, i.e., $f^{\rightarrow}(x_{(a,b,c)}) = (f(x))_{(a,b,c)}$.*

Theorem 4.9. *Let $F: LVNS(X)\rightarrow LVNS(Y)$ be an ordinary mapping. If*

(1) F preserves arbitrary unions

(2) F preserves LVN points with height,

then, there exists a unique mapping $f : X \to Y$ *such that* $F = f^{\to}$.

Proof:

We will first show that there exists a function $f : X \to Y$ such that $F(x_{(a,b,c)}) = (f(x))_{(a,b,c)}$, $\forall (a,b,c) \in L \setminus \{0_L\} \times L \setminus \{1_L\} \times L \setminus \{1_L\}$.
Since F preserves LVN points, $\forall x \in X$ and $\forall (a,b,c) \in L \setminus \{0_L\} \times L \setminus \{1_L\} \times L \setminus \{1_L\}$,
$\exists f_{a,b,c}(x) \in Y$ such that $F(x_{(a,b,c)}) = (f_{(a,b,c)}(x))_{(a,b,c)}$.
Fix an LVN point $x_{(a,b,c)}$,

$$\begin{aligned}
(f_{a,b,c}(x))_{a,b,c} &\subseteq \bigcup_{(d,e,f) \in L\setminus\{0_L\} \times L\setminus\{1_L\} \times L\setminus\{1_L\}} \{(f_{d,e,f}(x))_{d,e,f}\} \\
&= \bigcup_{(d,e,f) \in L\setminus\{0_L\} \times L\setminus\{1_L\} \times L\setminus\{1_L\}} \{F(x_{(d,e,f)})\} \\
&= F\Big(\bigcup_{(d,e,f) \in L\setminus\{0_L\} \times L\setminus\{1_L\} \times L\setminus\{1_L\}} \{x_{(d,e,f)}\} \Big) \\
&= F(x_{(1_L,0_L,0_L)})
\end{aligned}$$

Since $(f_{(a,b,c)}(x))_{(a,b,c)} \subseteq (f_{(1_L,0_L,0_L)})_{(1_L,0_L,0_L)}$ $\forall a,b,c \in L$, define $f(x) = f_{(1_L,0_L,0_L)}(x) \forall x \in X$.
Thus we have obtained an ordinary function $f : X \to Y$ such that

$$F(x_{(a,b,c)}) = (f(x))_{(a,b,c)} \forall x \in X \text{ and } \forall (a,b,c) \in L \setminus \{0_L\} \times L \setminus \{1_L\} \times L \setminus \{1_L\}.$$

Now to show that $F = f^{\to}$, let $A \in LVNS(X)$. It's easy to verify that $A = \bigcup_{x \in X} \{x_{(T_A(x), I_A(x), F_A(x))}\}$

$$F(A) = F(\bigcup_{x \in X} \{x_{(T_A(x), I_A(x), F_A(x))}\}) = \bigcup_{x \in X} (f(x))_{(T_A(x), I_A(x), F_A(x))}$$

Therefore, $\forall y \in Y$, $T_{F(A)}(y) = \bigvee_{f(z)=y} T_A(z)$, $I_{F(A)}(y) = \bigwedge_{f(z)=y} I_A(z)$ and $F_{F(A)}(y) = \bigwedge_{f(z)=y} F_A(z)$.
Thus, $F(A) = f^{\to}(A)$. The uniqueness of f is clear.

5. Lattice Valued Neutrosophic Topology

In this section, we define a new topological structure which is a generalization of the single valued neutrosophic topology. The lattice valued neutrosophic counterpart of the interior operator and closure operator are introduced and their basic properties are listed.

Definition 5.1. *A family τ of LVN subsets of a non-empty set X is said to be a Lattice valued neutrosophic topology(LVN topology) if the following conditions hold:*

(1) $(0_L,1_L,1_L)$, $(1_L,0_L,0_L) \in \tau$

(2) $A_1 \cap A_2 \in \tau$ *for* $A_1, A_2 \in \tau$

(3) $\cup A_i \in \tau$ *for* $\{A_i\} \subseteq \tau$.

The pair (X,τ) is called a LVN topological space and any LVN set in τ is known as a LVN open set in X. A LVN set B is said to be closed if and only if B^c is LVN open. Base and subbase for an LVN topology can be defined analogous to the base and subbase in crisp topology.

Remark 5.2. *A LVN topological space is a generalization of fuzzy toplogical spaces, L-Fuzzy topological spaces and single valued neutrosophic topological spaces. Lupianez*[14] *has already proved that an intuitionistic fuzzy topological space need not be a single valued neutrosophic topological space, and thus, an intuitionistic fuzzy topological space need not be a LVN topological space.*

Theorem 5.3. *Let* $\mathscr{A} \subseteq LVNS(X)$ *be a subcollection of LVN sets. Then,*

(1) $\mathscr{A}$ *is a base of a unique LVN topology iff* $\underset{A\in\mathscr{A}}{\cup} A = (1_L,0_L,0_L)$ *and* $A,B \in \mathscr{A} \implies A\cap B \in \mathscr{A}$

(2) $\mathscr{A}$ *is a subbase of a unique LVN topology on X iff* $\underset{A\in\mathscr{A}}{\cup} A = (1_L,0_L,0_L)$.

Proof:

(1) First we prove the necessity part. Let τ be the unique topology generated by $\mathscr{A}$. Then, the necessity part follows since $\mathscr{A}$ is a subset of a topology. Conversely, take $\tau = \{\cup\mathscr{C} : \mathscr{C} \subseteq \mathscr{A}\}$. Since $(1_L,0_L,0_L) = \underset{A\in\mathscr{A}}{\cup} A$, $(1_L,0_L,0_L) \in \tau$. The empty LVN set belongs to τ as the union of empty class from $\mathscr{A}$. By definition, τ is closed under arbitrary union. By 3.12, LVNS(X) is infinitely distributive. Thus, for $\underset{c\in\mathscr{C}}{\cup} c$, $\underset{d\in\mathscr{D}}{\cup} d \in \mathscr{A}$, $\underset{c\in\mathscr{C}}{\cup} c \cap \underset{d\in\mathscr{D}}{\cup} d = \underset{c\in\mathscr{C}, d\in\mathscr{D}}{\cup} (c\cap d)$, belongs to τ.

(2) Follows from (1).

Definition 5.4. *Let* (X,τ) *be a LVN topological space and* $A \in LVNS(X)$.

(1) The LVN interior of A is defined as $LNint(A) = \cup\{A_i : A_i \in \tau, A_i \subseteq A\}$

(2) The LVN closure of A is defined as $LNcl(A) = \cap\{A_i : A_i^c \in \tau, A_i \supseteq A\}$

Theorem 5.5. *Let* (X,τ) *be a lattice valued neutrosophic topological space and* $A, B \in LVNS(X)$. *The following properties hold:*

(1) $LNint(LNint(A)) = LNint(A)$

(2) $LNcl(LNcl(A)) = LNcl(A)$

(3) $LNint(A \cap B) = LNint(A) \cap LNint(B)$ *and* $LNint(A \cup B) \supseteq LNint(A) \cup LNint(B)$

(4) $LNcl(A \cup B) = LNcl(A) \cup LNcl(B)$ *and* $LNcl(A \cap B) = LNcl(A) \cap LNcl(B)$

(5) $A \subseteq B \implies LNint(A) \subseteq LNint(B)$ *and* $LNcl(A) \subseteq LNcl(B)$

Definition 5.6. *Let* (X,τ) *and* (Y,δ) *be two LVN topolgical spaces and* $f : X \to Y$ *a function. We define* f *to be continuous from* (X,τ) *to* (Y,δ) *if the inverse mapping* $f^{\leftarrow} : LVNS(Y) \to LVNS(X)$ *maps each set in* δ *to a set in* τ.

Remark 5.7. *Similar to the deviation of Chang Goguen L-fuzzy topological spaces from crisp topology, constant mappings in LVN topological spaces need not be continuous. Consider the following example:*

Example 5.8. *Let* $X = \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be a constant function. Take* $\tau_1 = \{(0_L, 1_L, 1_L), (1_L, 0_L, 0_L)\}$ *and* $\tau_2 = \{(0_L, 1_L, 1_L), (1_L, 0_L, 0_L), (a, 1_L, 1_L)\}$ *where* $a \neq 0_L$. *It is an easy observation that the constant function* f *is not a continuous function from* (X,τ_1) *to* (X,τ_2).

6. Conclusion

The study have introduced a generalization of single valued neutrosophic sets and a corresponding topological structure. Various properties of the introduced concepts were found to be in consistent with their conventional counterparts.

Investigating the neighbourhood structures of LVN topological spaces forms part of our future study. The relations of the generalized structures with their special cases will be examined in the light of category theory. Having defined the category of lattice valued neutrosophic sets, its relation with various subcategories can be investigated and a similar investigation for the category of LVN topological spaces can also be undertaken.

ACKNOWLEDGEMENTS

The first author wishes to thank CSIR, India for giving financial assistance under the Junior Research Fellowship awarded by order No. 08/528(0009)/2020-EMR-1 dated 22/07/2020.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20 (1986), 87-96.

[2] S. Broumi, F. Smarandache, Single valued neutrosophic trapezoid linguistic aggregation operators based multiattribute decision making, Bulletin of Pure and Applied Sciences- Mathematics and Statistics (2014), 135-155.

[3] C.L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182-190.

[4] I. Deli, M. Ali, F. Smarandache, Bipolar neutrosophic sets and their application based on multi-criteria decision making problems, in: 2015 International Conference on Advanced Mechatronic Systems (ICAMechS), IEEE, Beijing, China, 2015: pp. 249–254.

[5] M.K. EL Gayyar, Smooth neutrosophic topological spaces, Neutrosophic Sets Syst. 12 (2016), 65-72.

[6] G. Gierz , K.H. Hofmann, K. Keimel, et al. A compendium of continuous lattices, Springer, Berlin, 1980.

[7] J.A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967), 145-174 .

[8] J.A. Goguen, The fuzzy Tychonoff theorem, J. Math. Anal. Appl. 43 (1973), 734-742.

[9] A. Kharal, A neutrosophic multi-criteria decision making method, New Mathematics and Natural Computation 10(2) (2014), 143-162.

[10] J. Kim, P.K. Lim, J.G. Lee, K. Hur, Single valued neutrosophic relations, Ann. Fuzzy Math. Inform. 16 (2018), 201-221.

[11] J. Kim, F. Smarandache, J.G. Lee , K. Hur, Ordinary single valued neutrosophic topological spaces, Symmetry, 11 (2019), 1075.

[12] D. Koundal, S. Gupta, S. Singh, Applications of Neutrosophic Sets in Medical Image Denoising and Segmentation, In: F. Smarandache, S. Pramanik (eds) New Trends in Neutrosophic Theory and Applications, Pons Editions, Brussels, Belgium, 2016.

[13] Y.M. Liu, M.K. Luo, Fuzzy topology, World Scientific, Singapore, 1997.

[14] F.G. Lupiáñez, On neutrosophic sets and topology, Procedia Comput. Sci. 120 (2017), 975-982.

[15] P.K. Maji, Neutrosophic soft set, Ann. Fuzzy Math. Inform. 5 (2013), 157-168.

[16] P. Majumdar, Neutrosophic Sets and Its Applications to Decision Making, in: D.P. Acharjya, S. Dehuri, S. Sanyal (Eds.), Computational Intelligence for Big Data Analysis, Springer International Publishing, Cham, 2015: pp. 97–115.

[17] A.A. Salama, S.A. Alblowi, Neutrosophic set and neutrosophic topological spaces, IOSR J. Math. 3 (2012), 31-35.

[18] F. Smarandache, N-norm N-conorm in Neutrosophic logic and set, and the neutrosophic topologies, in: A unifying field in logics: Neutrosophic logic. Neutrosophy, Neutrosophic set, Neutrosophic probability (4th ed.) Amer. Res. Press, Rehoboth, 2005.

[19] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, Amer. Res. Press, Rehoboth, 1998.

[20] H. Wang, F. Smarandache, Y.Q. Zhang, R. Sunderraman, Single valued neutrosophic sets, Multispace Multistruct. 4 (2010), 410-413.

[21] H. Wang, Interval neutrosophic sets and logic: theory and applications in computing, Hexis, Phoenix, 2005.

[22] H.L. Yang, Z.L. Guo, X. Liao, On single valued neutrosophic relations, J. Intell. Fuzzy Syst. 30 (2016), 1045-1056.

[23] J. Ye, A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, J. Intell. Fuzzy Syst. 26 (2014), 2450-2466.

[24] L. A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338-353.