A NEW ACCURATE PROCEDURE FOR SOLVING NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a modified algorithm based on the residual power series procedure is employed to find an accurate approximate solution for nonlinear delay differential equations (NDDEs). This modification is considered as a powerful procedure for improving the efficiency of the residual power series method (RPSM) by using the Laplace transform and Padé approximant to be an effective procedure that has the ability to give accurate results closed to the exact solutions with easy computational work. Some numerical examples are presented to check the validity and the applicability of this modification and the results obtained are compared by the results obtained by other method in literature to illustrate and prove it is efficiency and reliability for solving this kind of equations.

Keywords: NDDEs; RPSM; MRPSM; Laplace transform; Padé approximant.

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1. INTRODUCTION

Delay differential equations are considered as an important type of functional differential equations. The derivative of the unknown function in these type of equations at a certain time is given in terms of the function at previous times. DDEs arise in several branch of applied

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science and engineering such as biological models, physics, medical and biochemical, and control system [1, 2, 3]. DDEs attracted the attention of the researchers in the last few years and a great efforts have been conducted to obtain accurate approximate solutions. High accurate approximate solutions are achieved by Gheng and Wang (2010) using variational iteration method (VIM). Evan and Raslan (2005) obtained appropriate approximate solutions for some classes of DDEs via Adomian decomposition method [5]. Anakira et al (2013) employed for the first time the optimal homotopy asymptotic method to solve these type of equations and appropriate results were obtained [6]. Recent years witnessed a great efforts and works from the researchers to develop a numerical or approximate analytical method which is valid for this type of equations and great efforts have been achieved using different methods [7, 8, 9, 10, 11, 12, 13].

The residual power series method is an effective and reliable procedure for solving a large classes of differential equations with easy computational work without needed to perturbation, discretization or linearization. This procedure give us the solutions in a form of polynomial. the last few years witnessed alot of works conducted by residual power series method for solving different classes and types of differential equations [14, 15, 16, 17, 18, 19, 20].

The main objective of this paper is to find accurate solutions which is closed to the exact one by applying a new modification based on the standard RPSM by applying the Laplace transformation to the truncated series obtained by RPSM, followed by employing the Padé approximants to convert the transformed series into a meromorphic function, and finally obtaining an analytic solution using the inverse Laplace transform. The capability of this modification is tested throughout several examples which provide us the solution in a form of series that is closed to the exact solution.

This paper formulated in the following form, Section 2 present a brief overview and some basic rules of RPSM, Padé approximants and Laplace transform. In Section the cabability of this procedure is tested throughout several examples to illustrates it is simplicity and efficiency. Lastly, the conclusion and discussion are in Section 4.

2. DESCRIPTION OF THE SOLUTIONS PROCEDURES

2.1. Residual Power Series Method RPSM. In this part, we will explain the solution procedure of RPSM [15]. The RPSM consists in expressing the solutions of of the given problem in

a form of a power series expansion about the initial point $t = t_0$ for the given problem

(1) $$u'(t) = f(t, u(t)), \quad u(0) = t_0, \quad t \in [0, a],$$

where $f : [0, a] \times R \rightarrow R$ are nonlinear continuous function, $u(t)$ are unknown functions of independent variable t to be determined, and $a > 0$. To reach our goal, we assume the solution in the following form

(2) $$u(t) = \sum_{m=0}^{\infty} u_m(t),$$

where $u_m(t)$ are terms of approximations, note that, when $m = 0$, we have $u_0(t) = u(t_0) = c_0$, which is the initial guess approximation, then we evaluate $u_m(t), \forall m = 1, 2, \dots$ and approximate the solution $u(t)$ of the given problem by k'th truncated series

(3) $$u_k(t) = \sum_{m=0}^{k} c_m(t)^m$$

To apply the RPSM, we write the given problem (1) in the following form:

(4) $$u'(t) - f(t, u(t) = 0.$$

Now, the kth residual function will be obtained by substituting the $k'th$ truncated series (3) into Eq. (4), as given below

(5) $$Res_k(t) = \sum_{m=1}^{k} m c_m t^{m-1} - f(t, \sum_{m=0}^{k} u_m(t)),$$

and the following ∞ 'th residual function:

(6) $$Res^{\infty}(t) = \lim_{k \to \infty} Res^k(t)$$

Clearly, it is easy to see that $Res^{\infty}(t) = 0$ for each $t \in (t_0, T)$, are infinitely differentiable functions at $t = t_0$. Moreover, $\frac{d^m}{dt^m} Res^{\infty}(t_0) = \frac{d^m}{dt^m} Res^k(t_0) = 0, m = 1, 2, \dots, k$, this relation is considered a basic rule in the RPSM and its applications.

Now, in order to obtain the first order-approximate solutions, we put $k = 1$, and substituting $t = 0$ into Eq. (5), and using the fact that $Res^{\infty}(0) = Res^1(0) = 0$, to evaluate $c_1 = f(0, c_0) =$

$f(0, u(0))$. Thus, using first-truncated series the first approximation for the given problem can be written as

$$(7) \qquad\qquad u(t) = u(t_0) + f(t_0, u(t_0))t$$

Similarly, the second-order approximation will be will be obtained by substituting $k = 2$ into Eq. (2) to be $\sum_{m=0}^{k=2} u_m(t)$ and by differentiate both sides of Eq. (5) with respect to t which yields to $\frac{d}{dt} Res^2(0) = 2c_2 - \frac{\partial}{\partial t} f(0, c_0) - c_1 \frac{\partial}{\partial u^2} f(0, c_0)$. In fact, $\frac{d}{dt} Res^2(0) = Res^\infty(0) = 0$.

Thus, we can write $c_2 = \frac{1}{2}(\frac{\partial}{\partial t} f(0, u(0)) + c_1 \frac{\partial}{\partial u^2} f(0, u(0)))$. Therefore, by consider the values of c_1 and c_2 into Eq.(3) when $k = 2$, the second-order approximate solution for the given problem becomes:

$$(8) \qquad\qquad u(t) = u(0) + c_1 t + c_2 t^2$$

The same proccess will be repeated to compute more components of the solution-order to obtain higher accuracy. The next theorem shows convergence of the RPS method.

Theorem 2.1. [15] *Suppose that $u(t)$ is the exact solution for the given problem (). Then, the approximate solution obtained by the RPS method is just the Maclaurin expansion of $u(t)$.*

Corollary 2.1.1. [15] *If $u(t)$ or some components of $u(t)$ is a polynomial, then the RPS method will be obtained the exact solution. It will be convenient to have a notation for the error in the approximation $u(t) \approx u_k(t)$. Accordingly, we will let $Rem^k(t)$ denote the difference between $u(t)$ and its k th Maclaurin polynomial; that is,*

$$(9) \qquad\qquad Rem^k(t) = u(t) - u_k(t) = \sum_{m=k+1}^{\infty} u_m(0)t^m.$$

The functions $Rem^k(t)$ are called the *kth* remainder for the Maclaurin series of $u(t)$. In fact, it often happens that the remainders $Rem^k(t)$ become smaller and smaller, approaching zero, as k gets large

2.2. Padé approximation. The $[L/M]$ Padé approximants of a function $u(x)$ [22, 23] is given by

$$\left[\frac{L}{M}\right] = \frac{P_L(t)}{Q_M(t)}$$

where $P_L(t)$ and $Q_M(t)$ are polynomials of degrees at most L and M, respectively. We know the formal power series

$$u(t) = \sum_{i=1}^{\infty} a_i t^i.$$

The coefficients of the polynomials $P_L(t)$ and $Q_M(t)$ are obtained from the equation

(10) $$u(t) - \frac{P_L(t)}{Q_M(t)} = O\left(t^{L+M+1}\right)$$

When the fraction of the numerator and denominator $\frac{P_L(t)}{Q_M(t)}$ is multiplying by a nonzero constant the fractional values remain unchanged, then we can define the normalization condition as

(11) $$Q_M(0) = 1.$$

Hence, we note that $P_L(t)$ and $Q_M(t)$ have no public factors. If we express the coefficient of $P_L(t)$ and $Q_M(t)$ as

(12) $$\begin{cases} P_L(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_L t^L \\ Q_M(t) = q_0 + q_1 t + q_2 t^2 + \cdots + q_M t^M \end{cases}$$

then, by Eq.s (17) and (18), we may multiply (16) by $Q_M(x)$, which linearizes the coefficient equations. We can write out Eq. (16) in more detail as

(13) $$\begin{cases} a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + \cdots + a_{L-M+2} q_M = 0 \\ \qquad\qquad . \\ \qquad\qquad . \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M = 0 \end{cases}$$

(14) $$\begin{cases} a_0 = p_0 \\ a_0 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ \qquad\qquad . \\ \qquad\qquad . \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L = p_L \end{cases}$$

To solve these equations, we start with Eq. (19), which is a set of linear equations for all the unknown $q's$. Once the $q's$ are known, then Eq. (20) gives an explicit formula for the unknown $p's$, which complete the solution.

If Eq. 19 and Eq. 20 are non-singular, then we can solve them directly and obtain Eq.21 [**?**], where Eq. 21 holds, and if the lower index on a sum exceeds the upper, the sum is replaced by zero:

$$
(15) \quad \left[\frac{L}{M}\right] = \frac{det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ \sum_{j=M}^{L} a_{j-M}x^j & \sum_{j=M-1}^{L} a_{j-M+1}x^j & \cdots & \sum_{j=0}^{L} a_j x^j \end{bmatrix}}{det \begin{bmatrix} a_{L-M+1} & a_{L-M+2} & \cdots & a_{L+1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_L & a_{L+1} & \cdots & a_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{bmatrix}}
$$

Now, we can obtain Padé approximants diagonal matrix of different order using software such as Mathematica, Matlab and son.

3. NUMERICAL RESULTS AND DISSECTIONS

This section is devoted to present some numerical examples to check the validity and performances of our procedure.

3.1. Numerical Results.

3.1.1. *Example 1.* Consider the following first-order nonlinear delay differential equation taken from Anakira et al [6]

$$
(16) \quad u'(t) = -2u^2\left(\frac{t}{2}\right) + 1, u(0) = 1
$$

To obtain approximate solution using RPSM, we consider the general form of the solution in the following kth-truncated series

$$(17) \qquad u_k(t) = \sum_{m=0}^{k} c_m t^m = 1 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_k t^k.$$

Now, the unknown coefficients $c_m, m = 1, 2, \cdots, k$ will be determined by constructing the following kth residual functions

$$(18) \qquad Res^m(t) = \sum_{m=1}^{k} m c_m t^{m-1} - 2\left(\sum_{m=0}^{k} c_m \left(\frac{t}{2}\right)^m\right)^2 + 1.$$

To find the first ten-order RPSM approximate solution, we substitute $k = 10$ into Eq. (17)

$$(19) \qquad Res^m(t) = \sum_{m=1}^{10} m c_m t^{m-1} - 2\left(\sum_{m=0}^{10} c_m \left(\frac{t}{2}\right)^m\right)^2 + 1.$$

Based on the description of the RPSM method formulated in section 2, the value of the $c_0 = u_0(0) = 1$ which is the initial guesses or the given initial condition. Now to find c_1, we use Eq.(24) to the evaluate the $Res^m(0) = 0$ and by consider $c_0 = 1$ we obtain $c_1 = 1$, Based on the fact that $\frac{d}{dt} Res^m(0) = 0$, and using $c_0 = 1$ and $c_1 = 1$ we have the value the constant $c_2 = 0$, then we find the value of the constant c_3 by using $\frac{d}{dt} Res^m(0) = 0$ and substituting $c_0 = 1$, $c_1 = 1$ and $c_2 = 0$, we obtain $c_3 = -\frac{1}{6}$, we follow the same procedure by applying $\frac{d^{k-1}}{dt^{k-1}} Res^m(0) = 0$ to find the constant $c_4, \cdots, c_{10}$. to obtain following 10th order-approximate solutions

$$(20) \qquad u_{10}(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!}$$

which gives the exact solution in the limit of infinity terms of the order of the approximate solutions. To improve the performance of the RPSM, we employ an effective and powerful procedure depend on RPSM truncated series solutions by taken the laplace transform to the first four terms of Eq. (27) to obtain

$$(21) \qquad L\{u_{10}(t)\} = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \frac{1}{s^{10}}$$

assume that $s = \frac{1}{z}$, yields to

$$(22) \qquad L\{u_{10}(t)\} = z^2 - z^4 + z^6 - z^8 + z^{10}.$$

Then, using the Padé approximants of $[\frac{2}{2}]$

$$(23) \qquad\qquad Lu \;=\; \frac{z^2}{z^2+1}.$$

Now use $z = \frac{1}{s}$, and applying the inverse of Laplace transform, we obtain the exact form of the given problem

$$(24) \qquad\qquad u(t) = Sin(t)$$

3.1.2. *Example 2.* The second example considered in this section, is the following second-order nonlinear delay differential equation

$$(25) \qquad\qquad u''(t) = 1 - 2u^2(\frac{t}{2}), \quad u(0) = 1, \quad u'(0) = 1.$$

To solve this problem by means of RPSM, we consider the general form of the solution in the following k'th truncated series

$$(26) \qquad\qquad u_k(t) = \sum_{m=0}^{k} c_m t^m = 1 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_k t^k.$$

The coefficients $c_m, m = 1, 2, \cdots, k$ will be determined by constructing the following k'th residual functions

$$(27) \qquad\qquad Res^m(t) = \sum_{m=2}^{k} m^2 c_m t^{m-2} + 2(\sum_{m=0}^{k} c_m (\frac{t}{2})^m)^2 - 1.$$

The first ten-order RPSM approximate solution will be obtained by using $k = 10$ into Eq. (27)

$$(28) \qquad\qquad Res^m(t) = \sum_{m=2}^{10} m^2 c_m t^{m-2} + 2(\sum_{m=0}^{10} c_m (\frac{t}{2})^m)^2 - 1.$$

depend on the RPSM method formulated in section 2, the value of the $c_0 = u_0(0) = 1$ which is the initial guesses or the given initial condition. Now to determined the value of c_1, we use Eq.(24) to the compute the $Res^m(0) = 0$ and by consider $c_0 = 1$, we have $c_1 = 0$, depend on the fact that $\frac{d}{dt}Res^m(0) = 0$, and using $c_0 = 1$ and $c_1 = 1$ we obtaine the value the constant $c_2 = -\frac{1}{2}$, then we find the value of the constant c_3 by using $\frac{d}{dt}Res^m(0) = 0$ and substituting $c_0 = 1, \quad c_1 = 1$ and $c_2 = 0$, we have $c_3 = 0$, the same procedurewill be repeated by applying $\frac{d^{k-1}}{dt^{k-1}}Res^m(0) = 0$ to compute the constant $c_4, \cdots, c_{10}$. to obtain following 10th order-approximate solutions

$$(29) \qquad\qquad u(t) = 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!}$$

which leads to the exact solution in the limit of infinity terms of the order of the itrations of the approximate solutions. To improve the performance of the RPSM, we apply an effective and powerful procedure depends the truncated series solutions of the RPSM by applying the laplace transform to the first four terms of Eq. (27) to obtain

$$(30) \qquad L\{u(t)\} = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \frac{1}{s^{10}}.$$

Let $s = \frac{1}{z}$, yields to

$$(31) \qquad L\{u_{10}(t)\} = z - z^3 + z^5 - z^7 + z^9$$

Then, using the Padé approximants of $\left[\frac{2}{2}\right]$

$$(32) \qquad Lu \quad = \quad \frac{z}{z^2 + 1}.$$

Using $z = \frac{1}{s}$, and applying the inverse of the Laplace transform, we obtain the exact solution

$$(33) \qquad u(t) = cos(t).$$

3.1.3. *Example 3.* In this example, the third-order nonlinear delay differential equation taken from Anakira et al [6] is given in the following form

$$(34) \qquad u'''(t) = 2u^2(\frac{t}{2}) - 1, \quad u(0) = 1, \quad u'(0) = 1, \quad u''(0) == 0, \quad x \in [0, 1].$$

To solve this problem by means RPSM, the general kth-truncated series form of the RPSM solution is considered

$$(35) \qquad u_k(t) = \sum_{m=0}^{k} c_m t^m = 1 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots + c_k t^k.$$

The coefficients $c_m, m = 1, 2, \cdots, k$ can be determined by constructing the following kth residual functions

$$(36) \qquad Res^m(t) = \sum_{m=3}^{k} m(m-1)(m-2)c_m t^{m-3} + 2(\sum_{m=0}^{k} c_m(\frac{t}{2})^m)^2 - 1.$$

To determine the the first ten-order RPSM approximate solution, we put $k = 10$ into Eq. (23)

$$(37) \qquad Res^m(t) = \sum_{m=3}^{10} m(m-1)(m-2)c_m t^{m-3} + 2(\sum_{m=0}^{10} c_m(\frac{t}{2})^m)^2 - 1.$$

depend on the description of the RPS method formulated in section 2, the value of the $c_0 = u_0(0) = 1, \quad c_1 = u'(0) = 1, c_2 = u''(0) == 0$, which are the initial guesses or the given

initial conditions. To find c_3, we use Eq.(42) to evaluate the $Res^m(0) = 0$ and by consider $c_0 = 1$, $c_1 = 1$ and $c_2 = 0$, we obtain $c_3 = -\frac{1}{6}$, depend on the fact that $\frac{d}{dt}Res^m(0) = 0$, and using $c_0 = 1$, $c_1 = 1$, $c_2 = 0$ and $c_3 = -\frac{1}{6}$ we have the value the constant $c_4 = 0$, The value of the constant c_5 will be obtained by using $\frac{d^2}{dt^2}Res^m(0) = 0$ and substituting $c_0 = 1$, $c_1 = 1$, $c_2 = 0$ $c_3 = -\frac{1}{6}$, and $c_4 = 0$, we obtain $c_5 = \frac{1}{120}$. The same procedure will be repeated by applying $\frac{d^{k-1}}{dt^{k-1}}Res^m(0) = 0$ to find the constant $c_6, \cdots, c_{10}$. to obtain following 10th order-approximate solutions

$$(38) \qquad u(t) = t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880}$$

which is converge to the exact solution in the limit of infinity terms of the order of the approximate solutions. To improve the accuracy of the RPSM, we will use an effective and powerful procedure based on RPSM truncated series solutions by applying the laplace transform to the first four terms of Eq. (27) to obtain

$$(39) \qquad L\{u_{10}(t)\} = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} + \frac{1}{s^{10}}$$

Consider $s = \frac{1}{z}$, yields

$$(40) \qquad L\{u_{10}(t)\} = z^2 - z^4 + z^6 - z^8 + z^{10}.$$

Then, using the Padé approximants of $\left[\frac{2}{2}\right]$

$$(41) \qquad Lu \;\; = \;\; \frac{z^2}{z^2 + 1}.$$

Now, we use $z = \frac{1}{s}$, and then applying the inverse of Laplace transform yields to the exact form solution

$$(42) \qquad u(t) = sin(t)$$

 The results obtained show that the prsented procedure give us accurate solutions closed to the analytic solution using a few number's of iterations of the RPSM solutions, this advantage overcomes the difficulties of calculating more iteration solutions to achive more accurate results and improve the efficiency of the standard RPSM that is observed from the absolute errors displayed in Figs.1-3.
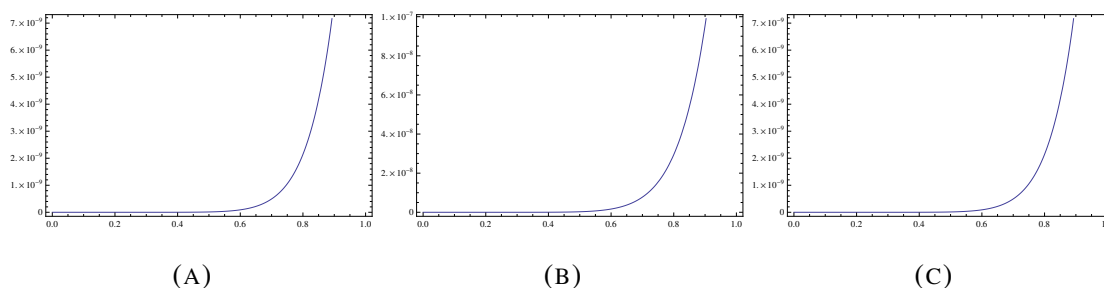
FIGURE 1. Absolute errors related to example 1, 2 and 3, respictively

4. CONCLUSIONS

In this study, an effictive and powerful procedure based on the RPSM has been employed successfully for the first time to find an accurate approximate solution for classes of NDDEs which is closed form of the analytic solution. The performance and efficiency of this procedure repsented by it is simplicity of application that are verfied and satisfied throughout the numerical results. This procedure converges rapidly to the exact solution and needs less computational work comparing by other methods which is consider one of the best advantage.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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