CRITICAL PATH METHOD FOR THE ANALYSIS OF BITUMINOUS ROAD TRANSPORT NETWORK UNDER FUZZY ENVIRONMENT OF VARIOUS FUZZY QUANTITIES

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Abstract: Determining the critical path in bituminous road transport network in the past period depended only on time and cost, whereas in current period the identifying the critical path has become complicated because of multi-criteria: time, distance, security, etc. We have proposed Integrated FAHP – FTOPSIS Methodology for determining the optimal critical path based on trapezoidal, hexagonal and octagonal fuzzy numbers. The aim of this research work is for transport department and society should have through the effective methodology to avoid critical path and for safe journey. Moreover, this methodology will carry for future course of action through Ministry of Road Transport to make safe roadways which is helpful and useful for effective transportations. We are exhibiting the proposed methodology by giving a numerical example and comparing trapezoidal, hexagonal and octagonal fuzzy numbers.

Keywords: trapezoidal fuzzy number; hexagonal fuzzy number; octagonal fuzzy number; fuzzy critical path (FCP); fuzzy analytical hierarchy process (FAHP); fuzzy technique for order preference by similarity to ideal solution

(FTOPSIS).

2010 AMS Subject Classification: 94D05, 97M40, 03E72.

1. INTRODUCTION

To reach from one place to another place, one medium is needed for carrying goods, animals or humans which is called transport. The day was there people utilized horse and oxen or they used to go everywhere they defined on foot and return from the destination in the same way as they were. This method and system for transport was felt difficult for the people especially for the village people. Days go by, transport also has been developed due to technology growth. In general sense, three modes of transportations air, water and land have got enormous growth.

Among three modes of transportations, land is considered as an important and inevitable mode. Because air and water are utilized only by big shots and bearocracy people but land, people who are not sound in financial position, influenced persons, can utilize. Railways and airways cannot enter into nook and corner of the village and rural areas whereas roadways can execute service door to door even in a remote place. Rail and airways services can be utilized for a long distance. Shortest distance coverage, personal service, flexible and for all purposes of service reaching only possible and acceptable mode is Road Transportations which is always under easy moving and covered budget. Hence Road Transport plays an important role in all developments which are seen for the betterment of national growth in the world arena.

Therefore, roadways are considered to be the most important for communication. It is clear that efficient management of road network never choose or desire the critical route because of the barriers and disturbances. They are nature of the road, poor maintenance, interior places without infrastructure resulted often accidents, delaying in moving as well as reaching the destination, congestion, etc. In the past years, the selection of the routes were calculated by using numerical methods as attributes of road transporting were considered as crisp values. Now, in modern time, technology has been highly improved. In that juncture, sometimes attributes might have given vague values are presented for which we cannot use or utilize numerical methods. Therefore, we use Fuzzy concept to solve the previous problematic experiences. When attributes

are seen vague, Fuzzy critical path involves to improve road transport network and it is proved a success one. First and foremost, Critical path was identified under fuzzy environment in 1970's and various techniques and also ranking methods have been proposed for determining fuzzy critical path by many researcher till now.

After that, methods for multi-criteria decision making were used and developed from classical decision-making methods such as AHP, Electre, etc. in which only time was an important criterion to determine the critical path. But in today's era, not only time but other criteria like safety, distance, are also considered for determining the critical path. Zammori et al. [4] in 2009 used some critical parameters such as time variation, risks in externals, etc. for identifying the critical path by using TOPSIS. In 2010 [1], Ahmadvand et al. developed a method for FCP using more critical parameters. In 2013 [5], Cristobal used PROMETHEE methodology for FCP using safety, time, etc. Mehlawat in 2016 [6] have applied TOPSIS method to find FCP with some criteria and also identified it based on weakness and strength indexes. In 2016 [7], Mahtab developed a reliable method for ranking the affecting factors of critical path using FAHP and SIR.VIKOR and found the optimal critical path. Riddhi et al. used fuzzy programming method for an exponential and linear membership functions to find the critical path and also compared with TOPSIS method in 2019 [10]. Recently, We have proposed two distinct algorithms for FCP and also proposed the defuzzification technique using centroid in 2020[8]. Also, we have proposed Integrated FAHP – FTOPSIS methodology to find the optimal critical path using trapezoidal fuzzy numbers in road transport with multi-criteria [9]. In this paper, we have extended the integrated FAHP – FTOPSIS methodology for hexagonal and octagonal fuzzy numbers and also we have introduced the distance function based on centroid of incenters to rank the alternatives.

The structure of this research paper is as follows: Section 1 gives the literature. In Section 2, the basic concepts of fuzzy set theory are reviewed. Section 3 describes an Integrated FAHP – FTOPSIS methodology to identify Fuzzy Critical Path and Section 4 illustrates the numerical example to demonstrate our proposed methodology. In Section 5, We have provided the results with discussion and Section 6 ends with conclusion.

2. PRELIMINARIES

L. A. Zadeh [12] introduced the Fuzzy set theory for uncertain environment and also modelling the fuzzy decision makings is its important role. There are some basic definitions in this section.

Definition 1 (Fuzzy Set). A non-empty subset A in a universal set X is a fuzzy set if it is defined by the membership function μ which assigns a value in $[0,1]$ and

$$\tilde{A} = \left\{(x,\ \mu_{\tilde{A}}(x)) : x \in X, \mu_{\tilde{A}}(x) \in [0,1]\right\}$$

Definition 2 (Trapezoidal Fuzzy Number). A trapezoidal fuzzy number $\tilde{A} = (\theta_1, \theta_2, \theta_3, \theta_4 ; \omega)$ is non-normal if

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\left(\dfrac{x - \theta_1}{\theta_2 - \theta_1}\right), & \theta_1 \le x \le \theta_2 \\ \omega, & \theta_2 \le x \le \theta_3 \\ \omega\left(\dfrac{x - \theta_4}{\theta_3 - \theta_4}\right), & \theta_3 \le x \le \theta_4 \\ 0, & otherwise \end{cases}$$

where $0 < \omega \le 1$ and it is normal when $\omega = 1$.

Definition 3 (Hexagonal Fuzzy Number). A hexagonal fuzzy number $\tilde{A} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 ; \omega)$ is non-normal if

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{\omega}{2}\left(\dfrac{x - \theta_1}{\theta_2 - \theta_1}\right), & \theta_1 \le x \le \theta_2 \\ \dfrac{\omega}{2} + \dfrac{\omega}{2}\left(\dfrac{x - \theta_2}{\theta_3 - \theta_2}\right), & \theta_2 \le x \le \theta_3 \\ \omega, & \theta_3 \le x \le \theta_4 \\ \omega - \dfrac{\omega}{2}\left(\dfrac{x - \theta_4}{\theta_5 - \theta_4}\right), & \theta_4 \le x \le \theta_5 \\ \dfrac{\omega}{2}\left(\dfrac{\theta_6 - x}{\theta_6 - \theta_5}\right), & \theta_5 \le x \le \theta_6 \\ 0, & otherwise \end{cases}$$

where $0 < \omega \le 1$ and it is normal when $\omega = 1$.

Definition 4 (Octagonal Fuzzy Number). $\tilde{A} = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7, \theta_8; \omega)$ is non-normal octagonal fuzzy number if

$$\mu_{\tilde{A}}(x) = \begin{cases} k\left(\dfrac{x - \theta_1}{\theta_2 - \theta_1}\right), & \theta_1 \le x \le \theta_2 \\ k, & \theta_2 \le x \le \theta_3 \\ k + (\omega - k)\left(\dfrac{x - \theta_3}{\theta_4 - \theta_3}\right), & \theta_3 \le x \le \theta_4 \\ \omega, & \theta_4 \le x \le \theta_5 \\ k + (\omega - k)\left(\dfrac{\theta_6 - x}{\theta_6 - \theta_5}\right), & \theta_5 \le x \le \theta_6 \\ k, & \theta_6 \le x \le \theta_7 \\ k\left(\dfrac{\theta_8 - x}{\theta_8 - \theta_7}\right), & \theta_7 \le x \le \theta_8 \\ 0, & otherwise \end{cases}$$

where $0 < \omega \le 1$ *and* $0 < k \le \omega$ and it is normal when $\omega = 1$.

3. PROPOSED FUZZIFICATION APPROACHES

Let the interval data be $I = [X, Y]$. Then, the tri-section, penta-section and hepta-section of I are $D = \dfrac{(Y - X)}{3}$, $D = \dfrac{(Y - X)}{5}$ *and* $D = \dfrac{(Y - X)}{7}$ respectively.

Thus, the trapezoidal, hexagonal and octagonal fuzzy numbers will be taken as

(1) $$(X, X + D, X + 2D, Y)$$

(2) $$(X, X + D, X + 2D, X + 3D, X + 4D, Y)$$

(3) $$(X, X + D, X + 2D, X + 3D, X + 4D, X + 4D, X + 5D, X + 6D, Y)$$

4. METHODOLOGY

4.1. Goal

The main aim of this proposed integrated methodology is to identify the critical path of the road transport network and an index based model by taking both qualitative and quantitative aspects is used to calculate the critical path.

4.2. Identification of Criteria and Sub-Criteria

We have presented the criteria and their sub-criteria in Table 1.

TABLE 1. Criteria and their Sub-criteria [9]

Criteria	Sub-criteria
Cost (C_1)	Fuel Cost(C_{11})
	Toll Cost(C_{12})
	Maintenance Cost(C_{13})
Time(C_2)	Running Time(C_{21})
	Standstill Time(C_{22})
	Unwind Time(C_{23})
Risk in Travel(C_3)	Distance(C_{31})
	Nature of the Road(C_{32})
	Climate Conditions(C_{33})
Non-availability of Facilities & Services(C_4)	Lighting Facilities(C_{41})
	Information Board(C_{42})
	Restaurant(C_{43})
	Medical Facilities(C_{44})
	Vehicle Service Stations(C_{45})
Insecurity(C_5)	Thefts(C_{51})
	Terror Attacks(C_{52})
	Threats caused by Animals(C_{53})

4.3. Proposed Ranking Method based on Centroid of Incenters for Trapezoidal Fuzzy Numbers [9]

Ordering fuzzy numbers is very important in optimization and decision making problems under uncertain environment. We have proposed centroid based ranking method to order trapezoidal fuzzy numbers for critical path selection. First, We divide the trapezoid into three triangles ABR, ARD and RCD and subsequently G_1, G_2 and G_3 are incenters of these three triangles. Join these three incenters and find the centroid G which is a point of inference for defining the ranking function to order trapezoidal fuzzy numbers which is shown in Fig. 1

Let the trapezoidal fuzzy number be $\tilde{A} = (a, b, c, d; \omega)$.

The incenters of ΔABR, ΔARD and ΔRCD are
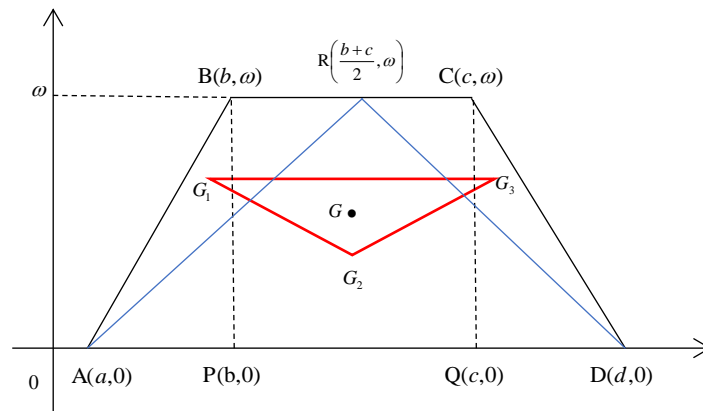
FIGURE 1. Centroid of incenters for trapezoidal fuzzy number

$$G_1 = (x_1, y_1) = \left(\frac{a\alpha_1 + b\beta_1 + \left(\frac{b+c}{2}\right)\gamma_1}{\alpha_1 + \beta_1 + \gamma_1}, \frac{\omega(\beta_1 + \gamma_1)}{\alpha_1 + \beta_1 + \gamma_1} \right)$$

$$G_2 = (x_2, y_2) = \left(\frac{a\alpha_2 + \left(\frac{b+c}{2}\right)\beta_2 + d\gamma_2}{\alpha_2 + \beta_2 + \gamma_2}, \frac{\omega\beta_2}{\alpha_2 + \beta_2 + \gamma_2} \right)$$

$$G_3 = (x_3, y_3) = \left(\frac{\left(\frac{b+c}{2} \right)\alpha_3 + c\beta_3 + d\gamma_3}{\alpha_3 + \beta_3 + \gamma_3}, \frac{\omega(\alpha_3 + \beta_3)}{\alpha_3 + \beta_3 + \gamma_3} \right)$$

Where

$$\alpha_1 = \frac{1}{2}\sqrt{(c-b)^2}, \beta_1 = \frac{1}{2}\sqrt{(b+c-2a)^2 + 4\omega^2}, \gamma_1 = \sqrt{(b-a)^2 + \omega^2},$$

$$\alpha_2 = \frac{1}{2}\sqrt{(2d-b-c)^2 + 4\omega^2}, \beta_2 = \sqrt{(d-a)^2}, \gamma_2 = \frac{1}{2}\sqrt{(b+c-2a)^2 + 4\omega^2},$$

$$\alpha_3 = \sqrt{(d-c)^2 + \omega^2}, \beta_3 = \frac{1}{2}\sqrt{(2d-b-c)^2 + 4\omega^2}, \gamma_3 = \frac{1}{2}\sqrt{(c-b)^2}.$$

These incenters G_1, G_2 and G_3 can form a triangle because they are not in the same line.

The centroid of $\Delta G_1 G_2 G_3$ of the trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is defined by

(4) $$G = (\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Thus, the ranking function of the trapezoidal fuzzy number $\tilde{A}$ is

(5) $$R(\tilde{A}) = \bar{x}$$

4.4. Proposed Ranking Method based on Centroid of Incenters for Hexagonal Fuzzy Numbers

Ordering fuzzy numbers is very important in optimization and decision making problems under uncertain environment. We have proposed a centroid based ranking method to order hexagonal fuzzy numbers for critical path selection. First, hexagon is divided into two triangles ABQ, SEF and two trapezoids BQSE, BCDE. Again, Trapezoid BQSE is divided into three triangles BQR, BRE and RES and trapezoid BCDE which is divided into three triangles BCU, BUE and UDE. Hence, We find the incenters of two triangles ABQ and SEF are G_1 and G_2. For trapezoid BQSE, the centroid of incenters of BQR, BRE and RES is G_3' and for trapezoid BCDE, the centroid of incenters of BCU, BUE and UDE is G_3'' and the centroid of G_3' and G_3'' is G_3. Join these three points G_1, G_2 and G_3 and find the centroid G which is a point of inference

A. HARI GANESH, A. HELEN SHOBANA

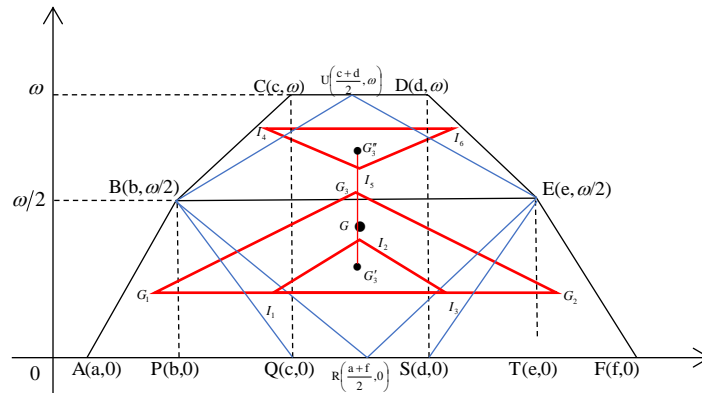for defining the ranking function to order hexagonal fuzzy numbers shown in Fig. 2

FIGURE 2. Centroid of incenters for hexagonal fuzzy number

Let the hexagonal fuzzy number be $\tilde{A} = (a,b,c,d,e,f;\omega)$.

The incenters of $\triangle ABQ$ and $\triangle SEF$ are

$$G_1 = (x_1, y_1) = \left(\frac{a\alpha_1 + b\beta_1 + c\gamma_1}{\alpha_1 + \beta_1 + \gamma_1}, \frac{\left(\frac{\omega}{2}\right)\beta_1}{\alpha_1 + \beta_1 + \gamma_1} \right)$$

$$G_2 = (x_2, y_2) = \left(\frac{d\alpha_2 + e\beta_2 + f\gamma_2}{\alpha_2 + \beta_2 + \gamma_2}, \frac{\left(\frac{\omega}{2}\right)\beta_2}{\alpha_2 + \beta_2 + \gamma_2} \right)$$

Where

$$\alpha_1 = \frac{1}{2}\sqrt{4(c-b)^2 + \omega^2}, \beta_1 = \sqrt{(c-a)^2}, \gamma_1 = \frac{1}{2}\sqrt{4(b-a)^2 + \omega^2},$$

$$\alpha_2 = \frac{1}{2}\sqrt{4(f-e)^2 + \omega^2}, \beta_2 = \sqrt{(f-d)^2}, \gamma_2 = \frac{1}{2}\sqrt{4(e-d)^2 + \omega^2}.$$

For trapezoid BQSE, the incenters of $\triangle BQR$, $\triangle BRE$ and $\triangle RES$ are

$$I_1 = (x_{I_1}, y_{I_1}) = \left(\frac{c\alpha_3 + \left(\frac{a+f}{2}\right)\beta_3 + b\gamma_3}{\alpha_3 + \beta_3 + \gamma_3}, \frac{\left(\frac{\omega}{2}\right)\gamma_3}{\alpha_3 + \beta_3 + \gamma_3} \right)$$

$$I_2 = (x_{I_2}, y_{I_2}) = \left(\frac{b\alpha_4 + \left(\frac{a+f}{2}\right)\beta_4 + e\gamma_4}{\alpha_4 + \beta_4 + \gamma_4}, \frac{\left(\frac{\omega}{2}\right)(\alpha_4 + \gamma_4)}{\alpha_4 + \beta_4 + \gamma_4} \right)$$

$$I_3 = (x_{I_3}, y_{I_3}) = \left(\frac{\left(\frac{a+f}{2}\right)\alpha_5 + e\beta_5 + d\gamma_5}{\alpha_5 + \beta_5 + \gamma_5}, \frac{\left(\frac{\omega}{2}\right)\beta_5}{\alpha_5 + \beta_5 + \gamma_5} \right)$$

Where

$$\alpha_3 = \frac{1}{2}\sqrt{(2b-a-f)^2 + \omega^2}, \ \beta_3 = \frac{1}{2}\sqrt{4(b-c)^2 + \omega^2}, \ \gamma_3 = \frac{1}{2}\sqrt{(a+f-2c)^2},$$

$$\alpha_4 = \frac{1}{2}\sqrt{(2e-a-f)^2 + \omega^2}, \ \beta_4 = \sqrt{(e-b)^2}, \gamma_4 = \frac{1}{2}\sqrt{(a+f-2b)^2 + \omega^2},$$

$$\alpha_5 = \frac{1}{2}\sqrt{4(d-e)^2 + \omega^2}, \ \beta_5 = \frac{1}{2}\sqrt{(2d-a-f)^2}, \gamma_5 = \frac{1}{2}\sqrt{(2e-a-f)^2 + \omega^2}.$$

Hence, the centroid of incenters I_1, I_2 and I_3 is

$$G_3' = (x_3', y_3') = \left(\frac{x_{I_1} + x_{I_2} + x_{I_3}}{3}, \frac{y_{I_1} + y_{I_2} + y_{I_3}}{3} \right)$$

For trapezoid BCDE, the incenters of $\triangle BCU$, $\triangle BUE$ and $\triangle UDE$ are

$$I_4 = (x_{I_4}, y_{I_4}) = \left(\frac{b\alpha_6 + c\beta_6 + \left(\frac{c+d}{2}\right)\gamma_6}{\alpha_6 + \beta_6 + \gamma_6}, \frac{\left(\frac{\omega}{2}\right)\alpha_6 + \omega(\beta_6 + \gamma_6)}{\alpha_6 + \beta_6 + \gamma_6} \right)$$

$$I_5 = (x_{I_5}, y_{I_5}) = \left(\frac{b\alpha_7 + \left(\frac{c+d}{2}\right)\beta_7 + e\gamma_7}{\alpha_7 + \beta_7 + \gamma_7}, \frac{\left(\frac{\omega}{2}\right)(\alpha_7 + \gamma_7) + \omega\beta_7}{\alpha_7 + \beta_7 + \gamma_7} \right)$$

$$I_6 = (x_{I_6}, y_{I_6}) = \left(\frac{\left(\frac{c+d}{2}\right)\alpha_8 + d\beta_8 + e\gamma_8}{\alpha_8 + \beta_8 + \gamma_8}, \frac{\omega(\alpha_5 + \beta_5) + \left(\frac{\omega}{2}\right)\gamma_8}{\alpha_8 + \beta_8 + \gamma_8} \right)$$

Where

$$\alpha_6 = \frac{1}{2}\sqrt{(d-c)^2}, \beta_6 = \frac{1}{2}\sqrt{(c+d-2b)^2 + \omega^2}, \gamma_6 = \frac{1}{2}\sqrt{4(b-c)^2 + \omega^2},$$

$$\alpha_7 = \frac{1}{2}\sqrt{(2e-c-d)^2 + \omega^2}, \beta_7 = \sqrt{(e-b)^2}, \gamma_7 = \frac{1}{2}\sqrt{(c+d-2b)^2 + \omega^2},$$

$$\alpha_8 = \frac{1}{2}\sqrt{4(e-d)^2 + \omega^2}, \beta_8 = \frac{1}{2}\sqrt{(2e-c-d)^2 + \omega^2}, \gamma_8 = \frac{1}{2}\sqrt{(d-c)^2}.$$

Hence, the centroid of incenters I₄, I₅ and I₆ is

$$G_3'' = (x_3'', y_3'') = \left(\frac{x_{I_4} + x_{I_5} + x_{I_6}}{3}, \frac{y_{I_4} + y_{I_5} + y_{I_6}}{3} \right)$$

Thus the centroid of G_3' and G_3'' is

$$G_3 = (x_3, y_3) = \left(\frac{x_3' + x_3''}{2}, \frac{y_3' + y_3''}{2} \right)$$

The centroid of $\Delta G_1 G_2 G_3$ of $\tilde{A} = (a,b,c,d,e,f;w)$ is

(6)
$$G = (\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Therefore, the ranking function of the hexagonal fuzzy number $\tilde{A}$ is

(7)
$$R(\tilde{A}) = \bar{x}$$

4.5. Proposed Ranking Method based on Centroid of Incenters for Octagonal Fuzzy Numbers

Ordering fuzzy numbers is very important in optimization and decision making problems under uncertain environment. We have proposed a centroid based ranking method to order octagonal fuzzy numbers for critical path selection. First, We divide the octagon into four trapezoids ABCR, TFHI, CRTF and CDEF. Again, We divide ABCR into three triangles ABY, AYR and YCR and the centroid of their incenters is G₁ ; Divide the trapezoid into three triangles TFW, TWI and WHI and centroid of their incenters is G₂ ; Divide the trapezoid CRTF into three triangles CRS, CSF and STF and the centroid of their incenters is G_3' ; Divide the trapezoid CDEF into three triangles CDX, CXF and XEF and the centroid of their incenters is G_3''. Then

the centroid of G_3' and G_3'' is G_3. Join these three points G_1, G_2 and G_3 and find the centroid G which is a point of inference for defining the ranking function to order octagonal fuzzy numbers shown in Fig. 3
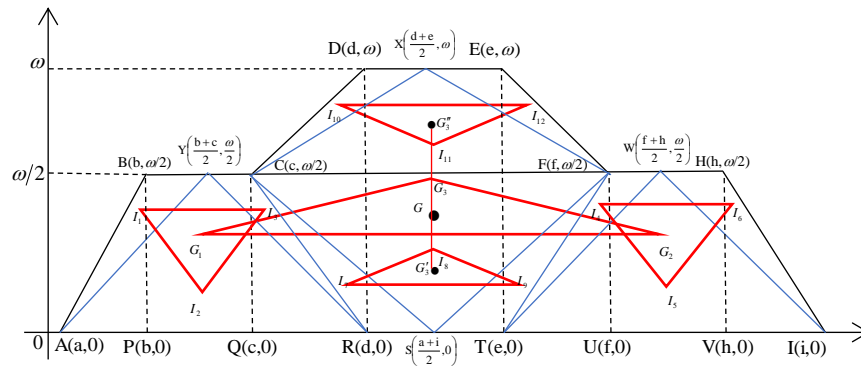

FIGURE 3. Centroid of incenters for octagonal fuzzy number

Let the octagonal fuzzy number be $\tilde{A} = (a,b,c,d,e,f,h,i;\omega)$.

The incenters of $\triangle ABY$, $\triangle AYR$ and $\triangle YCR$ are

$$I_1 = \left(x_{I_1}, y_{I_1}\right) = \left(\frac{a\alpha_1 + b\beta_1 + \left(\frac{b+c}{2}\right)\gamma_1}{\alpha_1 + \beta_1 + \gamma_1}, \frac{\left(\frac{\omega}{2}\right)(\beta_1 + \gamma_1)}{\alpha_1 + \beta_1 + \gamma_1} \right)$$

$$I_2 = \left(x_{I_2}, y_{I_2}\right) = \left(\frac{a\alpha_2 + \left(\frac{b+c}{2}\right)\beta_2 + d\gamma_2}{\alpha_2 + \beta_2 + \gamma_2}, \frac{\left(\frac{\omega}{2}\right)\beta_2}{\alpha_2 + \beta_2 + \gamma_2} \right)$$

$$I_3 = \left(x_{I_3}, y_{I_3}\right) = \left(\frac{\left(\frac{b+c}{2}\right)\alpha_3 + c\beta_3 + d\gamma_3}{\alpha_3 + \beta_3 + \gamma_3}, \frac{\left(\frac{\omega}{2}\right)(\alpha_3 + \beta_3)}{\alpha_3 + \beta_3 + \gamma_3} \right)$$

Where

$$\alpha_1 = \frac{1}{2}\sqrt{(c-b)^2}, \beta_1 = \frac{1}{2}\sqrt{(b+c-2a)^2 + \omega^2}, \gamma_1 = \frac{1}{2}\sqrt{4(b-a)^2 + \omega^2},$$

$$\alpha_2 = \frac{1}{2}\sqrt{(2d-b-c)^2 + \omega^2}\ ,\ \beta_2 = \sqrt{(d-a)^2}\ ,\ \gamma_2 = \frac{1}{2}\sqrt{(b+c-2a)^2 + \omega^2}\ ,$$

$$\alpha_3 = \frac{1}{2}\sqrt{4(d-c)^2 + \omega^2}\ ,\ \beta_3 = \frac{1}{2}\sqrt{(2d-b-c)^2 + \omega^2}\ ,\ \gamma_3 = \frac{1}{2}\sqrt{(c-b)^2}\ .$$

The centroid of incenters I_1, I_2 and I_3 is

$$G_1 = (x_1, y_1) = \left(\frac{x_{I_1} + x_{I_2} + x_{I_3}}{3}, \frac{y_{I_1} + y_{I_2} + y_{I_3}}{3} \right)$$

The incenters of $\triangle TFW$, $\triangle TWI$ and $\triangle WHI$ are

$$I_4 = (x_{I_4}, y_{I_4}) = \left(\frac{e\alpha_4 + f\beta_4 + \left(\dfrac{f+h}{2}\right)\gamma_4}{\alpha_4 + \beta_4 + \gamma_4}, \frac{\left(\dfrac{\omega}{2}\right)(\beta_4 + \gamma_4)}{\alpha_4 + \beta_4 + \gamma_4} \right)$$

$$I_5 = (x_{I_5}, y_{I_5}) = \left(\frac{e\alpha_5 + \left(\dfrac{f+h}{2}\right)\beta_5 + i\gamma_5}{\alpha_5 + \beta_5 + \gamma_5}, \frac{\left(\dfrac{\omega}{2}\right)\beta_5}{\alpha_5 + \beta_5 + \gamma_5} \right)$$

$$I_6 = (x_{I_6}, y_{I_6}) = \left(\frac{\left(\dfrac{f+h}{2}\right)\alpha_6 + h\beta_6 + i\gamma_6}{\alpha_6 + \beta_6 + \gamma_6}, \frac{\left(\dfrac{\omega}{2}\right)(\alpha_6 + \beta_6)}{\alpha_6 + \beta_6 + \gamma_6} \right)$$

Where

$$\alpha_4 = \frac{1}{2}\sqrt{(h-f)^2}\ ,\ \beta_4 = \frac{1}{2}\sqrt{(f+h-2e)^2 + \omega^2}\ ,\ \gamma_4 = \frac{1}{2}\sqrt{4(f-e)^2 + \omega^2}\ ,$$

$$\alpha_5 = \frac{1}{2}\sqrt{(2i-f-h)^2 + \omega^2}\ ,\ \beta_5 = \sqrt{(i-e)^2}\ ,\ \gamma_5 = \frac{1}{2}\sqrt{(f+h-2e)^2 + \omega^2}\ ,$$

$$\alpha_6 = \frac{1}{2}\sqrt{4(i-h)^2 + \omega^2}\ ,\ \beta_6 = \frac{1}{2}\sqrt{(2i-f-h)^2 + \omega^2}\ ,\ \gamma_6 = \frac{1}{2}\sqrt{(h-f)^2}\ .$$

The centroid of incenters I_4, I_5 and I_6 is

$$G_2 = (x_2, y_2) = \left(\frac{x_{I_4} + x_{I_5} + x_{I_6}}{3}, \frac{y_{I_4} + y_{I_5} + y_{I_6}}{3} \right)$$

The incenters of $\triangle CRS$, $\triangle CSF$ and $\triangle SFT$ are

$$I_7 = \left(x_{I_7}, y_{I_7}\right) = \left(\frac{c\alpha_7 + d\beta_7 + \left(\frac{a+i}{2}\right)\gamma_7}{\alpha_7 + \beta_7 + \gamma_7}, \frac{\left(\frac{\omega}{2}\right)\alpha_7}{\alpha_7 + \beta_7 + \gamma_7}\right)$$

$$I_8 = \left(x_{I_8}, y_{I_8}\right) = \left(\frac{c\alpha_8 + \left(\frac{a+i}{2}\right)\beta_8 + f\gamma_8}{\alpha_8 + \beta_8 + \gamma_8}, \frac{\left(\frac{\omega}{2}\right)(\alpha_8 + \gamma_8)}{\alpha_8 + \beta_8 + \gamma_8}\right)$$

$$I_9 = \left(x_{I_9}, y_{I_9}\right) = \left(\frac{\left(\frac{a+i}{2}\right)\alpha_9 + f\beta_9 + e\gamma_9}{\alpha_9 + \beta_9 + \gamma_9}, \frac{\left(\frac{\omega}{2}\right)\beta_9}{\alpha_9 + \beta_9 + \gamma_9}\right)$$

Where

$$\alpha_7 = \frac{1}{2}\sqrt{(a+i-2d)^2}, \beta_7 = \frac{1}{2}\sqrt{(a+i-2c)^2 + \omega^2}, \gamma_7 = \frac{1}{2}\sqrt{4(d-c)^2 + \omega^2},$$

$$\alpha_8 = \frac{1}{2}\sqrt{(2f-a-i)^2 + \omega^2}, \beta_8 = \sqrt{(f-c)^2}, \gamma_8 = \frac{1}{2}\sqrt{(a+i-2c)^2 + \omega^2},$$

$$\alpha_9 = \frac{1}{2}\sqrt{4(e-f)^2 + \omega^2}, \beta_9 = \frac{1}{2}\sqrt{(2e-a-i)^2}, \gamma_9 = \frac{1}{2}\sqrt{(2f-a-i)^2 + \omega^2}.$$

The centroid of incenters I_7, I_8 and I_9 is

$$G_3' = \left(x_3', y_3'\right) = \left(\frac{x_{I_7} + x_{I_8} + x_{I_9}}{3}, \frac{y_{I_7} + y_{I_8} + y_{I_9}}{3}\right)$$

The incenters of $\triangle CDX$, $\triangle CXF$ and $\triangle XEF$ are

$$I_{10} = \left(x_{I_{10}}, y_{I_{10}}\right) = \left(\frac{c\alpha_{10} + d\beta_{10} + \left(\frac{d+e}{2}\right)\gamma_{10}}{\alpha_{10} + \beta_{10} + \gamma_{10}}, \frac{\left(\frac{\omega}{2}\right)\alpha_{10} + \omega(\beta_{10} + \gamma_{10})}{\alpha_{10} + \beta_{10} + \gamma_{10}}\right)$$

$$I_{11} = \left(x_{I_{11}}, y_{I_{11}}\right) = \left(\frac{c\alpha_{11} + \left(\frac{d+e}{2}\right)\beta_{11} + f\gamma_{11}}{\alpha_{11} + \beta_{11} + \gamma_{11}}, \frac{\left(\frac{\omega}{2}\right)(\alpha_{11} + \gamma_{11}) + \omega\beta_{11}}{\alpha_{11} + \beta_{11} + \gamma_{11}}\right)$$

$$I_{12} = (x_{I_{12}}, y_{I_{12}}) = \left(\frac{\left(\frac{d+e}{2}\right)\alpha_{12} + f\beta_{12} + e\gamma_{12}}{\alpha_{12} + \beta_{12} + \gamma_{12}}, \frac{\omega(\alpha_{12} + \gamma_{12}) + \left(\frac{\omega}{2}\right)\beta_{12}}{\alpha_{12} + \beta_{12} + \gamma_{12}} \right)$$

Where

$$\alpha_{10} = \frac{1}{2}\sqrt{(e-d)^2}, \beta_{10} = \frac{1}{2}\sqrt{(d+e-2c)^2 + \omega^2}, \gamma_{10} = \frac{1}{2}\sqrt{4(d-c)^2 + \omega^2},$$

$$\alpha_{11} = \frac{1}{2}\sqrt{(2f-d-e)^2 + \omega^2}, \beta_{11} = \sqrt{(f-c)^2}, \gamma_{11} = \frac{1}{2}\sqrt{(d+e-2c)^2 + \omega^2},$$

$$\alpha_{12} = \frac{1}{2}\sqrt{4(e-f)^2 + \omega^2}, \beta_{12} = \frac{1}{2}\sqrt{(e-d)^2}, \gamma_{12} = \frac{1}{2}\sqrt{(2f-d-e)^2 + \omega^2}.$$

The centroid of incenters I_{10}, I_{11} and I_{12} is

$$G_3'' = (x_3'', y_3'') = \left(\frac{x_{I_{10}} + x_{I_{11}} + x_{I_{12}}}{3}, \frac{y_{I_{10}} + y_{I_{11}} + y_{I_{12}}}{3} \right)$$

Thus the centroid of G_3' and G_3'' is

$$G_3 = (x_3, y_3) = \left(\frac{x_3' + x_3''}{2}, \frac{y_3' + y_3''}{2} \right)$$

The centroid of $\Delta G_1 G_2 G_3$ of the octagonal fuzzy number $\tilde{A} = (a,b,c,d,e,f,h,i;w)$ is

(8)
$$G = (\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

Therefore, the ranking function of the octagonal fuzzy number $\tilde{A}$ is

(9)
$$R(\tilde{A}) = \bar{x}$$

4.6. Distance between two Fuzzy Numbers

If $\tilde{X} = (\theta_1, \theta_2, \theta_3, \theta_4; \omega_A)$ and $\tilde{Y} = (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4; \omega_B)$ are two trapezoidal (or hexagonal or octagonal) fuzzy numbers, then the distance between $\tilde{X}$ and $\tilde{Y}$ is

(10)
$$dis(\tilde{X}, \tilde{Y}) = \left| R(\tilde{X}) - R(\tilde{Y}) \right|$$

4.7. Proposed Integrated FAHP – FTOPSIS Methodology

There are various techniques for Multi-criteria decision making in which two important familiar techniques are AHP and TOPSIS. Saaty [11] in 1980's introduced AHP and in 1996, Chang [2] was the first to introduce FAHP using triangular fuzzy numbers and also extent analysis method was used to compute criteria's weight. In 1992 [3], FTOPSIS was introduced by Chen and Hwang for decision making problems with ambiguity and used to evaluate the alternative with respect to criteria. In the proposed methodology, We integrate Fuzzy AHP and Fuzzy TOPSIS to rank the alternatives based on chosen criteria and their sub-criteria.

The steps of our proposed integrated FAHP and FTOPSIS method for determining the critical path are given below.

Step 1: Criteria C_i and their sub-criteria $C_{ij}, i, j = 1, 2, ..., m$ are defined to identify the critical path

Step 2: We have formed the comparison matrix and also the linguistic variables are assigned for criteria and their sub-criteria in the form of trapezoidal fuzzy numbers. That is,

(11)
$$X_K = \begin{bmatrix} a_{ij1} & b_{ij1} & c_{ij1} & d_{ij1} \\ a_{ij2} & b_{ij2} & c_{ij2} & d_{ij2} \\ ... & ... & ... & ... \\ a_{ijk} & b_{ijk} & c_{ijk} & d_{ijk} \end{bmatrix}$$

is the comparison matrix where $1 \le i \le m, 1 \le j \le m$ and k – no. of experts and all elements of

X_K is taken as the trapezoidal fuzzy numbers and also used the pairwise comparison presented

for comparing itself in Table 2.

(or) the comparison matrix is

(12)
$$X_K = \begin{bmatrix} a_{ij1} & b_{ij1} & c_{ij1} & d_{ij1} & e_{ij1} & f_{ij1} \\ a_{ij2} & b_{ij2} & c_{ij2} & d_{ij2} & e_{ij2} & f_{ij2} \\ ... & ... & ... & ... & ... & ... \\ a_{ijk} & b_{ijk} & c_{ijk} & d_{ijk} & e_{ijk} & f_{ijk} \end{bmatrix}$$

(13)
$$X_K = \begin{bmatrix} a_{ij1} & b_{ij1} & c_{ij1} & d_{ij1} & e_{ij1} & f_{ij1} & g_{ij1} & h_{ij1} \\ a_{ij2} & b_{ij2} & c_{ij2} & d_{ij2} & e_{ij2} & f_{ij2} & g_{ij2} & h_{ij2} \\ \ldots & \ldots & & \ldots & \ldots & & \ldots & \ldots & \ldots \\ a_{ijk} & b_{ijk} & c_{ijk} & d_{ijk} & e_{ijk} & f_{ijk} & g_{ijk} & h_{ijk} \end{bmatrix}$$

where all elements of X_K will be taken as the hexagonal and octagonal fuzzy numbers and also used the pairwise comparison presented in Tables 3 and 4 for comparing itself.

TABLE 2. Trapezoidal Fuzzy Conversion Scale for Pair-wise Comparison

Linguistic Variables	Scale of Relative Important of AHP Interval Numbers		Trapezoidal Fuzzy Numbers
Absolutely Less Important (A-L-I)	1	(0, 2)	$\left(0, \frac{2}{3}, \frac{4}{3}, 2\right)$
Very Strong Less Important (V-S-L-I)	2	(1, 3)	$\left(1, \frac{5}{3}, \frac{7}{3}, 3\right)$
Strong Less Important (S-L-I)	3	(2, 4)	$\left(2, \frac{8}{3}, \frac{10}{3}, 4\right)$
Weakly Less Important (W-L-I)	4	(3, 5)	$\left(3, \frac{11}{3}, \frac{13}{3}, 5\right)$
Equally Important (E-I)	5	(4, 6)	$\left(4, \frac{14}{3}, \frac{16}{3}, 6\right)$
Weakly More Important (W-M-I)	6	(5, 7)	$\left(5, \frac{17}{3}, \frac{19}{3}, 7\right)$
Strong More Important (S-M-I)	7	(6, 8)	$\left(6, \frac{20}{3}, \frac{22}{3}, 8\right)$
Very Strong More Important (V-S-M-I)	8	(7, 9)	$\left(7, \frac{23}{3}, \frac{25}{3}, 9\right)$
Absolutely More Important (A-M-I)	9	(8, 10)	$\left(8, \frac{26}{3}, \frac{28}{3}, 10\right)$

TABLE 3. Hexagonal Fuzzy Conversion Scale for Pair-wise Comparison

Linguistic Variables	Scale of Relative Important	of AHP Interval Numbers	Hexagonal Fuzzy Numbers
Absolutely Less Important (A-L-I)	1	(0, 2)	$\left(0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, 2\right)$
Very Strong Less Important (V-S-L-I)	2	(1, 3)	$\left(1, \frac{7}{5}, \frac{9}{5}, \frac{11}{5}, \frac{13}{5}, 3\right)$
Strong Less Important (S-L-I)	3	(2, 4)	$\left(2, \frac{12}{5}, \frac{14}{5}, \frac{16}{5}, \frac{18}{5}, 4\right)$
Weakly Less Important (W-L-I)	4	(3, 5)	$\left(3, \frac{17}{5}, \frac{19}{5}, \frac{21}{5}, \frac{23}{5}, 5\right)$
Equally Important (E-I)	5	(4, 6)	$\left(4, \frac{22}{5}, \frac{24}{5}, \frac{26}{5}, \frac{28}{5}, 6\right)$
Weakly More Important (W-M-I)	6	(5, 7)	$\left(5, \frac{27}{5}, \frac{29}{5}, \frac{31}{5}, \frac{33}{5}, 7\right)$
Strong More Important (S-M-I)	7	(6, 8)	$\left(6, \frac{32}{5}, \frac{34}{5}, \frac{36}{5}, \frac{38}{5}, 8\right)$
Very Strong More Important (V-S-M-I)	8	(7, 9)	$\left(7, \frac{37}{5}, \frac{39}{5}, \frac{41}{5}, \frac{43}{5}, 9\right)$
Absolutely More Important (A-M-I)	9	(8, 10)	$\left(8, \frac{42}{5}, \frac{44}{5}, \frac{46}{5}, \frac{48}{5}, 10\right)$

Step 3: The arithmetic operation for aggregation is

$$\left(a_{ij}, b_{ij}, c_{ij}, d_{ij}\right) = \left(a_{ij1} + a_{ij2} + ... + a_{ijk}, b_{ij1} + b_{ij2} + ... + b_{ijk}, c_{ij1} + c_{ij2} + ... + c_{ijk}, d_{ij1} + d_{ij2} + ... + d_{ijk}\right) \quad \text{(or)}$$

$$\left(a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}\right) = \left(a_{ij1} + a_{ij2} + ... + a_{ijk}, b_{ij1} + b_{ij2} + ... + b_{ijk}, c_{ij1} + c_{ij2} + ... + c_{ijk}, d_{ij1} + d_{ij2} + ... + d_{ijk},\right.$$
$$\left. e_{ij1} + e_{ij2} + ... + e_{ijk}, f_{ij1} + f_{ij2} + ... + f_{ijk}\right)$$

(or)

$$\left(a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij}\right) = \left(a_{ij1} + a_{ij2} + \ldots + a_{ijk}, b_{ij1} + b_{ij2} + \ldots + b_{ijk}, c_{ij1} + c_{ij2} + \ldots + c_{ijk}, d_{ij1} + d_{ij2} + \ldots + d_{ijk},\right.$$
$$\left. e_{ij1} + e_{ij2} + \ldots + e_{ijk}, f_{ij1} + f_{ij2} + \ldots + f_{ijk}, g_{ij1} + g_{ij2} + \ldots + g_{ijk}, h_{ij1} + h_{ij2} + \ldots + h_{ijk}\right)$$

TABLE 4. Octagonal Fuzzy Conversion Scale for Pair-wise Comparison

Linguistic Variables	Scale of Relative Important of AHP Interval Numbers		Octagonal Fuzzy Numbers
Absolutely Less Important (A-L-I)	1	(0, 2)	$\left(0, \frac{2}{7}, \frac{4}{7}, \frac{6}{7}, \frac{8}{7}, \frac{10}{7}, \frac{12}{7}, 2\right)$
Very Strong Less Important (V-S-L-I)	2	(1, 3)	$\left(1, \frac{9}{7}, \frac{11}{7}, \frac{13}{7}, \frac{15}{7}, \frac{17}{7}, \frac{19}{7}, 3\right)$
Strong Less Important (S-L-I)	3	(2, 4)	$\left(2, \frac{16}{7}, \frac{18}{7}, \frac{20}{7}, \frac{22}{7}, \frac{24}{7}, \frac{26}{7}, 4\right)$
Weakly Less Important (W-L-I)	4	(3, 5)	$\left(3, \frac{23}{7}, \frac{25}{7}, \frac{27}{7}, \frac{29}{7}, \frac{31}{7}, \frac{33}{7}, 5\right)$
Equally Important (E-I)	5	(4, 6)	$\left(4, \frac{30}{7}, \frac{32}{7}, \frac{34}{7}, \frac{36}{7}, \frac{38}{7}, \frac{40}{7}, 6\right)$
Weakly More Important (W-M-I)	6	(5, 7)	$\left(5, \frac{37}{7}, \frac{39}{7}, \frac{41}{7}, \frac{43}{7}, \frac{45}{7}, \frac{47}{7}, 7\right)$
Strong More Important (S-M-I)	7	(6, 8)	$\left(6, \frac{44}{7}, \frac{46}{7}, \frac{48}{7}, \frac{50}{7}, \frac{52}{7}, \frac{54}{7}, 8\right)$
Very Strong More Important (V-S-M-I)	8	(7, 9)	$\left(7, \frac{51}{7}, \frac{53}{7}, \frac{55}{7}, \frac{57}{7}, \frac{59}{7}, \frac{61}{7}, 9\right)$
Absolutely More Important (A-M-I)	9	(8, 10)	$\left(8, \frac{58}{7}, \frac{60}{7}, \frac{62}{7}, \frac{64}{7}, \frac{66}{7}, \frac{68}{7}, 10\right)$

Step 4: Synthetic extent value of criteria $SE(C_i)$ is defined as

(14)
$$SE(C_i) = \sum_{j=1}^{m} S_{ij} \times \left(\sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} \right)^{-1}$$

Where

$$\sum_{j=1}^{m} S_{ij} = \left(a_{i1} + a_{i2} + \ldots + a_{im}, b_{i1} + b_{i2} + \ldots + b_{im}, c_{i1} + c_{i2} + \ldots + c_{im}, d_{i1} + d_{i2} + \ldots + d_{im} \right)$$

$$= \left(a_i, b_i, c_i, d_i \right)_{i=1,2,3,\ldots m}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} = \left(a_1 + a_2 + \ldots + a_m, b_1 + b_2 + \ldots + b_m, c_1 + c_2 + \ldots + c_m, d_1 + d_2 + \ldots + d_m \right)$$

$$= \left(a, b, c, d \right)$$

Therefore, $\left(\sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} \right)^{-1} = \left(\dfrac{1}{d}, \dfrac{1}{c}, \dfrac{1}{b}, \dfrac{1}{a} \right)$

(or) $\left(\sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} \right)^{-1} = \left(\dfrac{1}{f}, \dfrac{1}{e}, \dfrac{1}{d}, \dfrac{1}{c}, \dfrac{1}{b}, \dfrac{1}{a} \right)$

(or) $\left(\sum_{i=1}^{m} \sum_{j=1}^{m} S_{ij} \right)^{-1} = \left(\dfrac{1}{h}, \dfrac{1}{g}, \dfrac{1}{f}, \dfrac{1}{e}, \dfrac{1}{d}, \dfrac{1}{c}, \dfrac{1}{b}, \dfrac{1}{a} \right)$

Similarly, We calculate synthetic value of sub-criteria $SE(C_{ij})$

Step 5: We have used the degree of possibility for finding comparison between criteria (or between sub-criteria) which is defined as

(15)
$$V\left(SE(C_i) \geq SE(C_j)\right)\left(or\ V\left(SE(C_{ij}) \geq SE(C_{ik})\right)\right) = \begin{cases} 1 & c \geq f \\ 0 & e > d \\ \dfrac{e-d}{e-d+c-f} & otherwise \end{cases}$$

Where $SE(C_i)\left(or\ SE(C_{ij})\right) = (a,b,c,d)$ and $SE(C_j)\left(or\ SE(C_{ik})\right) = (e,f,g,h)$ (or)

(16)
$$V\left(SE(C_i) \geq SE(C_j)\right)\left(or\ V\left(SE(C_{ij}) \geq SE(C_{ik})\right)\right) = \begin{cases} 1 & d \geq i \\ 0 & g > f \\ \dfrac{g-f}{g-f+d-i} & otherwise \end{cases}$$

Where $SE(C_i)(or\ SE(C_{ij}))=(a,b,c,d,e,f)$ and $SE(C_j)(or\ SE(C_{ik}))=(g,h,i,j,k,l)$ (or)

(17)
$$V(SE(C_i)\geq SE(C_j))(or\ V(SE(C_{ij})\geq SE(C_{ik})))=\begin{cases} 1 & e\geq l \\ 0 & i>h \\ \dfrac{i-h}{i-h+e-l} & otherwise \end{cases}$$

Where $SE(C_i)(or\ SE(C_{ij}))=(a,b,c,d,e,f,g,h)$ and $SE(C_j)(or\ SE(C_{ik}))=(i,j,k,l,m,n,o,p)$

Step 6: Using FAHP, the weights of criteria & sub-criteria are found as follows. Let

(18)
$$d'(C_i)=\min V(SE(C_i)\geq SE(C_j))$$

(19)
$$d'(C_{ij})=\min V(SE(C_{ij})\geq SE(C_{ik}))$$

Thus, the criteria's and sub-criteria's weight vectors are

(20)
$$W'=(d'(C_1),d'(C_2),...d'(C_i))^t$$

(21)
$$W'=(d'(C_{i1}),d'(C_{i2}),...d'(C_{ij}))^t$$

Where $i,j,k=1,2,...,m,\ i\neq j\neq k$

Step 7: Normalize the weight vectors of criteria and sub-criteria

For criteria and sub-criteria,

(22)
$$W=(d(C_1),d(C_2),...d(C_i))^t$$

(23)
$$W=(d(C_{i1}),d(C_{i2}),...d(C_{ij}))^t$$

Step 8: Fuzzy project network is constructed to use FTOPSIS technique

Step 9: The fuzzy decision matrix is constructed as

(24)
$$\tilde{Y}=\begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1q} \\ y_{21} & y_{22} & \cdots & y_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ y_{p1} & y_{p2} & \cdots & y_{pq} \end{bmatrix}$$

Where the trapezoidal fuzzy number $\tilde{y}_{ij}=(a_{ij},b_{ij},c_{ij},d_{ij})$ represents linguistic variables shown in Table 5.

TABLE 5. Linguistic Variables for Trapezoidal Fuzzy Numbers

Linguistic Variables	Trapezoidal Fuzzy Numbers
Very Low (VLO)	(0, 1, 2, 3)
Low (LO)	(1, 2, 3, 4)
Medium (ME)	(3, 4, 5, 6)
High (HI)	(5, 6, 7, 8)
Very High (VHI)	(7, 8, 9, 10)

(or) the hexagonal fuzzy number $\tilde{y}_{ij} = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij} \right)$ represents linguistic variables shown in Table 6.

TABLE 6. Linguistic Variables for Hexagonal Fuzzy Numbers

Linguistic Variables	Hexagonal Fuzzy Numbers
Very Low(VLO)	$\left(0, \dfrac{3}{5}, \dfrac{6}{5}, \dfrac{9}{5}, \dfrac{12}{5}, 3 \right)$
Low(LO)	$\left(1, \dfrac{8}{5}, \dfrac{11}{5}, \dfrac{14}{5}, \dfrac{17}{5}, 4 \right)$
Medium(ME)	$\left(3, \dfrac{18}{5}, \dfrac{21}{5}, \dfrac{24}{5}, \dfrac{27}{5}, 6 \right)$
High(HI)	$\left(5, \dfrac{28}{5}, \dfrac{31}{5}, \dfrac{34}{5}, \dfrac{37}{5}, 8 \right)$
Very High(VHI)	$\left(7, \dfrac{38}{5}, \dfrac{41}{5}, \dfrac{44}{5}, \dfrac{47}{5}, 10 \right)$

(or) the octagonal fuzzy number $\tilde{y}_{ij} = \left(a_{ij}, b_{ij}, c_{ij}, d_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij} \right)$ represents linguistic variables given in Table 7.

TABLE 7. Linguistic Variables for Octagonal Fuzzy Numbers

Linguistic Variables	Octagonal Fuzzy Numbers
Very Low(VLO)	$\left(0,\dfrac{3}{7},\dfrac{6}{7},\dfrac{9}{7},\dfrac{12}{7},\dfrac{15}{7},\dfrac{18}{7},3\right)$
Low(LO)	$\left(1,\dfrac{10}{7},\dfrac{13}{7},\dfrac{16}{7},\dfrac{19}{7},\dfrac{22}{7},\dfrac{25}{7},4\right)$
Medium(ME)	$\left(3,\dfrac{24}{7},\dfrac{27}{7},\dfrac{30}{7},\dfrac{33}{7},\dfrac{36}{7},\dfrac{39}{7},6\right)$
High(HI)	$\left(5,\dfrac{38}{7},\dfrac{41}{7},\dfrac{44}{7},\dfrac{47}{7},\dfrac{50}{7},\dfrac{53}{7},8\right)$
Very High(VHI)	$\left(7,\dfrac{52}{7},\dfrac{55}{7},\dfrac{58}{7},\dfrac{61}{7},\dfrac{64}{7},\dfrac{67}{7},10\right)$

Step 10: Normalization of fuzzy decision matrix is computed using

$$(25) \qquad \tilde{N}_{FDM} = \left[\tilde{n}_{ij}\right]_{p \times q}$$

Where

$$\tilde{n}_{ij} = \left[\frac{a_{ij}}{d_j^+}, \frac{b_{ij}}{d_j^+}, \frac{c_{ij}}{d_j^+}, \frac{d_{ij}}{d_j^+}\right], \ j \in BC$$

$$\tilde{n}_{ij} = \left[\frac{a_j^-}{d_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right], \ j \in CC$$

Where BC and CC are set of benefit & cost criteria respectively and $a_j^- = \min_i\left(a_{ij}\right)$ & $d_j^+ = \max_i\left(d_{ij}\right)$

$$(or) \quad \tilde{n}_{ij} = \left[\frac{a_{ij}}{f_j^+}, \frac{b_{ij}}{f_j^+}, \frac{c_{ij}}{f_j^+}, \frac{d_{ij}}{f_j^+}, \frac{e_{ij}}{f_j^+}, \frac{f_{ij}}{f_j^+}\right], \ j \in BC$$

$$\tilde{n}_{ij} = \left[\frac{a_j^-}{f_{ij}}, \frac{a_j^-}{e_{ij}}, \frac{a_j^-}{d_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right], \ j \in CC$$

Where $a_j^- = \min_i(a_{ij})$ and $f_j^+ = \max_i(f_{ij})$

$$(\text{or}) \quad \tilde{n}_{ij} = \left[\frac{a_{ij}}{h_j^+}, \frac{b_{ij}}{h_j^+}, \frac{c_{ij}}{h_j^+}, \frac{d_{ij}}{h_j^+}, \frac{e_{ij}}{h_j^+}, \frac{f_{ij}}{h_j^+}, \frac{g_{ij}}{h_j^+}, \frac{h_{ij}}{h_j^+}\right], \quad j \in BC$$

$$\tilde{n}_{ij} = \left[\frac{a_j^-}{h_{ij}}, \frac{a_j^-}{g_{ij}}, \frac{a_j^-}{f_{ij}}, \frac{a_j^-}{e_{ij}}, \frac{a_j^-}{d_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right], \quad j \in CC$$

Where $a_j^- = \min_i(a_{ij})$ and $h_j^+ = \max_i(h_{ij})$)

Step11: The weights of normalized fuzzy decision matrix is computed using

$$(26) \qquad \tilde{V}_{FDM} = \left[\tilde{v}_{ij}\right]_{p \times q} \ where \ \tilde{v}_{ij} = \tilde{N}_{FDM} \times W$$

Step 12: Rank the paths based on sub-criteria

The possible paths P_i are ranked based on sub-criteria C_{ij} by using eq.(10).

Step 13: From step 11, We define the Fuzzy Positive Ideal Solution and Fuzzy Negative Ideal Solution as

$$(27) \qquad \tilde{P}^+ = \left(\tilde{u}_1^+, \tilde{u}_2^+, ..., \tilde{u}_q^+\right) and \ \tilde{P}^- = \left(\tilde{u}_1^-, \tilde{u}_2^-, ..., \tilde{u}_q^-\right)$$

Where $\tilde{u}_j^+ = \max_i(\tilde{v}_{ij})$ and $\tilde{u}_j^- = \min_i(\tilde{v}_{ij})$

Step 14: The distance of each path from $\tilde{P}_i^+$ and $\tilde{P}_i^-$ is defined as

$$(28) \qquad DIS_i^+ = \sum_{j=1}^q d\left(\tilde{v}_{ij}, \tilde{u}_j^+\right) and \ DIS_i^- = \sum_{j=1}^q d\left(\tilde{v}_{ij}, \tilde{u}_j^-\right)$$

Step 15: Finally, Path's closeness coefficient is computed by using

$$(29) \qquad PC_i = \frac{DIS_i^-}{DIS_i^- + DIS_i^+}$$

Step 16: All possible paths are ranked based on closeness coefficient which is given in eq. (29) and the path with the highest rank will be considered as the fuzzy critical path

5. NUMERICAL EXAMPLE

Consider six states namely, U, V, W, X, Y and Z and suppose a bituminous road is laid among these six states shown in Fig. 4. In this, U and Z are the origin and terminus. This road to

be laid represents the connection between one state to another state and there are four ways to reach F. Using our proposed methodology, find out which of these four ways is the critical one.
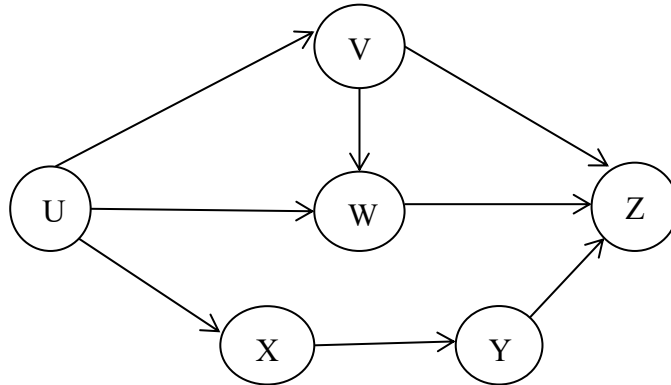

FIGURE 4. Bituminous Road Transport Network

5.1. Calculation for Trapezoidal Fuzzy Numbers

First, FAHP is used to find the weights of criteria and their sub-criteria

Weights of Criteria and their Sub-Criteria

Decision makers give the comparative judgments between criteria and also between sub-criteria using linguistic variables presented in Tables 8 - 13.

TABLE 8. Fuzzy Pairwise Comparison Matrix for Criteria

Factors	C_1	C_2	C_3	C_4	C_5
C_1	E-I	V-S-L-I	S-L-I	W-M-I	W-L-I
C_2	A-M-I	E-I	S-M-I	A-M-I	S-M-I
C_3	S-M-I	W-L-I	E-I	S-M-I	W-M-I
C_4	W-L-I	V-S-L-I	S-L-I	E-I	W-L-I
C_5	W-M-I	S-L-I	W-L-I	S-M-I	E-I

TABLE 9. Fuzzy Pairwise Comparison Matrix for Sub-Criteria of C_1

C_1	C_{11}	C_{12}	C_{13}
C_{11}	E-I	A-M-I	S-M-I
C_{12}	V-S-L-I	E-I	S-L-I
C_{13}	S-L-I	V-S-M-I	E-I

Table 10. Fuzzy Pairwise Comparison Matrix for Sub-Criteria of C_2

C_2	C_{21}	C_{22}	C_{23}
C_{21}	E-I	V-S-M-I	A-M-I
C_{22}	V-S-L-I	E-I	S-M-I
C_{23}	A-L-I	W-L-I	E-I

TABLE 11. Fuzzy Pairwise Comparison Matrix for Sub-Criteria of C_3

C_3	C_{31}	C_{32}	C_{33}
C_{31}	E-I	V-S-M-I	A-M-I
C_{32}	S-L-I	E-I	S-M-I
C_{33}	A-L-I	W-L-I	E-I

.TABLE 12. Fuzzy Pairwise Comparison Matrix for Sub-Criteria of C_4

C_4	C_{41}	C_{42}	C_{43}	C_{45}	C_{45}
C_{41}	E-I	S-L-I	W-M-I	V-S-M-I	S-M-I
C_{42}	S-M-I	E-I	V-S-M-I	A-M-I	V-S-M-I
C_{43}	S-L-I	V-S-L-I	E-I	V-S-M-I	S-M-I
C_{44}	S-L-I	V-S-L-I	S-L-I	E-I	W-L-I
C_{45}	S-L-I	V-S-L-I	W-L-I	W-M-I	E-I

TABLE 13. Fuzzy Pairwise Comparison Matrix for Sub-Criteria of C_5

C_5	C_{51}	C_{52}	C_{53}
C_{51}	E-I	S-M-I	A-M-I
C_{52}	V-S-L-I	E-I	S-M-I
C_{53}	A-L-I	S-L-I	E-I

Using Tables 8 – 13 and our proposed methodology, we find

(i) weight of criteria is

$$W = (0.0959, 0.3243, 0.3158, 0.0403, 0.2237)^t$$

Hence, We prioritize the criteria in the order Time, Risk in travel, Insecurity, Cost and Non-availability of facilities and services.

Also,

(ii) weight of sub-criteria of C_1 is

$$W = (0.4877, 0.0592, 0.4532)^t$$

(iii) weight of sub-criteria of C_2 is

$$W = (0.6168, 0.3628, 0.0204)^t$$

(iv) weight of sub-criteria of C_3 is

$$W = (0.5812, 0.4094, 0.0094)^t$$

(v) weight of sub-criteria of C_4 is

$$W = (0.3244, 0.3322, 0.2301, 0.0136, 0.0997)^t$$

(vi) weight of sub-criteria of C_5 is

$$W = (0.5786, 0.4110, 0.0104)^t$$

Secondly, We use FTOPSIS to determine the critical path based on closeness co-efficient.

Closeness Co-efficient

Fuzzy decision matrix is formed using alternatives and sub-criteria presented in Table 14.

TABLE 14. Rating of Activity based on Sub-Criteria [9]

Activity	U – V	U – W	U – X	X – Y	V – W	V – Z	W – Z	Y – Z
C_{11}	HI	VHI	LO	ME	ME	VHI	VHI	ME
C_{12}	ME	HI	VLO	LO	LO	HI	LO	VLO
C_{13}	ME	HI	LO	LO	ME	VHI	ME	LO
C_{21}	HI	VHI	LO	ME	ME	HI	HI	VLO
C_{22}	HI	VHI	LO	LO	ME	HI	ME	VLO
C_{23}	ME	HI	VLO	LO	LO	LO	ME	LO
C_{31}	ME	HI	VHI	HI	HI	ME	VHI	ME
C_{32}	ME	HI	HI	ME	HI	HI	ME	ME
C_{33}	VLO	ME	ME	ME	HI	HI	HI	LO
C_{41}	HI	HI	ME	VLO	ME	ME	HI	ME
C_{42}	HI	VHI	ME	LO	ME	HI	HI	ME
C_{43}	ME	ME	LO	ME	VHI	HI	ME	HI
C_{44}	VLO	LO	VLO	LO	HI	HI	LO	LO
C_{45}	LO	ME	LO	ME	HI	ME	ME	LO
C_{51}	ME	HI	VHI	LO	ME	LO	HI	ME
C_{52}	LO	ME	HI	LO	HI	ME	ME	ME
C_{53}	LO	ME	LO	VLO	ME	LO	ME	LO

Using Table 5 and using eq. (24) – (29), closeness co-efficient is computed presented in Table 15

TABLE 15. Closeness Co-efficient PC_i

Paths	DIS_i^{+}	DIS_i^{-}	PC_i
U – V – Z	2.5034	1.3229	0.3457
U – V – W – Z	3.4562	0.3701	0.0967
U – W – Z	3.004	0.8223	0.2149
U – X – Y – Z	2.6962	1.1301	0.2954

From Table 15, four paths are ranked as $PC_1 > PC_4 > PC_3 > PC_2$ and the path U – V – Z with maximum closeness co-efficient is the optimal fuzzy critical path

5.2. Calculation for Hexagonal Fuzzy Numbers

Weights of Criteria and their Sub-Criteria

Using FAHP method, We find

(i) weight of criteria is

$$W = (0.0900, 0.3643, 0.2974, 0.0379, 0.2104)^t$$

(ii) weight of sub-criteria of C_1 is

$$W = (0.5318, 0.0539, 0.4142)^t$$

(iii) weight of sub-criteria of C_2 is

$$W = (0.6578, 0.3241, 0.0182)^t$$

(iv) weight of sub-criteria of C_3 is

$$W = (0.6236, 0.3679, 0.0084)^t$$

(v) weight of sub-criteria of C_4 is

$$W = (0.3050, 0.3725, 0.2163, 0.0127, 0.0935)^t$$

(vi) weight of sub-criteria of C_5 is

$$W = (0.6211, 0.3696, 0.0093)^t$$

Closeness Co-efficient

We find the closeness co-efficient for hexagonal fuzzy numbers using FTOPSIS presented in Table 16.

TABLE 16. Closeness Co-efficient PC_i

Paths	DIS_i^+	DIS_i^-	PC_i
U – V – Z	2.4736	1.3480	0.3527
U – V – W – Z	3.4476	0.3740	0.0979
U – W – Z	2.9915	0.8301	0.2172
U – X – Y – Z	2.6597	1.1619	0.3040

From Table 16, four paths are ranked as $PC_1 > PC_4 > PC_3 > PC_2$ and the path U – V – Z with

maximum closeness co-efficient is the optimal fuzzy critical path

5.3. Calculation for Octagonal Fuzzy Numbers

Weights of Criteria and their Sub-Criteria

Using FAHP method, We find

(i) weight of each criteria is

$$W = (0.0877, 0.3799, 0.2902, 0.0369, 0.2052)^t$$

(ii) weight of sub-criteria of C_1 is

$$W = (0.5485, 0.0519, 0.3995)^t$$

(iii) weight of sub-criteria of C_2 is

$$W = (0.6728, 0.3099, 0.0174)^t$$

(iv) weight of sub-criteria of C_3 is

$$W = (0.6393, 0.3527, 0.0081)^t$$

(v) weight of sub-criteria of C_4 is

$$W = (0.2973, 0.3883, 0.2109, 0.0124, 0.0911)^t$$

(vi) weight of sub-criteria of C_5 is

$$W = (0.6368, 0.3543, 0.0089)^t$$

Closeness Co-efficient

We calculate the closeness co-efficient for octagonal fuzzy numbers using FTOPSIS presented in

Table 17.

TABLE 17. Closeness Co-efficient PC_i

Paths	DIS_i^{+}	DIS_i^{-}	PC_i
U – V – Z	2.4958	1.3241	0.3466
U – V – W – Z	3.4596	0.3603	0.0943
U – W – Z	3.0097	0.8102	0.2121
U – X – Y – Z	2.7156	1.1043	0.2891

From Table 17, four paths are ranked as $PC_1 > PC_4 > PC_3 > PC_2$ and the path U – V – Z with maximum closeness co-efficient is the optimal fuzzy critical path.

6. RESULTS AND DISCUSSION

The comparison among trapezoidal, hexagonal and octagonal fuzzy numbers is presented in Table 17 and Fig. 5.

TABLE 17. Comparison among Trapezoidal, Hexagonal and Octagonal Fuzzy Numbers

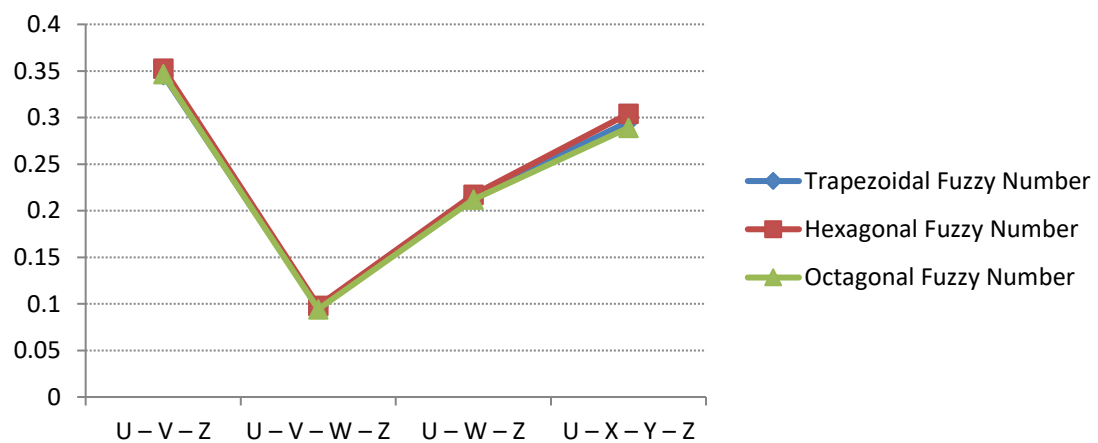| Paths	Closeness Co-efficient		
	Trapezoidal fuzzy number	Hexagonal fuzzy number	Octagonal fuzzy number
U – V – Z	0.3457	0.3527	0.3466
U – V – W – Z	0.0967	0.0979	0.0943
U – W – Z	0.2149	0.2172	0.2121
U – X – Y – Z	0.2954	0.3040	0.2891

FIGURE 5. Comparison among Trapezoidal, Hexagonal and Octagonal Fuzzy Numbers

7. CONCLUSION

In this paper, critical path in bituminous road transport is identified using proposed Integrated FAHP – FTOPSIS and more criteria and sub – criteria are used to identify it.

Moreover, We have used trapezoidal, hexagonal and octagonal fuzzy numbers as parameters and the comparison have been made among them. The numerical example illustrated for providing benefits to the decision makers by analyzing the integrated technique for bituminous road transport system and also hexagonal fuzzy number is more effective than trapezoidal and octagonal fuzzy numbers for identifying the critical path based on our ranking technique.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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