Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 5, 5181-5195
https://doi.org/10.28919/jmcs/5929
ISSN: 1927-5307

FIXED POINT EXISTENCE RESULTS OF F-KANNAN TYPE CONTRACTION ON A METRIC SPACE WITH DIRECTED GRAPH

RAKESH BATRA[1], RUCHI GUPTA[2], PRIYANKA SAHNI[2,*]

[1]Department of Mathematics, Hans Raj College, University of Delhi, New Delhi, 110007, India

[2]Department of Mathematics, Manav Rachna University, Faridabad, 121004, India

Abstract. In this article, we have attained some interesting results on existence of fixed points for a newly developed larger class of self mappings called $F_{\mathcal{G}}$ -Kannan defined on a metric space equipped with a special type of graph called $\mho$-orbitally connected graph. This extended class is a merger of the most recently developed Kannan type mapping called F-Kannan contractions defined in a metric space and $\mathcal{G}$-Kannan mapping defined in a metric space with an underlying graph. It is highlighted through example that such a graph condition is sufficient to study fixed points of $F_{\mathcal{G}}$ -Kannan mappings. Several interesting examples are illustrated which justify that our obtained results are more general and further, many previously developed fixed point results related to Kannan mappings are encompassed in our main result. The article closes by raising some open problems of this work.

Keywords: Kannan; contraction; connected; graph; complete; fixed point.

2010 AMS Subject Classification: Primary 47H10; Secondary 47H09, 54H25.

1. Introduction and Preliminaries

Jachymski in 2008 [7] initiated the idea of a contraction mapping called $\mathcal{G}$-contraction mapping in metric spaces which are somehow equipped with a directed graph $\mathcal{G}$. He proved a few fixed point results for such contraction mappings. Later in 2012, Wardowski [3] considered a

*Corresponding author

E-mail address: pri.sah25.phd@gmail.com

Received April 26, 2021

special type of real-valued function $\digamma$ defined on $\mathbb{R}^+$. This function satisfied some additional properties and was used to define $\digamma$-contractions on a metric space. He proved that such mappings do possess a fixed point under the completeness of the space. Since then many authors have worked on fixed point results of such mappings and their possible extensions([1], [4], [6], [8], [9]).

Extending the work in this direction, Batra and Vashistha [11] in 2014 generalized the above two contractions to produce a new type of contraction mapping defined as $\digamma$-$\mathcal{G}$-contraction which carried features of both. He proved some fixed point results which extended Jachymski's and Wardowski's work.

In 2012, Bojor [5] also introduced a new class of mappings called $\mathcal{G}$-Kannan mapping and gave some results on existence of fixed points of such mappings on a metric space endowed with a graph $\mathcal{G}$. Very recently in 2020, Batra *et. al.* [10] introduced $\digamma$-Kannan mappings and established some fixed point existence theorems for such mappings. The present article is a merger of the concepts of $\mathcal{G}$-Kannan and $\digamma$-Kannan mappings to form an extended class of $\digamma_{\mathcal{G}}$ -Kannan mappings on a metric spaced with a graph $\mathcal{G}$. Throughout the paper $(\mathcal{W},\zeta)$ denotes a metric space.

Definition 1.1. *([10])* Consider a function $\digamma : \mathbb{R}^+ \to \mathbb{R}$ with the following properties

($\bar{\mathfrak{F}}1$) $\digamma(\omega_1) < \digamma(\omega_2)$ for every $\omega_1 < \omega_2 \in \mathbb{R}^+$.

($\bar{\mathfrak{F}}2$) $lim_{n\to\infty}\beta_n = 0 \Leftrightarrow lim_{n\to\infty}\digamma(\beta_n) = -\infty,\ \beta_n > 0$.

($\bar{\mathfrak{F}}3$) For some positive number $r < 1$, following holds

$$lim_{\beta\to 0^+}\beta^r\digamma(\beta) = 0.$$

A mapping $S : \mathcal{W} \to \mathcal{W}$ is defined to be an $\digamma$-Kannan mapping if:

(1) $S\bar{\xi} \neq S\bar{\rho} \Rightarrow S\bar{\xi} \neq \bar{\xi}$ or $S\bar{\rho} \neq \bar{\rho}$.

(2) $\exists\, \Upsilon > 0$ such that

$$\Upsilon + \digamma[\zeta(S\bar{\xi}, S\bar{\rho})] \leq \digamma\left[\frac{\zeta(\bar{\xi}, S\bar{\xi}) + \zeta(\bar{\rho}, S\bar{\rho})}{2}\right]$$

for all $\bar{\xi}, \bar{\rho} \in \mathcal{W}$ and $S\bar{\xi} \neq S\bar{\rho}$.

Definition 1.2. *([5])* Let $(\mathcal{W},\zeta)$ be equipped with graph $\mathcal{G}$. The mapping $S:\mathcal{W}\to\mathcal{W}$ is defined to be $\mathcal{G}$-Kannan mapping if :

(a) S is edge preserving, that is:

$$(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}\Rightarrow(S\bar{\xi},S\bar{\rho})\in E_{\mathcal{G}}.$$

(b) $\exists\, k\in[0,\frac{1}{2})$ such that :

$$\zeta(S\bar{\xi},S\bar{\rho})\leq k\zeta(\bar{\xi},S\bar{\xi})+k\zeta(\bar{\rho},S\bar{\rho}) \tag{1}$$

for all $(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}$.

An equivalent form of (1) is

$$\zeta(S\bar{\xi},S\bar{\rho})\leq\frac{k}{2}[\zeta(\bar{\xi},S\bar{\xi})+\zeta(\bar{\rho},S\bar{\rho})] \tag{2}$$

for some $k\in[0,1)$ and all $(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}$.

2. Main Results

We now introduce $\digamma_{\mathcal{G}}$ -Kannan mapping by composing above definitions. Consider the space $(\mathcal{W},\zeta)$ endowed with a graph $\mathcal{G}$.

Definition 2.1. Consider a mapping $\digamma$ with properties ($\bar{\mathcal{F}}$1)-($\bar{\mathcal{F}}$3). Consider an operator $\mho:\mathcal{W}\to\mathcal{W}$ satisfying the following conditions:

(K1) $\mho$ preserves the edges of $\mathcal{G}$, that is

$$(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}\Rightarrow(\mho\bar{\xi},\mho\bar{\rho})\in E_{\mathcal{G}}. \tag{3}$$

(K2)

$$(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}\text{ and }\mho\bar{\xi}\neq\mho\bar{\rho}\Rightarrow\mho\bar{\xi}\neq\bar{\xi}\text{ or }\mho\bar{\rho}\neq\bar{\rho}. \tag{4}$$

(K3) $\exists\,\Upsilon>0$ such that

$$\Upsilon+\digamma[\zeta(\mho\bar{\xi},\mho\bar{\rho})]\leq\digamma\left[\frac{\zeta(\bar{\xi},\mho\bar{\xi})+\zeta(\bar{\rho},\mho\bar{\rho})}{2}\right] \tag{5}$$

for all $(\bar{\xi},\bar{\rho})\in E_{\mathcal{G}}$ and $\mho\bar{\xi}\neq\mho\bar{\rho}$. Then $\mho$ is called an $\digamma_{\mathcal{G}}$ -Kannan mapping.

There are various types of the mapping $\digamma$ satisfying ($\bar{\mathcal{F}}$1)-($\bar{\mathcal{F}}$3) and graphs $\mathcal{G}$ on the space $\mathcal{W}$ which generate a variety of $\digamma_{\mathcal{G}}$ -Kannan mappings (for examples of mappings $\digamma$, refer [7]). In the underlying example, it is proved that a necessary and sufficient condition for a mapping to be $\mathcal{G}$-Kannan ([5]) mapping is it must be $\ln_{\mathcal{G}}$-Kannan mapping. Thus, this class of mappings as considered by Bojor [5] are a special case of our defined class.

Example 2.1. Let $\digamma_1 : \mathbb{R}^+ \to \mathbb{R}$ be defined as $\digamma_1(\alpha) = ln\alpha,\ \alpha > 0$. Then clearly ($\bar{\mathcal{F}}$1), ($\bar{\mathcal{F}}$2) and ($\bar{\mathcal{F}}$3) are satisfied by $\digamma_1$. Indeed condition ($\bar{\mathcal{F}}$3) is true for every $k \in (0,1)$. Also, equation (5) becomes :

$$\zeta(\mho\bar{\xi},\mho\bar{\rho}) \leq e^{-\Upsilon}\left[\frac{\zeta(\bar{\xi},\mho\bar{\xi})+\zeta(\bar{\rho},\mho\bar{\rho})}{2}\right] \tag{6}$$

for all $(\bar{\xi},\bar{\rho}) \in E_{\mathcal{G}}$ and $\mho\bar{\xi} \neq \mho\bar{\rho}$.

Let $\mho : \mathcal{W} \to \mathcal{W}$ be a $\mathcal{G}$-Kannan mapping. If $k = 0$ in (2), then $\mho\bar{\xi} = \mho\bar{\rho}$ for all $(\bar{\xi},\bar{\rho}) \in E_{\mathcal{G}}$ and therefore (K2) and (K3) hold vacuously i.e, $\mho$ is also $\digamma_{1\mathcal{G}}$ -Kannan mapping. Further, if $k > 0$ in (2), then (6) holds with $\Upsilon = ln(1/k)$. Also whenever $(\bar{\xi},\bar{\rho}) \in E_{\mathcal{G}}$ and $\mho\bar{\xi} \neq \mho\bar{\rho}$ then (2) implies $\mho\bar{\xi} \neq \bar{\xi}$ or $\mho\bar{\rho} \neq \bar{\rho}$ that is, (K2) is also true. Thus, $\mho$ is again $\digamma_{1\mathcal{G}}$ -Kannan mapping.

On the other hand, assume that the mapping $\mho$ is $\digamma_{1\mathcal{G}}$ -Kannan. We show it is $\mathcal{G}$-Kannan as well. Indeed, if $(\bar{\xi},\bar{\rho}) \in E_{\mathcal{G}}$ and $\mho\bar{\xi} = \mho\bar{\rho}$ then we may choose any $k \in [0,1)$ so that (2) holds. If $\mho\bar{\xi} \neq \mho\bar{\rho}$ then from (6), by choosing $k = e^{-\Upsilon} \in [0,1)$, again (2) holds.

Example 2.2. Let $\mathcal{G}_{diag}$ be the graph such that $V(\mathcal{G}_{diag}) = \mathcal{W}$ and $E(\mathcal{G}_{diag}) = \Delta$. Then every self map on $\mathcal{W}$ is vacuously an $\digamma_{\mathcal{G}_{diag}}$-Kannan mapping for all $\digamma$ satisfying ($\bar{\mathcal{F}}$1)-($\bar{\mathcal{F}}$3).

Remark 2.1. For any $\digamma$ satisfying ($\bar{\mathcal{F}}$1)-($\bar{\mathcal{F}}$3), there always exists a graph $\mathcal{G}$ on $\mathcal{W}$ and a self mapping $\mho$ on $\mathcal{W}$ which is $\digamma_{\mathcal{G}}$ -Kannan.

Remark 2.2. It follows from above example that in general, an $\digamma_{\mathcal{G}}$ -Kannan mapping need not be always continuous on $\mathcal{W}$.

Following two examples are an illustrative of the fact that the above defined class of $\digamma_{\mathcal{G}}$ - Kannan mapping is universal with respect to the class of $\digamma$-Kannan mapping [10] as well as the class of $\mathcal{G}$-Kannan mapping as defined by Bojor [5]. For every fixed graph $\mathcal{G}$ on $\mathcal{W}$, there exists a self mapping $\mho$ on $\mathcal{W}$ and a function $\digamma$ satisfying ($\bar{\mathcal{F}}1$)-($\bar{\mathcal{F}}3$) such that $\mho$ is an $\digamma_{\mathcal{G}}$ -Kannan mapping but not $\mathcal{G}$-Kannan mapping.

Example 2.3. Let $\mathcal{W} = \{y_n : n \in \mathbb{N}\}$ where $y_n = (-1)^n n,\ n \geq 2$ and $y_1 = 0$ with euclidean distance on $\mathcal{W}$. Define $\mho : \mathcal{W} \to \mathcal{W}$ as $\mho y_1 = y_1$ and $\mho y_n = y_{n-1},\ n \geq 2$. Choose $\digamma(\bar{\xi}) = \ln(\bar{\xi}) + \bar{\xi},\ \bar{\xi} > 0$.

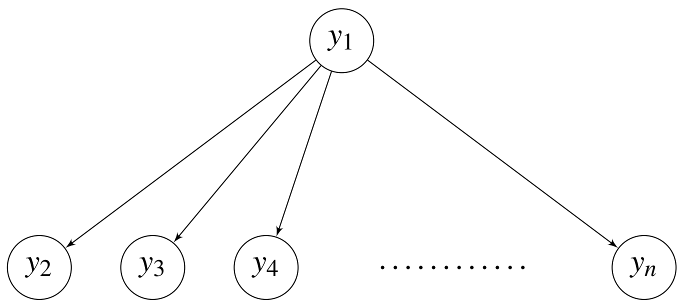

FIGURE 1

Also define a graph $\mathcal{G}$ on $\mathcal{W}$ as $V_{\mathcal{G}} = \mathcal{W}$ and $E_{\mathcal{G}} = \Delta \cup \{(y_1, y_n) : n \geq 2\}$ (refer Fig 1) . Then clearly $\mho$ preserves edges of $\mathcal{G}$.

Equation (5) becomes :

$$\frac{\zeta(\mho\bar{\xi}, \mho\bar{\rho})}{(\zeta(\bar{\xi}, \mho\bar{\xi}) + \zeta(\bar{\rho}, \mho\bar{\rho}))/2} e^{\zeta(\mho\bar{\xi}, \mho\bar{\rho}) - \left\{\frac{\zeta(\bar{\xi}, \mho\bar{\xi}) + \zeta(\bar{\rho}, \mho\bar{\rho})}{2}\right\}} \leq e^{-\Upsilon} \tag{7}$$

for all $(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}}$ and $\mho\bar{\xi} \neq \mho\bar{\rho}$. We now prove that equation (7) is satisfied by $\mho$ but not equation (2). Also $\{(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}} : \mho\bar{\xi} \neq \mho\bar{\rho}\} = \{(y_1, y_n) : n \geq 3\}$ and thus, correspondingly,

$$\begin{aligned} &\frac{\zeta(\mho\bar{\xi}, \mho\bar{\rho})}{(\zeta(\bar{\xi}, \mho\bar{\xi}) + \zeta(\bar{\rho}, \mho\bar{\rho}))/2} e^{\zeta(\mho\bar{\xi}, \mho\bar{\rho}) - \left\{\frac{\zeta(\bar{\xi}, \mho\bar{\xi}) + \zeta(\bar{\rho}, \mho\bar{\rho})}{2}\right\}} \\ &\qquad = \frac{\zeta(y_1, y_{n-1})}{\zeta(y_n, y_{n-1})/2} e^{\zeta(y_1, y_{n-1}) - \frac{\zeta(y_n, y_{n-1})}{2}} \\ &\qquad\qquad = \frac{n-1}{(2n-1)/2} e^{(n-1) - \frac{(2n-1)}{2}} = \frac{n-1}{(2n-1)/2} e^{-\frac{1}{2}} < e^{-\frac{1}{2}}. \end{aligned}$$

By making a choice of $\Upsilon = 1/2$, it is proved that equation (7) is satisfied by $\mho$. For any $(\bar{\xi}, \bar{\rho}) \in \Delta$, (2) holds. Also for $(y_1, y_2) \in E_{\mathcal{G}}$, (2) again holds but for $(y_1, y_n) \in E_{\mathcal{G}}, n \geq 3$, we have as above, $\frac{\zeta(\mho\bar{\xi},\mho\bar{\rho})}{(\zeta(\bar{\xi},\mho\bar{\xi})+\zeta(\bar{\rho},\mho\bar{\rho}))/2} = \frac{n-1}{(2n-1)/2} \to 1$ as $n \to \infty$. Hence $\mho$ fails to satisfy equation (2). Thus $\mho$ is an $F_{\mathcal{G}}$ -Kannan mapping but not $\mathcal{G}$-Kannan.

Consider any mapping F which obeys properties ($\bar{\mathcal{F}}1$)-($\bar{\mathcal{F}}3$). Then, there is always a self mapping $\mho : \mathcal{W} \to \mathcal{W}$ and a graph $\mathcal{G}$ such that $\mho$ is $F_{\mathcal{G}}$ -Kannan mapping but not F-Kannan mapping. Indeed, consider the following example.

Example 2.4. Letting $\mathcal{G} = \mathcal{G}_{diag}$, then by Example 2.2, every self mapping on $\mathcal{W}$ becomes $F_{\mathcal{G}_{diag}}$-Kannan mapping. In particular, consider $\mho : [0,1] \to [0,1]$ as:

$$\mho\bar{\xi} = \begin{cases} 0.99 & \text{if } 0 \leq \bar{\xi} < 1 \\ 0.8 & \text{if } \bar{\xi} = 1 \end{cases}$$

Under the euclidean metric on $[0,1]$, it can be seen that $F[\zeta(T0.9,T1)] = F[0.19] > F[0.145] = F\left[\frac{\zeta(0.9,T0.9)+\zeta(1,T1)}{2}\right]$. This justifies that this mapping $\mho$ is not F-Kannan. It is interesting to note here that $\mho$ has exactly one fixed point.

Definition 2.2. *([11])* Sequences $\{a_n\}$ and $\{b_n\}$ in $(\mathcal{W},\zeta)$ are said to be equivalent, if $\lim_{n\to\infty} \zeta(a_n,b_n) = 0$.

Proposition 2.1. Let $S : \mathcal{W} \to \mathcal{W}$ satisfy (K1) (respectively (K2) and (K3)) for graph $\mathcal{G}$, then S will also satisfy (K1) (respectively (K2) and (K3)) for graphs $\mathcal{G}^{-1}$ and $\tilde{\mathcal{G}}$. Further, consider the following statements

(i) S is $F_{\mathcal{G}}$ -Kannan mapping.
(ii) S is $F_{\mathcal{G}^{-1}}$ -Kannan mapping.
(iii) S is $F_{\tilde{\mathcal{G}}}$ -Kannan mapping.

Then (i) and (ii) are equivalent and (i) implies (iii).

Proof. Proof follows by symmetry of ζ. □

Definition 2.3. Let $\mho$ be a self map on $\mathcal{W}$. A graph $\mathcal{G}$ on $\mathcal{W}$ is said to be $\mho$-orbitally connected if for each $\bar{\xi} \in \mathcal{W}$, there is at least one $k \in \mathbb{N}$ such that $(\mho^{k-1}\bar{\xi}, \mho^{k}\bar{\xi}) \in E_{\mathcal{G}}$.

Remark 2.3. $\mathcal{G}$ is $\mho$-orbitally connected $\Rightarrow$ $\tilde{\mathcal{G}}$ is $\mho$-orbitally connected.

The following example illuminates that the condition of $\mathcal{G}$ being $\mho$-orbitally connected is necessary to study the existence of fixed points of $\mathcal{F}_{\mathcal{G}}$ -Kannan mapping $\mho$.

Example 2.5. Consider the set $\mathcal{W} = \mathbb{N} \setminus \{1\}$ equipped with euclidean metric. Define $\mho : \mathcal{W} \to \mathcal{W}$ as $\mho\bar{\xi} = 2x$. Then $\mho$ has no fixed point in $\mathcal{W}$. Therefore condition (K2) holds trivially, independent of graph $\mathcal{G}$ on $\mathcal{W}$ and mapping $\mathcal{F}$.

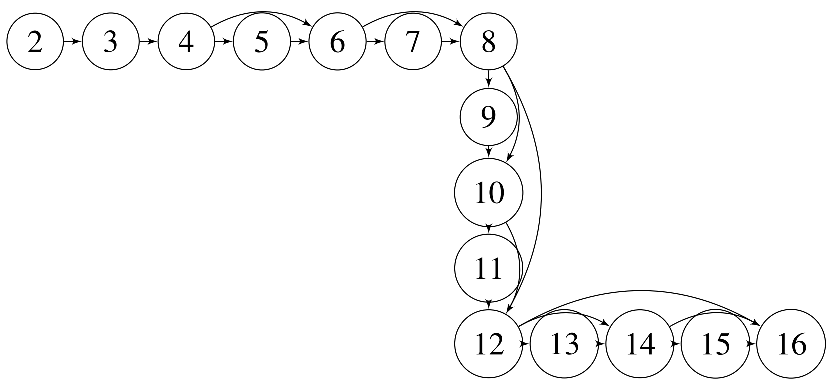

FIGURE 2

Suppose $\mathcal{W}$ is equipped with a graph $\mathcal{G}$ where $V_{\mathcal{G}} = \mathcal{W}$ and $E_{\mathcal{G}} = \{(2^k n, 2^k(n+1)) : n \in \mathcal{W}, k \in \{0,1,2...\}\} \cup \Delta$ (refer Fig 2). Then clearly $(\mho\bar{\xi}, \mho\bar{\rho}) \in E_{\mathcal{G}}$ whenever $(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}}$.

Additionally, $(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}}$ and $\mho\bar{\xi} \neq \mho\bar{\rho}$ if and only if $(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}} \setminus \Delta$. We prove that $\mho$ is $\ln_{\mathcal{G}}$ -Kannan mapping. Consider any $(\bar{\xi}, \bar{\rho}) \in E_{\mathcal{G}} \setminus \Delta$ say $\bar{\xi} = 2^k n$ and $\bar{\rho} = 2^k(n+1)$, $k \in \{0,1,2...\}$ and $n \in \mathcal{W}$.

Then,

$$\frac{\zeta(\bar{\xi}, \mho\bar{\xi}) + \zeta(\bar{\rho}, \mho\bar{\rho})}{2d(\mho\bar{\xi}, \mho\bar{\rho})} = \frac{\bar{\xi} + \bar{\rho}}{4|\bar{\xi} - \bar{\rho}|}$$
$$= \frac{2n+1}{4}.$$

Since $n \geq 2$, choosing $\Upsilon = ln(\frac{5}{4})$, condition (K3) is also true.

Finally it remains to prove that $\mathcal{G}$ is not $\mho$-orbitally connected. For $\bar{\xi} = 2 \in \mathcal{W}$, $(\mho^{k-1}\bar{\xi}, \mho^k\bar{\xi}) = (2^k, 2^{k+1}) \notin E_(\mathcal{G})$ for any $k \in \{0,1,2,...\}$.

In the remainder of the article $(\mathcal{W}, \zeta)$ is a metric space equipped with a graph $\mathcal{G}$ which is $\mho$-orbitally connected with $\mho$ being any self mapping on $\mathcal{W}$.

Lemma 2.1. If $\mho$ is an $\mathcal{F}_{\mathcal{G}}$ -Kannan mapping on $\mathcal{W}$ then, for any $\bar{\xi} \in \mathcal{W}$ and $\bar{\rho} \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$, we have $\zeta(\mho^n \bar{\xi}, \mho^n \bar{\rho}) \to 0$ as $n \to \infty$.

Proof. By Proposition 2.1 $\mho$ is also $\mathcal{F}_{\tilde{\mathcal{G}}}$-Kannan mapping. Let $\bar{\xi} \in \mathcal{W}$ and $\bar{\rho} \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$. Thus a path exists from $\bar{\xi}$ to $\bar{\rho}$ in $\tilde{\mathcal{G}}$ say $(\bar{\xi}_j)_{i=0}^{N}$ where $\bar{\xi}_0 = \bar{\xi}$, $\bar{\xi}_N = \bar{\rho}$ and $(\bar{\xi}_j, \bar{\xi}_{j+1}) \in E_{\tilde{\mathcal{G}}}$ for all $j = 0, 1, \cdots, N-1$.

Inductively applying (K1) gives, for each $n \in \mathbb{N}$ and $j = 0, 1, \ldots, N-1$,

$$(\mho^n \bar{\xi}_j, \mho^n \bar{\xi}_{j+1}) \in E_{\tilde{\mathcal{G}}} \tag{8}$$

Consider for any $n \in \mathbb{N}$,

$$\begin{aligned} \zeta(\mho^n \bar{\xi}, \mho^n \bar{\rho}) &= \zeta(\mho^n \bar{\xi}_0, \mho^n \bar{\xi}_N) \\ &\leq \sum_{i=0}^{N-1} \zeta(\mho^n \bar{\xi}_j, \mho^n \bar{\xi}_{j+1}) \end{aligned} \tag{9}$$

If $\mho^m \bar{\xi}_j = \mho^m \bar{\xi}_{j+1}$ for some $j \in \{0, 1, \ldots, N-1\}$ and $m \in \mathbb{N}$, the following is true for all $n \geq m$

$$\begin{aligned} \zeta(\mho^n \bar{\xi}_j, \mho^n \bar{\xi}_{j+1}) &= 0 \\ &\leq \frac{\zeta(\mho^{n-1} \bar{\xi}_j, \mho^n \bar{\xi}_j) + \zeta(\mho^{n-1} \bar{\xi}_{j+1}, \mho^n \bar{\xi}_{j+1})}{2} \end{aligned} \tag{10}$$

Now consider $j \in \{0, 1, \ldots, N-1\}$ and $n \in \mathbb{N}$ for which $\mho^n \bar{\xi}_j \neq \mho^n \bar{\xi}_{j+1}$.

Also by equation (8), $(\mho^{n-1} \bar{\xi}_j, \mho^{n-1} \bar{\xi}_{j+1}) \in E_{\tilde{\mathcal{G}}}$. Therefore, by (K3), there exists $\Upsilon > 0$ such that

$$\Upsilon + \mathcal{F}[\zeta(\mho^n \bar{\xi}_j, \mho^n \bar{\xi}_{j+1})] \leq \mathcal{F}\left[\frac{\zeta(\mho^{n-1} \bar{\xi}_j, \mho^n \bar{\xi}_j) + \zeta(\mho^{n-1} \bar{\xi}_{j+1}, \mho^n \bar{\xi}_{j+1})}{2}\right]$$

or,

$$\begin{aligned} \mathcal{F}[\zeta(\mho^n \bar{\xi}_j, \mho^n \bar{\xi}_{j+1})] &\leq \mathcal{F}\left[\frac{\zeta(\mho^{n-1} \bar{\xi}_j, \mho^n \bar{\xi}_j) + \zeta(\mho^{n-1} \bar{\xi}_{j+1}, \mho^n \bar{\xi}_{j+1})}{2}\right] - \Upsilon \\ &\leq \mathcal{F}\left[\frac{\zeta(\mho^{n-1} \bar{\xi}_j, \mho^n \bar{\xi}_j) + \zeta(\mho^{n-1} \bar{\xi}_{j+1}, \mho^n \bar{\xi}_{j+1})}{2}\right] \end{aligned}$$

Since $\digamma$ is non-decreasing, so above inequality implies

$$\zeta(\mho^n\bar{\xi}_j,\mho^n\bar{\xi}_{j+1}) \leq \frac{\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)+\zeta(\mho^{n-1}\bar{\xi}_{j+1},\mho^n\bar{\xi}_{j+1})}{2} \tag{11}$$

From equations (9), (10) and (11), we obtain

$$\zeta(\mho^n\bar{\xi},\mho^n\bar{\rho}) \leq \sum_{j=0}^{N-1}\frac{\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)+\zeta(\mho^{n-1}\bar{\xi}_{j+1},\mho^n\bar{\xi}_{j+1})]}{2} \tag{12}$$

for all $n \in \mathbb{N}$.

Consider any fixed but arbitrary $j \in \{0,1,\cdots,N\}$.

If there exists some $k \in \mathbb{N}$ such that $\mho^{k-1}\bar{\xi}_j = \mho^k\bar{\xi}_j$, then the sequence $\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)$ is eventually zero sequence. Hence, $\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j) \to 0$. On the other hand assume that $\mho^{n-1}\bar{\xi}_j \neq \mho^n\bar{\xi}_j$ for all $n \in \mathbb{N}$. Also, since the graph $\mathcal{G}$ is $\mho$-orbitally connected, there exists some $k_j \in \mathbb{N}$ such that $(\mho^{k_j-1}\bar{\xi}_j,\mho^k_j\bar{\xi}_j) \in E_{\tilde{\mathcal{G}}}$. Moreover, $(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j) \in E_{\tilde{\mathcal{G}}}$ for every $n \geq k_j$, equivalently, $(\mho^{n-2}\bar{\xi}_j,\mho^{n-1}\bar{\xi}_j) \in E_{\tilde{\mathcal{G}}}$ for every $n \geq k_j+1$. Using (K3) and property ($\bar{\mathcal{F}}1$) of $\digamma$, we have for each $n \geq k_j+1$,

$$\begin{aligned}\digamma[\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)] &\leq \digamma[\frac{\zeta(\mho^{n-2}\bar{\xi}_j,\mho^{n-1}\bar{\xi}_j)+\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)}{2}]-\Upsilon \\ &\leq \digamma[\zeta(\mho^{n-2}\bar{\xi}_j,\mho^{n-1}\bar{\xi}_j)]-\Upsilon\end{aligned}$$

Recursively using the above inequality, we get

$$\digamma[\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j)] \leq \digamma[\zeta(\mho^{k_j-1}\bar{\xi}_j,\mho^{k_j}\bar{\xi}_j)]-(n-k_j)\Upsilon$$

Letting $n\to\infty$ and using property ($\bar{\mathcal{F}}2$) of $\digamma$, we again have $\zeta(\mho^{n-1}\bar{\xi}_j,\mho^n\bar{\xi}_j) \to 0$. The proof now follows by letting $n\to\infty$ in inequality (12).

□

Theorem 2.1. The statements below are equivalent:

(i) $\mathcal{G}$ is a weakly connected graph.

(ii) Sequences $\{\mho^n\bar{\xi}\}$ and $\{\mho^n\bar{\rho}\}$ are equivalent and Cauchy for any $\bar{\xi},\bar{\rho}\in\mathcal{W}$ and self mapping $\mho$ on $\mathcal{W}$ which is $\digamma_{\mathcal{G}}$ -Kannan mapping.

(iii) $Card(Fix\mho)\leq 1$ for any $\digamma_{\mathcal{G}}$ -Kannan mapping $\mho$ on $\mathcal{W}$, where $Fix\mho$ represents the collection of all fixed points of $\mho$.

Proof. (i) $\Rightarrow$ (ii)

The graph $\mathcal{G}$ being weakly connected, we have $\mathcal{W}=[\bar{\xi}]_{\tilde{\mathcal{G}}}$. Consider a mapping $\mho$ on $\mathcal{W}$ which is $\digamma_{\mathcal{G}}$ -Kannan and $\bar{\xi},\bar{\rho}\in\mathcal{W}$. If there exists $k\in\mathbb{N}$ such that $\mho^{k-1}\bar{\xi}=\mho^k\bar{\xi}$, then the sequence $\{\mho^n\bar{\xi}\}$ is eventually a constant sequence and therefore Cauchy. Now assume $\mho^{k-1}\bar{\xi}\neq\mho^k\bar{\xi}$ for all $k\in\mathbb{N}$. Hence, $p_k=\zeta(\mho^{k-1}\bar{\xi},\mho^k\bar{\xi})>0$ for all $k\in\mathbb{N}$. Also, since $\mho\bar{\xi}\in[\bar{\xi}]_{\tilde{\mathcal{G}}}$, we can find a path $(\bar{\xi}_i)_{i=0}^N$ such that $\bar{\xi}_0=\bar{\xi},\bar{\xi}_N=\mho\bar{\xi}$ and $(\bar{\xi}_{i-1},\bar{\xi}_i)\in E_{\tilde{\mathcal{G}}}$ for every $i=1,2,...,N$. Let $p_{k,i}=\zeta(\mho^{k-1}\bar{\xi}_{i-1},\mho^{k-1}\bar{\xi}_i)$ and $q_{k,i}=\zeta(\mho^{k-1}\bar{\xi}_i,\mho^k\bar{\xi}_i)$. Then for any $k\in\mathbb{N}$, by triangle inequality, we have

$$p_k\leq\sum_{i=1}^{N}p_{k,i} \tag{13}$$

We will prove that series $\sum_{k=1}^{\infty}p_{k,i}$ converges for every $i=1,2,...,N$. Consider any fixed $i\in\{1,2,...,N\}$. If for some k, $\mho^{k-1}\bar{\xi}_{i-1}=\mho^{k-1}\bar{\xi}_i$ then $\{p_{k,i}\}_k$ is eventually zero sequence. Therefore this implies that $\sum_{k=1}^{\infty}p_{k,i}$ converges, being a sum of finitely many terms. So now assume that $\mho^{k-1}\bar{\xi}_{i-1}\neq\mho^{k-1}\bar{\xi}_i$ for any k. Also since $(\bar{\xi}_{i-1},\bar{\xi}_i)\in E_{\tilde{\mathcal{G}}}$, using (K1) inductively, we have $(\mho^{k-2}\bar{\xi}_{i-1},\mho^{k-2}\bar{\xi}_i)\in E_{\tilde{\mathcal{G}}}$ for all $k\geq 2$. By (K3) we get

$$\digamma[\zeta(\mho^{k-1}\bar{\xi}_{i-1},\mho^{k-1}\bar{\xi}_i)]\leq\digamma\left[\frac{\zeta(\mho^{k-2}\bar{\xi}_{i-1},\mho^{k-1}\bar{\xi}_{i-1})+\zeta(\mho^{k-2}\bar{\xi}_i,\mho^{k-1}\bar{\xi}_i)}{2}\right]-\Upsilon$$

or

$$\digamma[p_{k,i}]\leq\digamma\left[\frac{q_{k-1,i-1}+q_{k-1,i}}{2}\right]-\Upsilon$$

Since $\digamma$ is non decreasing, we obtain

$$p_{k,i}\leq\frac{q_{k-1,i-1}+q_{k-1,i}}{2} \tag{14}$$

By similar arguments as in Lemma 2.1, we have for each $i=0,1,...,N$, $q_{k,i}\to 0$ as $k\to\infty$ and

$$F[q_{k,i}] \leq F[\zeta(\mho^{k_i-1}\bar{\xi}_i, \mho^{k_i}\bar{\xi}_i)] - (k-k_i)\Upsilon \tag{15}$$

Since $q_{k,i} \to 0$ as $k \to \infty$, by property ($\bar{\mathcal{F}}3$), there exists $r_i \in (0,1)$ such that

$$\lim_{k\to\infty} q_{k,i}^{r_i} F(q_{k,i}) = 0. \tag{16}$$

(15), (16) gives $\lim_{k\to\infty}(k-k_i)q_{k,i}^{r_i} = 0$ and hence $\lim_{k\to\infty} kq_{k,i}^{r_i} = 0$. So, there exists $m_i \in \mathbb{N}$ such that $kq_{k,i}^{r_i} < 1$ for all $k \geq m_i$ or, $q_{k,i} < \frac{1}{k^{1/r_i}}$ for all $k \geq m_i$. This implies that series $\sum_{k=1}^{\infty} q_{k,i}$ converges and therefore inequality (14) implies that $\sum_{k=1}^{\infty} p_{k,i}$ converges in this case also. Hence, inequality (13) clearly implies $\sum_{k=1}^{\infty} p_k$ also converges. Now for any $n > m$, we have $\zeta(\mho^m x, \mho^n \bar{\xi}) \leq \zeta(\mho^m x, \mho^{m+1}\bar{\xi}) + \zeta(\mho^{m+1}\bar{\xi}, \mho^{m+2}\bar{\xi}) + \cdots + \zeta(\mho^{n-1}\bar{\xi}, \mho^n\bar{\xi}) = p_{m+1} + p_{m+2} + \cdots + p_n \to 0$ as $m,n \to \infty$. Thus, $\{\mho^n\bar{\xi}\}$ is a Cauchy sequence. Similarly $\{\mho^n\bar{\rho}\}$ is also a Cauchy sequence. From lemma 2.1, since $\bar{\rho} \in \mathcal{W} = [\bar{\xi}]_{\tilde{\mathcal{G}}}$, we have $\zeta(\mho^n\bar{\xi}, \mho^n\bar{\rho}) \to 0$ as $n \to \infty$.

(ii) $\Rightarrow$ (iii) Let $\mho : \mathcal{W} \to \mathcal{W}$ be any $F_{\mathcal{G}}$ -Kannan mapping and if $\bar{\xi}, \bar{\rho}$ are any two fixed points of $\mho$, then by (ii) sequence $\{\mho^n\bar{\xi}\} = \{\bar{\xi}\}$ and $\{\mho^n\bar{\rho}\} = \{\bar{\rho}\}$ are equivalent and thus $\zeta(\bar{\xi}, \bar{\rho}) = 0$, i.e $\bar{\xi} = \bar{\rho}$.

(iii) $\Rightarrow$ (i) Let if possible $\mathcal{G}$ is not weakly connected. So there must exist some $\bar{\xi}_0 \in \mathcal{W}$ such that $[\bar{\xi}_0]_{\tilde{\mathcal{G}}} \subset \mathcal{W}$. This implies the existence of some element $\bar{\rho}_0 \in \mathcal{W}$ but $\bar{\rho}_0 \notin [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$. Define $\mho : \mathcal{W} \to \mathcal{W}$ as

$$\mho(\bar{\xi}) = \begin{cases} \bar{\xi}_0 & \text{if } \bar{\xi} \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}} \\ \bar{\rho}_0 & \text{if } \bar{\xi} \notin [\bar{\xi}_0]_{\tilde{\mathcal{G}}} \end{cases}$$

Then by definition of $\mho$, we observe that $Fix(\mho) = \{\bar{\xi}_0, \bar{\rho}_0\}$. Also we prove $\mho$ is $F_{\mathcal{G}}$ -Kannan mapping. In fact, if $(a,b) \in E_{\mathcal{G}}$ then $[a]_{\tilde{\mathcal{G}}} = [b]_{\tilde{\mathcal{G}}}$. Thus, either both $a,b \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ or both $a,b \notin [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$. In either case, we have $\mho a = \mho b$ thereby $(\mho a, \mho b) \in E_{\mathcal{G}}$ since $E_{\mathcal{G}}$ contains all loops. Thus $\mathcal{G}$ is edge preserving. Also it can be noted that whenever $(a,b) \in E_{\mathcal{G}}$, then $\mho a = \mho b$. Thus conditions (K2) and (K3) hold vacuously for $\mho$ as defined. Simultaneously condition (iii) is violated since $\mho$ has two fixed points. Hence, graph $\mathcal{G}$ must be weakly connected. $\square$

Corollary 2.1. If $(\mathcal{W},\zeta)$ is complete, then

$\mathcal{G}$ is a weakly connected graph $\iff$ For every $\digamma_{\mathcal{G}}$ -Kannan mapping $\mho$ on $\mathcal{W}$, there exists a unique $\bar{\xi}_* \in \mathcal{W}$ such that $lim_{n\to\infty}\mho^n\bar{\xi} = \bar{\xi}_*$ for every $\bar{\xi} \in \mathcal{W}$.

Lemma 2.2. Suppose condition (K1) holds for a mapping $\mho : \mathcal{W} \to \mathcal{W}$ with respect to graph $\mathcal{G}$. Then $\mho\bar{\xi} \in [\mho\bar{\rho}]_{\mathcal{G}}$ for every $\bar{\xi} \in [\bar{\rho}]_{\mathcal{G}}$.

Proof. The proof follows by edge preserving property (K1) and definition of equivalence class on $\mathcal{W}$. □

Theorem 2.2. If $\mho : \mathcal{W} \to \mathcal{W}$ be an $\digamma_{\mathcal{G}}$ -Kannan mapping such that $\mho\bar{\xi}_0 \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ for some $\bar{\xi}_0 \in \mathcal{W}$. Then

(i) The set $[\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ is invariant under $\mho$.
(ii) $\mho|_{[\bar{\xi}_0]_{\tilde{\mathcal{G}}}}$ is an $\digamma_{\tilde{\mathcal{G}}_{\bar{\xi}_0}}$-Kannan mapping.
(iii) $\tilde{\mathcal{G}}_{\bar{\xi}_0}$ is $\mho|_{[\bar{\xi}_0]_{\tilde{\mathcal{G}}}}$-orbitally connected.
(iv) For any $\bar{\xi},\bar{\rho} \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ the sequnces $\{\mho^n\bar{\xi}\}_{n\in\mathbb{N}}$ and $\{\mho^n\bar{\rho}\}_{n\in\mathbb{N}}$ are Cauchy and equivalent sequences .

Proof. By proposition 2.1, $\mho$ is also $\digamma_{\tilde{\mathcal{G}}}$-Kannan mapping.

(i) Since $\mho\bar{\xi}_0 \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$, we have $[\bar{\xi}_0]_{\tilde{\mathcal{G}}} = [\mho\bar{\xi}_0]_{\tilde{\mathcal{G}}}$. Consider any $\bar{\xi} \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$. Then by previous lemma, $\mho\bar{\xi} \in [\mho\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ or, $\mho\bar{\xi} \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$.

(ii) Let $(a,b) \in E(\tilde{\mathcal{G}}_{\bar{\xi}_0}) \subseteq E_{\tilde{\mathcal{G}}}$ and since $\mho$ is $\digamma_{\tilde{\mathcal{G}}}$-Kannan mapping, we have $(\mho a,\mho b) \in E_{\tilde{\mathcal{G}}}$. Also $a,b \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$, so by part (i), $\mho a, \mho b \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}} = V(\tilde{\mathcal{G}}_{\bar{\xi}_0})$. Since $E(\tilde{\mathcal{G}}_{\bar{\xi}_0}) \subseteq E_{\tilde{\mathcal{G}}}$ and $\mho$ is $\digamma_{\tilde{\mathcal{G}}}$-Kannan mapping so conditions (K2) and (K3) are true for graph $\tilde{\mathcal{G}}_{\bar{\xi}_0}$ also.

(iii) If $a \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}} \subseteq \mathcal{W}$, so there exists some $k \in \mathbb{N}$ such that $(\mho^{k-1}a,\mho^k a) \in E_{\mathcal{G}}$. It remains to justify that $\mho^{k-1}a$ and $\mho^k a \in [\bar{\xi}_0]_{\tilde{\mathcal{G}}} = V(\tilde{\mathcal{G}}_{\bar{\xi}_0})$. This holds by (i) applied recursively.

(iv) Since $\tilde{\mathcal{G}}_{\bar{\xi}_0}$ is $\mho$-orbitally connected on $[\bar{\xi}_0]_{\tilde{\mathcal{G}}}$ and is also connected, so the proof follows from above Theorem.

□

Theorem 2.3. Let $(\mathcal{W},\zeta)$ be complete. Assume the following property **(A)** for the triplet $(\mathcal{W},\zeta,\mathcal{G})$

If a sequence $\bar{\xi}_k$ converges to $\bar{\xi}$ in $\mathcal{W}$, such that $(\bar{\xi}_k, \bar{\xi}_{k+1}) \in E_{\mathcal{G}}$ for each $k \in \mathbb{N}$, then there exists some sub sequence $\{\bar{\xi}_{n_k}\}_k$ such that $(\bar{\xi}_{n_k}, \bar{\xi}) \in E_{\mathcal{G}}$ for all $k \in \mathbb{N}$.

Let $\mho : \mathcal{W} \to \mathcal{W}$ is $F_{\mathcal{G}}$ -Kannan mapping and $\mathcal{W}_{\mho} = \{\bar{\xi} \in \mathcal{W} : (\bar{\xi}, \mho\bar{\xi}) \in E_{\mathcal{G}}\}$. Then following statements hold:

(i) $Card(Fix\mho) = Card\{[\bar{\xi}]_{\tilde{\mathcal{G}}} : \bar{\xi} \in \mathcal{W}_{\mho}\}$.

(ii) $Fix\mho \neq \emptyset \Leftrightarrow \mathcal{W}_{\mho} \neq \emptyset$.

(iii) There exists a unique fixed point of $\mho \Leftrightarrow$ there exists some $\bar{\xi}_0 \in \mathcal{W}_{\mho}$ satisfying $\mathcal{W}_{\mho} \subseteq [\bar{\xi}_0]_{\tilde{\mathcal{G}}}$.

(iv) $\mho|_{[\bar{\xi}]_{\tilde{\mathcal{G}}}}$ is a *PO* for any $\bar{\xi} \in \mathcal{W}_{\mho}$.

(v) If $\mathcal{W}' = \cup\{[\bar{\xi}]_{\tilde{\mathcal{G}}} : \bar{\xi} \in \mathcal{W}_{\mho}\}$ then $\mho|_{\mathcal{W}'}$ is a *WPO*.

(vi) If $\mathcal{W}_{\mho} \neq \emptyset$ and $\mathcal{G}$ is weakly connected then $\mho$ is a *PO*.

(vii) If $\mho \subseteq E_{\mathcal{G}}$ then $\mho$ is a *WPO* on $\mathcal{W}$.

Proof. We first prove (iv). For this, consider any $\bar{\xi} \in \mathcal{W}_{\mho}$. Then, $(\bar{\xi}, \mho\bar{\xi}) \in E_{\mathcal{G}}$. Therefore $\mho\bar{\xi} \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$. Hence, for any $\bar{\rho} \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$, we have by Theorem 2.2, sequences $\{\mho^n\bar{\xi}\}$ and $\{\mho^n\bar{\rho}\}$ are both Cauchy and are equivalent. Sine the metric space $\mathcal{W}$ is complete, there exists $\bar{\xi}_* \in \mathcal{W}$ such that $\mho^n\bar{\xi} \to \bar{\xi}_*$ and $\mho^n\bar{\rho} \to \bar{\xi}_*$ as $n \to \infty$. Also by (K1) applied inductively to $(\bar{\xi}, \mho\bar{\xi})$, we have $(\mho^n\bar{\xi}, \mho^{n+1}\bar{\xi}) \in E_{\mathcal{G}}$ for all $n \in \mathbb{N}$. By property **(A)**, there exists a sub sequence $\{\mho^{k_n}\bar{\xi}\}$ such that $(\mho^{k_n}\bar{\xi}, \bar{\xi}_*) \in E_{\mathcal{G}}$ for every $n \in \mathbb{N}$. Also by condition (K3), we must have $\zeta(\mho^{k_n+1}\bar{\xi}, \mho\bar{\xi}_*) \leq \frac{\zeta(\mho^{k_n}\bar{\xi}, \mho^{k_n+1}\bar{\xi}) + \zeta(\bar{\xi}_*, \mho\bar{\xi}_*)}{2}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ gives $\zeta(\bar{\xi}_*, \mho\bar{\xi}_*) = 0$. Hence $\bar{\xi}_*$ is a fixed point of $\mho$. Moreover, $\bar{\xi}_* \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$ since there is a path from $\bar{\xi}$ to $\bar{\xi}_*$ namely $\bar{\xi}, \mho\bar{\xi}, \mho^2\bar{\xi}, ..., \mho^{k_1}\bar{\xi}, \bar{\xi}_*$ in $\mathcal{G}$ (and hence in $\tilde{\mathcal{G}}$). Thus, $\mho|_{[\bar{\xi}]_{\tilde{\mathcal{G}}}}$ is a *PO*.

Clearly (v) follows from (iv).

Next we prove (vi). If $\mathcal{G}$ is weakly connected, then $[\bar{\xi}]_{\tilde{\mathcal{G}}} = \mathcal{W}$. So the proof follows from (iv).

To prove (vii), it is noted that if $\mho \subseteq E_{\mathcal{G}}$, then $\mathcal{W}_{\mho} = \mathcal{W}$ and thus $\mathcal{W}' = \mathcal{W}$ which implies by (v) $\mho$ is a *WPO*.

Now we prove (i). Let $C = \{[\bar{\xi}]_{\tilde{\mathcal{G}}} : \bar{\xi} \in \mathcal{W}_{\mho}\}$. Define a mapping $\pi : Fix\mho \to C$ as $\pi(\bar{\xi}) = [\bar{\xi}]_{\tilde{\mathcal{G}}}$. Then since $Fix\mho \subseteq \mathcal{W}_{\mho}$, so π is well defined. It is required to show that π is bijective. Consider any arbitrary $\bar{\xi} \in \mathcal{W}_{\mho}$. By (iv), $\mho|_{[\bar{\xi}]_{\tilde{\mathcal{G}}}}$ is a *PO*. Hence, there exists a unique fixed point of $\mho$,

$\bar{\xi}_* \in [\bar{\xi}]_{\tilde{\mathcal{G}}}$ and $\mho^n\bar{\xi} \to \bar{\xi}_*$ as $n \to \infty$. So, $[\bar{\xi}]_{\tilde{\mathcal{G}}} = [\bar{\xi}_*]_{\tilde{\mathcal{G}}} = \pi\bar{\xi}_*$. Thus π is surjective. Now, assume $\bar{\xi}_1$ and $\bar{\xi}_2 \in Fix\mho \subseteq \mathcal{W}_\mho$ such that $[\bar{\xi}_1]_{\tilde{\mathcal{G}}} = [\bar{\xi}_2]_{\tilde{\mathcal{G}}}$. By part(iv), $\mho|_{[\bar{\xi}_1]_{\tilde{\mathcal{G}}}}$ is a PO. Then there must exist a unique fixed point of $\mho$ say $a_* \in [\bar{\xi}_1]_{\tilde{\mathcal{G}}}$. But since $\bar{\xi}_1$ is also a fixed point of $\mho$ in $[\bar{\xi}_1]_{\tilde{\mathcal{G}}}$, so $a_* = \bar{\xi}_1$. Now, since $\bar{\xi}_2 \in [\bar{\xi}_1]_{\tilde{\mathcal{G}}}$ we have $\mho^n\bar{\xi}_2 \to a_* = \bar{\xi}_1$. But $\mho^n\bar{\xi}_2 \to \bar{\xi}_2$. So we must have $\bar{\xi}_2 = \bar{\xi}_1$. Thus π is injective also.

Proofs of (ii) and (iii) are followed from (i). □

Corollary 2.2. The following statements are equivalent in a complete metric space $(\mathcal{W}, \zeta)$ satisfying property **(A)** :

(i) For any $\mathcal{F}_\mathcal{G}$ -Kannan mapping $\mho : \mathcal{W} \to \mathcal{W}$, $Card(Fix\mho) \leq 1$.

(ii) $\mathcal{G}$ is weakly connected.

(iii) Every $\mathcal{F}_\mathcal{G}$ -Kannan mapping $\mho : \mathcal{W} \to \mathcal{W}$ such that $(\bar{\xi}_0, \mho\bar{\xi}_0) \in E_\mathcal{G}$ for some $\bar{\xi}_0 \in \mathcal{W}$ is a PO.

Proof. (i)⇒(ii) Follows from Theorem 2.1.

(ii)⇒(iii) This is followed from above Theorem 2.3(vi).

(iii)⇒(i) Let $\mho : \mathcal{W} \to \mathcal{W}$ be an $\mathcal{F}_\mathcal{G}$ -Kannan mapping. By Theorem 2.3 (ii), if $\mathcal{W}_\mho = \emptyset$ then so is $Fix\mho$ and thus $Card(Fix\mho) = 0$. If $\mathcal{W}_\mho \neq \emptyset$, then by hypothesis, $Card(Fix\mho) = 1$. In both cases $Card(Fix\mho) \leq 1$.

□

Open Problems:

(1) Can the underlying graph property of being $\mho$-orbitally connected be weakened?

(2) In Theorem 2.3, can property **(A)** be replaced by some other property?

(3) Prove the analogue of chatterjea contraction.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A. Lukacs, S. Kajanto, Fixed point theorems for various types of F-contractions in complete b-metric spaces, Fixed Point Theory, 19(1) (2018), 321-334.

[2] A. Petrusel, I.A. Rus, Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134 (2006), 411-418.

[3] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94.

[4] D. Wardowski, N.V. Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstr. Math. 47 (2014), 146-155.

[5] F. Bojor, Fixed points of Kannan mappings in metric spaces endowed with a graph, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. 20(1) (2012), 31-40.

[6] J. Górnicki, Fixed point theorems for F-expanding mappings, Fixed Point Theory Appl. 2017 (2016), 9.

[7] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008), 1359-1373.

[8] N.A. Secelean, Weak F-contractions and some fixed point results, Bull. Iran. Math. Soc. 42 (2016), 779-798.

[9] N. Goswami, N. Haokip, V.N. Mishra, F-contractive type mappings in b-metric spaces and some related fixed point results, Fixed Point Theory Appl. 2019 (2019), 13.

[10] R. Batra, R. Gupta, P. Sahni, A new extension of Kannan contractions and related fixed point results, J. Anal. 28 (2020), 1143–1154.

[11] R. Batra, S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, Int. J. Comput. Math. 91 (2014), 2483-2490.