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A NEW COMBINED BRACKETING METHOD FOR SOLVING NONLINEAR EQUATIONS

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Abstract: An improved method based on a combination of bisection, regula falsi, and parabolic interpolation has been developed. A new algorithm has been established. The number of iterations ensures convergence with respect to many examples published in the literature.

Keywords: Nonlinear equations, Parabolic interpolation, Muller's method, Bisection, Regula falsi.

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1. Introduction

Finding roots of nonlinear equations efficiently has widespread applications in numerical mathematics and applied mathematics. If the function is real, continuous, and changes sign over a known interval, the method of regula falsi (or false position) can be used. To speed up convergence, a bisection method utilizes to divide the search interval in half with each iteration. The speed of convergence of the bisection method can be improved by fitting a quadratic polynomial to the function at the endpoints and midpoint of the search interval. The root of this bisected quadratic then becomes one of the endpoints of the new search interval in the next iteration. The function and the associated quadratic will become more linear over the search interval, thus improving convergence speed even more. This method of root finding, called *bisected direct quadratic regula falsi* [1]. A switching mechanism between the bisection and regula falsi prevents the slow convergence of the proposed algorithm [2].

Muller [3] used a quadratic polynomial interpolation instead of a linear interpolation for the root determination, The Muller's method can also be found in the

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combination of the bisection or inverse quadratic interpolation [4-5]. Recently, several new algorithms and methods were published [6 - 12]. In the present paper we use a combination of the bisection and regula falsi with the second order polynomial interpolation technique based on one end point, midpoint and regula falsi point, this method called BRFC. Numerical tests on a variety of functions show that BRFC requires fewer iterations than other regula falsi or bisection methods.

2. Proposed zero finding methods

Firstly, it will be assumed that the function $f(x)$ is continuous on a closed interval $[x_a, x_b]$. In this interval, the function has a root and the following inequality holds $f(x_a) f(x_b) < 0$, we aim to determine the root x^* of the nonlinear equation with least numerical evaluations as possible:

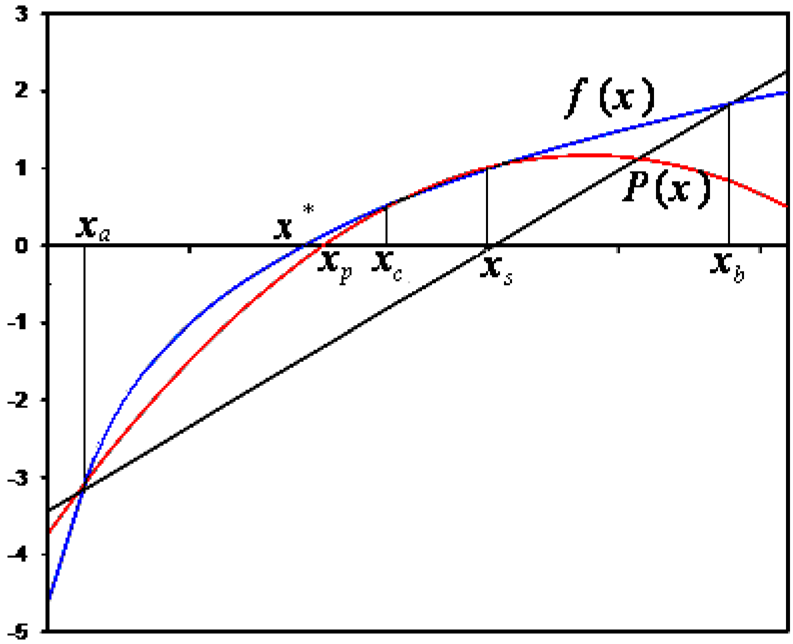

Figure 1: Root finding by Bisected and regula falsi and curve fitting *P(x)*

$$f(x) = 0. \tag{1}$$

A new iterative value on the closed interval is calculated by fitting a parabola to the three points of the function $f(x)$. The first point is the interval border point

$(x_a,(x_a))$, the second point $(x_c, f(x_c))$; $x_c \in (x_a, x_b)$ is calculated by using the bisection Fig. 1.

$$x_c = \frac{x_a + x_b}{2} \tag{2}$$

while the third point $(x_s, f(x_s))$; $x_s \in (x_a, x_b)$ is calculated by using regula falsi algorithm (Fig. 1)

$$x_s = \frac{f(x_a)x_b - f(x_b)x_a}{x_a - x_b} \tag{3}$$

If $x_s = x_c$ we replace x_s by x_b.

After calculating x_c and x_s, three points $(x_a, f(x_a))$, $(x_c, f(x_c))$ and $(x_s, f(x_s))$ are finally available and through these points, a second order polynomial can be constructed as follows:

$$p(x) = A(x - x_c)(x - x_s) + B(x - x_a)(x - x_s) + C(x - x_a)(x - x_c) \tag{4}$$

The constants *A*, *B*, and *C* can be determined from the following conditions:

$$\begin{aligned} P(x_a) &= f(x_a) = A(x_a - x_c)(x_a - x_s), \\ P(x_c) &= f(x_c) = B(x_c - x_a)(x_c - x_s), \\ P(x_s) &= f(x_s) = C(x_s - x_a)(x_s - x_c). \end{aligned} \tag{5}$$

Then, *A*, *B* and *C* can be determined from:

$$A = \frac{f(x_a)}{(x_a - x_c)(x_a - x_s)}, \quad B = \frac{f(x_c)}{(x_c - x_a)(x_c - x_s)}, \quad C = \frac{f(x_s)}{(x_s - x_a)(x_s - x_c)} \tag{6}$$

To determine the next approximation x_p by considering the intersection of the *x-axis* with the parabola defined in equation (4). The zeros of the parabola can be calculated from:

$$x_{1,2} = x_s - \frac{2c}{b \pm \sqrt{b^2 - 4ac}} \tag{7}$$

where the three parameters *a*, *b* and *c* can be determined from:

$$\begin{aligned} a &= A + B + C, \\ b &= A(x_s - x_c) + B(x_s - x_a) + C(2x_s - x_a - x_c), \\ c &= f(x_s). \end{aligned} \tag{8}$$

Eq. (7) gives two possibilities for $x_{1,2}$ depending on the sign preceding the radical term, therefore, Muller's method chooses the sign to agree with the sign of *b*. Chosen in this manner, the denominator will be the largest in magnitude and will result in $x_{1,2}$ being selected as the closed zeros of $P(x)$ to x_s. Thus we have:

$$x_p = x_s - \frac{2c}{b + sign(b)\sqrt{b^2 - 4ac}} \tag{9}$$

Equation (7) can be rewritten in an iterative form by introducing $x_i = x_s$ as an approximate value of the function zero in the current iteration. A new calculated value is then $x_{i+1} = x_p$.

$$x_{i+1} = x_i - \frac{2c}{b + sign(b)\sqrt{b^2 - 4ac}} \tag{10}$$

Because x_c and x_s are very closer to x^*, asymptotic superlinear convergence for a simple root x^* of a nonlinear equation $f(x) = 0$ and the proposed method is very effective with respect to the regula falsi-bisection-parabolic (RBP) method [2]. Since the convergence order of the bisection and regula-falsi methods is 2 then the convergence order of MRBP is also 2. See [2]

3. The new Bisection-Regula Falsi Algorithm

Procedure of Bisection- regula falsi to find a solution to $f(x) = 0$ given x_a, x_b, with $f(x_a)f(x_b) < 0$. Calculation precision ε and maximum number of iterations N_{max}

Start MRBP algorithm from i =1.

INPUT x_a, x_b, ε, $f(x)$ and maximum number of iterations N_{max} .

OUTPUT Approximate solution x^* or message of failure.

Step 1. Compute the initial bisection: $x_c = \dfrac{x_a + x_b}{2}$

Step 2. Compute the initial regula falsi: $x_s = \dfrac{f(x_a)x_b - f(x_b)x_a}{x_a - x_b}$

Step 3. If $x_s = x_c$ then set $x_s = x_b$

Step 4. Calculate Function values: $f(x_a)$, $f(x_c)$, $f(x_s)$ and $f(x_b)$

Step 5. if $f(x_a) = 0$ or $f(x_c) = 0$ or $f(x_s) = 0$ or $f(x_b) = 0$ print $x^*, f(x^*)$ and stop

Step 6. Set

$$A = \frac{f(x_a)}{(x_a - x_c)(x_a - x_s)}, \quad B = \frac{f(x_c)}{(x_c - x_a)(x_c - x_s)}, \quad C = \frac{f(x_s)}{(x_s - x_a)(x_s - x_c)}$$

Step 7. Set

$$a = A + B + C,$$
$$b = A(x_s - x_c) + B(x_s - x_a) + C(2x_s - x_a - x_c),$$
$$c = f(x_s).$$

Step 8. Set $x_p = x_s - \dfrac{2c}{b + sign(b)\sqrt{b^2 - 4ac}}$

Step 9. Calculate Function values: $f\left(x_p\right)$

Step 10. if $f\left(x_p\right) = 0$ or $\left|x_p - x_s\right| < \varepsilon$ print $x^*, f\left(x^*\right)$ and stop

Step 11. Set $\overline{X}$ =sort $(x_a, x_s, x_c, x_p, x_b)$

Step 12. for j = 1 to 4 do

If $f(\overline{X}_j)\, f(\overline{X}_{j+1}) < 0$ then $x_a = \overline{X}_a$, $x_b = \overline{X}_{j+1}$ end if

End for

Step 13. if $i < N_{max}$ then $i = i$ +1 and go to step 1. else print $x^*, f\left(x^*\right)$ and stop

End of the MRBP algorithm.

4. Numerical examples

We now compare the performance of the presented BRFC method with different methods. The number of iterations *n* is calculated for the bisection (Bis), regula falsi (Reg) and Suhadolnik's (Suh) algorithms. The comparisons are based on the number of iterations *n* which are presented in Table 1. This table contains the test functions used to test the performance of both methods. The numerical computations listed in the Table 1 were performed on Maple 15.

5. Conclusions

This paper presents a bracketing algorithm for the iterative zero finding of nonlinear equations. The algorithm is based on the combination of the bisection and regula falsi

Table 1. Different bracketing methods by presenting the number of iterations n.

$\varepsilon = 10^{-15}, N_{\max} = 10^5$			Bis	Reg	Suh	Present BRFC
No	$f(x)$	$[x_a, x_a]$	*Number of iterations n*			
1	$\ln x$	$[0.5, 5]$	52	29	6	4
2	$(10-x)e^{10x} - x^{10} + 1$	$[0.5, 8]$	53	$>10^5$	10	7
3	$e^{\sin x} - x - 1$	$[1, 4]$	52	33	5	4
4	$11x^{11} - 1$	$[0.5, 1]$	49	108	7	4
5	$2\sin x - 1$	$[0.1, \frac{\pi}{3}]$	50	15	4	3
6	$x^2 + \sin\frac{x}{10} - 25$	$[0, 1]$	50	34	3	3
7	$(x-1)e^{-x}$	$[0, 1.5]$	51	74	5	4
8	$\cos x - x$	$[0, 1.7]$	51	18	4	3
9	$(x-1)^3 - 1$	$[1.5, 3]$	51	61	5	3
10	$e^{x^2+7x-30} - 1$	$[2.6, 3.5]$	50	4020	7	4
11	$\arctan x - 1$	$[1, 8]$	53	27	6	4
12	$e^x - 2x - 1$	$[0.2, 3]$	52	157	6	4
13	$e^{-x} - x - \sin x$	$[0, 5]$	49	13	4	3
14	$x^3 - 1$	$[0.1,1.5]$	51	36	5	4
15	$x^2 - \sin^2 x - 1$	$[-1, 2]$	52	34	5	4
16	$\sin x - \frac{x}{2}$	$\left[\frac{\pi}{2}, \pi\right]$	51	33	4	3
17	x^3	$\left[-0.5, \frac{1}{3}\right]$	48	$>10^5$	44	6
18	x^5	$\left[-0.5, \frac{1}{3}\right]$	48	$>10^5$	49	5

with second order polynomial interpolation technique. The convergence of the algorithm is superlinear. The proposed algorithm can be used as a good substitute for well-known bracketing methods. The strength of the algorithm are presented on some typical examples and a comparison with other methods is given.

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