CONDITIONAL EXPECTATION FORMULA OF COPULAS FOR HIGHER DIMENSIONS AND ITS APPLICATION

ATINA AHDIKA[1,2], DEDI ROSADI[1,*], ADHITYA RONNIE EFFENDIE[1], GUNARDI[1]

[1]Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, Indonesia

[2]Department of Statistics, Universitas Islam Indonesia, Yogyakarta, Indonesia

Abstract. Due to its simple form, linear regression is the most commonly used model when dealing with a predictive model. However, there are some limitations to the model, such as the constraint of only being able to model variables that have a linear relationship, the assumption of normality on its error, and the multi-collinearity between independent variables which should not occur. One of the alternative models that is free from these limitations is the copula-based regression model defined by the conditional expectation formula of copulas. Leong and Valdez [Claims prediction using copula models, Insurance Math. Econom., 2005] [15] developed a conditional expectation formula of copulas for higher dimensions in the implicit form with bivariate case examples. Crane and Hoek [Conditional expectation formulae for copulas, Aust. N.Z.J. Stat, 2008] [5] provided conditional expectation formula of copulas explicitly for two dimensions with its examples. However, in practice, a predictive model often involves more than two variables, i.e. one dependent variable with more than one independent variable, including a copula-based regression model. With regard to these problems and the limitations of dimension in previous studies, our contribution in this study is extending the copula-based regression model for higher dimensions for class of Farlie-Gumbel-Morgenstern, elliptical, and Archimedean copula. We obtain a closed-form of conditional expectation formula of Farlie-Gumbel-Morgenstern, Gaussian, Student-t, and Clayton copula for n dimensions and provide the formula for Gumbel copula up to four dimensions. We apply our extended formula to estimate

KRW/USD currency based on its association with CNY/USD and JPY/USD, and found that the extended function can be used quite accurately.

Keywords: archimedean; conditional expectation; copula; elliptical; Farlie-Gumbel-Morgenstern; higher dimension.

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1. INTRODUCTION

One of the simplest and most widely used predictive models is a linear regression model. It is used to predict the data that have a linear relationship pattern. In addition, there are assumptions of normality that must be met and multi-collinearity that must be avoided. Meanwhile, there exist a possibility that there is a non-linear relationship between independent variables and the dependent variable. Besides, the distribution of the data is not always normal. In such cases, the linear regression model cannot be used. There is a function that can describe the relationship between some variables both linear and non-linear, namely copula function, first introduced by Abe Sklar in 1959. It is widely used to model the dependency structure between some variables with any non-linear relationship. The function's advantage lies in its ability to identify any structure of dependency, as well as there are no certain assumptions that the marginal distribution of the variables whose dependencies will be tested must satisfy.

The copula function is developed as an alternative predictive model for the linear regression model. Some studies mention that copula-based regression model can generally be written as a function of independent variables, where the model is defined as the conditional expectation of the dependent variable given by the independent variables [15, 5, 9, 11, 8, 6].

$$(1) \qquad\qquad\qquad E(Y|X) = h(X)$$

The main idea of the construction of a simple copula-based regression model using conditional expectation formula was introduced by Leong and Valdez [15] and Crane and Hoek [5]. Leong and Valdez [15] found the predictive claim in the form of a conditional expectation formula of copulas. They provided some formulas for multivariate copulas implicitly with the examples for bivariate cases. Crane and Hoek [5] derived conditional expectation formula for Farlie-Gumbel-Morgenstern (FGM), Iterated FGM, Gaussian, and Archimedean copula for

two and three-dimensional cases. The formula for the two dimensions model has been applied to several examples such as exchange-rate data, male waist size given by male forearm size, and male chest size given by male waist size. Kolev and Paiva [9] discussed the copula-based regression model for Gaussian copula, transitional regression model, longitudinal model, and Archimedean copula up to three dimensions cases. Parsa and Klugman [11] presented the formulas and algorithms necessary for conducting the Normal copula-based regression model and provided the examples for three-dimensional cases. The simulations describe that copula regression provides a good alternative to OLS and GLM. Noh *et al.* [8] studied the estimator for different copula-based regression models which are asymptotically normally distributed. Masarotto and Varin [6] made a package in R to implement the Gaussian copula-based regression model.

Based on the previous researches, the copula-based regression model gives more flexible result compared to the linear regression model because it can accommodate possible non-linear relationship that cannot be captured by the linear regression model. The basic foundation of the copula-based regression model is the conditional expectation formula of the dependent variable given by the independent variables. However, in the studies mentioned earlier, the copula-based regression model used is still focused on two-dimensional cases. Therefore, in this paper, our contribution is on the extension of the conditional expectation formula discussed by Leong and Valdez [15] and Crane and Hoek [5] to the higher dimensions for FGM, elliptical, consists of Gaussian and Student-t, and Archimedean copulas, consists of Clayton and Gumbel copula. This is interesting because in many cases, the predictive model is widely used to solve problems related to many independent variables. We obtain a closed-form of conditional expectation formula of FGM, Gaussian, Student-t, and Clayton copula for n dimensions and provide the formula for Gumbel copula up to four dimensions. To obtain a better understanding regarding the extended formula of the conditional expectation of copulas for higher dimensions, we provide an application on the financial case. We apply the three-dimensional conditional expectation of copula to estimate KRW/USD currency based on its association with CNY/USD and JPY/USD. For visualization purpose, we plot the estimation results against the original data in a three-dimensional plot.

The rest of this paper is organized as follows. In Section 2, we discuss the conditional expectation formula for multivariate copula and provide general functions for FGM, elliptical, and Archimedean copulas. In Section 3 we explain the estimation procedures for marginal and copula modeling. In Section 4, we apply the three-dimensional conditional expectation formula of copula to estimate KRW/USD currency based on its association with CNY/USD and JPY/USD. We also provide the three-dimensional plot of the estimation results for visualization purpose. The last Section provide the conclusion of our research.

2. CONDITIONAL EXPECTATION FORMULA FOR MULTIVARIATE COPULA

Suppose that Y is a random variable which represent the dependent variable and the sequence $\{X_1, X_2, \ldots, X_n\}$ are random variables which represent the independent variables, the conditional expectation of Y given by $\{X_1, X_2, \ldots, X_n\}$ defined by [5].

$$(2) \qquad E(Y|X_1, X_2, \ldots, X_n) = \int_R y \frac{\partial}{\partial y} P(Y \leq y | X_1, X_2, \ldots, X_n) dy$$

For bivariate cases, suppose that X represents the independent variable and Y represents the dependent variable. Suppose that $u = F_Y(y)$ and $v = G_X(x)$, H is the joint distribution function of X and Y, then

$$(3) \qquad H(F^{-1}(u), G^{-1}(v)) = C(u, v)$$

and the conditional distribution function is [14]

$$
\begin{aligned}
P(Y \leq y | X = x) &= \lim_{\Delta x \to 0} P(Y \leq y | x < X \leq x + \Delta x) \\
&= \lim_{\Delta x \to 0} \frac{P(Y \leq y, x < X \leq x + \Delta x)}{P(x < X \leq x + \Delta x)} \\
&= \lim_{\Delta x \to 0} \frac{H_{YX}(y, x + \Delta x) - H_{YX}(y, x)}{G_X(x + \Delta x) - G_X(x)} \\
&= \lim_{\Delta x \to 0} \frac{C(F_Y(y), G_X(x + \Delta x)) - C(F_Y(y), G_X(x))}{G_X(x + \Delta x) - G_X(x)} \\
(4) \qquad &= C_2(F_Y(y), G_X(x))
\end{aligned}
$$

The index 2 in Eq.(4) represents the partial derivative for the second variable, that is the independent variable.

Furthermore, for the three-dimensional cases, suppose that Y is the dependent variable and X_1 and X_2 are the independent variables with $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$ (see the illustration in Figure 1).
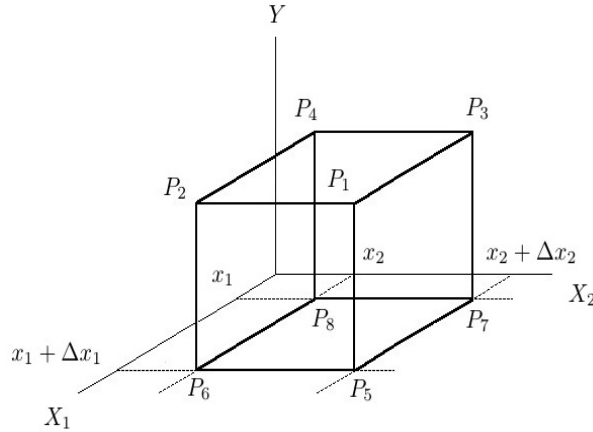
Figure 1. Definition Area for Three-Dimensional Cases

The multivariate distribution function of $P_1, P_2, \ldots, P_8$ can be expressed as the copula function as follows

$(i) \quad P_1(Y \leq y, X_1 \leq x_1 + \Delta x_1, X_2 \leq x_2 + \Delta x_2) = C_1(F(y), G_1(x_1 + \Delta x_1), G_2(x_2 + \Delta x_2))$

$(ii) \quad P_2(Y \leq y, X_1 \leq x_1 + \Delta x_1, X_2 \leq x_2) = C_2(F(y), G_1(x_1 + \Delta x_1), G_2(x_2))$

$(iii) \quad P_3(Y \leq y, X_1 \leq x_1, X_2 \leq x_2 + \Delta x_2) = C_3(F(y), G_1(x_1), G_2(x_2 + \Delta x_2))$

$(iv) \quad P_4(Y \leq y, X_1 \leq x_1, X_2 \leq x_2) = C_4(F(y), G_1(x_1), G_2(x_2))$

$(v) \quad P_5(X_1 \leq x_1 + \Delta x_1, X_2 \leq x_2 + \Delta x_2) = C_5(1, G_1(x_1 + \Delta x_1), G_2(x_2 + \Delta x_2))$

$(vi) \quad P_6(X_1 \leq x_1 + \Delta x_1, X_2 \leq x_2) = C_6(1, G_1(x_1 + \Delta x_1), G_2(x_2))$

$(vii) \quad P_7(X_1 \leq x_1, X_2 \leq x_2 + \Delta x_2) = C_7(1, G_1(x_1), G_2(x_2 + \Delta x_2))$

$(viii) \quad P_8(X_1 \leq x_1, X_2 \leq x_2) = C_8(1, G_1(x_1), G_2(x_2))$

Based on the definition area for the three-dimensional cases in Figure 1 and the relationship between multivariate distribution function and copula function, then the conditional distribution function of Y given by $X_1 = x_1$ and $X_2 = x_2$ can be derived as follows [5]

$$P(Y \leq y | X_1 = x_1, X_2 = x_2)$$

$$
\begin{aligned}
&= \lim_{(\Delta x_1, \Delta x_2) \to (0^+, 0^+)} P(Y \leq y | x_1 < X_1 \leq x_1 + \Delta x_1, x_2 < X_2 \leq x_2 + \Delta x_2) \\
&= \lim_{(\Delta x_1, \Delta x_2) \to (0^+, 0^+)} \frac{P(Y \leq y, x_1 < X_1 \leq x_1 + \Delta x_1, x_2 < X_2 \leq x_2 + \Delta x_2)}{P(x_1 < X_1 \leq x_1 + \Delta x_1, x_2 < X_2 \leq x_2 + \Delta x_2)} \\
&= \lim_{(\Delta x_1, \Delta x_2) \to (0^+, 0^+)} \frac{P_1 - P_2 - P_3 + P_4}{P_5 - P_6 - P_7 + P_8} \\
&= \lim_{(\Delta x_1, \Delta x_2) \to (0^+, 0^+)} \frac{C_1 - C_2 - C_3 + C_4}{C_5 - C_6 - C_7 + C_8} \\
&= \lim_{(\Delta x_1, \Delta x_2) \to (0^+, 0^+)} \frac{C_1 - C_2 - C_3 + C_4}{\{G_1(x_1 + \Delta x_1) - G_1(x_1)\}\{G_2(x_2 + \Delta x_2) - G_2(x_2)\}} \\
& \vdots \frac{C_5 - C_6 - C_7 + C_8}{\{G_1(x_1 + \Delta x_1) - G_1(x_1)\}\{G_2(x_2 + \Delta x_2) - G_2(x_2)\}}
\end{aligned}
$$

$$
\tag{5} = \frac{C_{23}(F(y), G_1(x_1), G_2(x_2))}{C_{23}(1, G_1(x_1), G_2(x_2))}
$$

Generally, for n independent variables, suppose that $u_1 = F(y), u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, where its multivariate distribution function can be expressed in the copula form below

$$
\tag{6} H(F^{-1}(u_1), G_1^{-1}(u_2), G_2^{-1}(u_3), \ldots, G_n^{-1}(u_{n+1})) = C(u_1, u_2, \ldots, u_{n+1})
$$

Then the conditional distribution function can be expressed as [5]

$$
\tag{7} P(Y \leq y | X_1 = x_1, \ldots, X_n = x_n) = \frac{C_{23\ldots n(n+1)}(F(y), G_1(x_1), \ldots, G_n(x_n))}{C_{23\ldots n(n+1)}(1, G_1(x_1), \ldots, G_n(x_n))}
$$

Therefore, the conditional expectation formula for multivariate cases is defined by [5]

$$
\begin{aligned}
E(Y | X_1 = x_1, \ldots, X_n = x_n) &= \int_R y \frac{\partial}{\partial y} P(Y \leq y | X_1 = x_1, \ldots, X_n = x_n) dy \\
&= \int_R y \frac{\partial}{\partial y} \frac{C_{23\ldots n(n+1)}(F(y), G_1(x_1), \ldots, G_n(x_n))}{C_{23\ldots n(n+1)}(1, G_1(x_1), \ldots, G_n(x_n))} dy \\
\tag{8} &= \int_R y \frac{C_{123\ldots n(n+1)}(F(y), G_1(x_1), \ldots, G_n(x_n))}{C_{23\ldots n(n+1)}(1, G_1(x_1), \ldots, G_n(x_n))} F'(y) dy
\end{aligned}
$$

2.1. Multivariate Farlie-Gumbel-Morgenstern Copula.

Farlie-Gumbel-Morgenstern (FGM) copula is one of the parametric copulas which is widely used because of its simple form. Multivariate FGM copula is defined by [13, 7]

$$
\tag{9} C(u_1, u_2, \ldots, u_d) = \prod_{k=1}^{d} u_k \left(1 + \sum_{k=2}^{d} \sum_{1 \leq j_1 < \ldots < j_k \leq d} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k} (1 - u_{j_i}) \right), \theta \in [-1, 1]
$$

Based on [7], multivariate FGM copula has $2^d - d - 1$ parameters.

By doing an algebraic derivation according to Eq.(8) using FGM d-copula from Eq.(9), we obtain the following theorem.

Theorem 2.1. *Suppose that Y represents a dependent variable and a sequence $\{X_1, X_2, \ldots, X_n\}$ represent the independent variables, with $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, then the conditional expectation of Y given by $\{X_1, X_2, \ldots, X_n\}$ for multivariate FGM copula is defined by*

$$(10) \qquad E(Y|X_1 = x_1, \ldots, X_n = x_n) = E(Y) + \eta(x_1, \ldots, x_n) \int_R y(1 - 2F(y))F'(y)dy$$

with

$$\eta(x_1, \ldots, x_n) = \frac{\sum_{k=1}^n \sum_{2 \leq j_1 < j_2 < \ldots < j_k \leq n+1} \theta_{1j_1j_2\ldots j_k} \prod_{i=1}^k (1 - 2u_{j_i})}{1 + \sum_{k=2}^n \sum_{2 \leq j_1 < j_2 < \ldots < j_k \leq n+1} \theta_{j_1j_2\ldots j_k} \prod_{i=1}^k (1 - 2u_{j_i})}$$

and

$$u_{j_i} = G_k(x_k), j_i = 2, 3, \ldots, n+1, \ i = 1, 2, \ldots, k, \ k = 1, 2, \ldots, n$$

Proof. Based on Eq.(9), we have $(n+1)$ variables consist of Y and $\{X_1, \ldots, X_n\}$ with $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, therefore we have the following equations

$$(11) \qquad C(u_1, u_2, \ldots, u_{n+1}) = \prod_{k=1}^{n+1} u_k + \prod_{k=1}^{n+1} u_k \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1j_2\ldots j_k} \prod_{i=1}^k (1 - u_{j_i})$$

and

$$(12) \qquad C(1, u_2, \ldots, u_{n+1}) = \prod_{k=2}^{n+1} u_k + \prod_{k=2}^{n+1} u_k \sum_{k=2}^{n} \sum_{2 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1j_2\ldots j_k} \prod_{i=1}^k (1 - u_{j_i})$$

Using Eq.(8), we have to derive Eq.(11) and (12) partially to $(n+1)$ and n-th order as follows

$$C_{123\ldots n(n+1)}(u_1, u_2, \ldots, u_{n+1}) = \frac{\partial^{n+1}}{\partial u_1 \ldots \partial u_{n+1}} C(u_1, u_2, \ldots, u_{n+1})$$

$$(13) \qquad\qquad = 1 + \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1j_2\ldots j_k} \prod_{i=1}^k (1 - 2u_{j_i})$$

$$C_{23\ldots n(n+1)}(1, u_2, \ldots, u_{n+1}) = \frac{\partial^n}{\partial u_2 \ldots \partial u_{n+1}} C(1, u_2, \ldots, u_{n+1})$$

$$(14) \qquad\qquad = 1 + \sum_{k=2}^{n} \sum_{2 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1j_2\ldots j_k} \prod_{i=1}^k (1 - 2u_{j_i})$$

Substitute Eq.(13) and (14) into Eq.(8), we obtain

$$
\begin{aligned}
& E(Y|X_1 = x_1, \ldots, X_n = x_n) \\
= {} & \int_R y \frac{C_{123\ldots n(n+1)}(F(y), G_1(x_1), \ldots, G_n(x_n))}{C_{23\ldots n(n+1)}(1, G_1(x_1), \ldots, G_n(x_n))} F'(y) dy \\
= {} & \int_R y \frac{C_{123\ldots n(n+1)}(u_1, u_2, \ldots, u_{n+1})}{C_{23\ldots n(n+1)}(1, u_2, \ldots, u_{n+1})} F'(y) dy \\
= {} & \int_R y \frac{1 + \sum_{k=2}^{n+1} \sum_{1 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})}{1 + \sum_{k=2}^{n} \sum_{2 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})} F'(y) dy \\
= {} & \int_R y \left(\frac{1 + \sum_{k=2}^{n} \sum_{2 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})}{1 + \sum_{k=2}^{n} \sum_{2 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})} \right) F'(y) dy + \\
& \int_R y \left(\frac{\sum_{k=1}^{n} \sum_{2 \le j_1 < j_2 < \ldots < j_k \le n+1} \theta_{1 j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})}{1 + \sum_{k=2}^{n} \sum_{2 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})} \right) (1 - 2u_1) F'(y) dy \\
= {} & E(Y) + \eta(x_1, \ldots, x_n) \int_R y(1 - 2F(y)) F'(y) dy
\end{aligned}
$$

with

$$
\eta(x_1, \ldots, x_n) = \frac{\sum_{k=1}^{n} \sum_{2 \le j_1 < j_2 < \ldots < j_k \le n+1} \theta_{1 j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})}{1 + \sum_{k=2}^{n} \sum_{2 \le j_1 < \ldots < j_k \le n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k}(1 - 2u_{j_i})}
$$

and

$$
u_{j_i} = G_k(x_k), j_i = 2, 3, \ldots, n+1, \ i = 1, 2, \ldots, k, \ k = 1, 2, \ldots, n
$$

$\square$

Example 2.2. For two-dimensional case, the conditional expectation of Y given by $X = x$ for bivariate FGM copula is

$$
(15) \qquad E(Y|X = x) = E(Y) + \theta(1 - 2G(x)) \int_R y(1 - 2F(y)) F'(y) dy
$$

Example 2.3. For three-dimensional case, suppose that there are three variables; Y that represents the dependent variable and X_1 and X_2 represent the independent variables, with $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$, then the FGM 3-copula has $2^3 - 3 - 1 = 4$ parameters and defined by

$$
C(u_1, u_2, u_3) = u_1 u_2 u_3 [1 + \theta_{12}(1 - u_1)(1 - u_2) + \theta_{13}(1 - u_1)(1 - u_3)
$$

$$(16) \qquad\qquad +\theta_{23}(1-u_2)(1-u_3)+\theta_{123}(1-u_1)(1-u_2)(1-u_3)]$$

The conditional expectation of Y given by $X_1 = x_1$ and $X_2 = x_2$ by using Theorem 2.1 is

$$(17) \qquad E(Y|X_1 = x_1, X_2 = x_2) = E(Y) + \eta(x_1,x_2)\int_R y(1-2F(y))F'(y)dy$$

with

$$\eta(x_1,x_2) =$$
$$\frac{\theta_{12}(1-2G_1(x_1))+\theta_{13}(1-2G_2(x_2))+\theta_{123}(1-2G_1(x_1))(1-2G_2(x_2))}{1+\theta_{23}(1-2G_1(x_1))(1-2G_2(x_2))}$$

Example 2.4. This example provides the conditional expectation formula for four-dimensional FGM copula. Suppose that Y represents the dependent variable and X_1, X_2, and X_3 are independent variables with $u_1 = F(y)$, $u_2 = G_1(x_1)$, $u_3 = G_2(x_2)$, and $u_4 = G_3(x_3)$, then the FGM 4-copula has $2^4 - 4 - 1 = 11$ parameters and defined by

$$
\begin{aligned}
C(u_1,\ldots,u_4) \;=\; & u_1u_2u_3u_4[1+\theta_{12}(1-u_1)(1-u_2)+\theta_{13}(1-u_1)(1-u_3)\\
&+\; \theta_{14}(1-u_1)(1-u_4)+\theta_{23}(1-u_2)(1-u_3)\\
&+\; \theta_{24}(1-u_2)(1-u_4)+\theta_{34}(1-u_3)(1-u_4)\\
&+\; \theta_{123}(1-u_1)(1-u_2)(1-u_3)+\theta_{124}(1-u_1)(1-u_2)(1-u_4)\\
&+\; \theta_{134}(1-u_1)(1-u_3)(1-u_4)+\theta_{234}(1-u_2)(1-u_3)(1-u_4)\\
&+\; \theta_{1234}(1-u_1)(1-u_2)(1-u_3)(1-u_4)]
\end{aligned}
$$

$$(18)$$

Conditional expectation formula for FGM 4-copula is defined by

$$(19)\quad E(Y|X_1 = x_1, X_2 = x_2, X_3 = x_3) \;=\; E(Y)+\eta(x_1,x_2,x_3)\int_R y(1-2F(y))F'(y)dy$$

with

$$
\begin{aligned}
\eta(x_1,x_2,x_3) \;=\; & \frac{a(x_1,x_2,x_3)}{b(x_1,x_2,x_3)}\\
a(x_1,x_2,x_3) \;=\; & \theta_{12}(1-2G_1(x_1))+\theta_{13}(1-2G_2(x_2))+\theta_{14}(1-2G_3(x_3))\\
&+\; \theta_{123}(1-2G_1(x_1))(1-2G_2(x_2))\\
&+\; \theta_{124}(1-2G_1(x_1))(1-2G_3(x_3))
\end{aligned}
$$

$$+ \quad \theta_{134}(1 - 2G_2(x_2))(1 - 2G_3(x_3))$$

$$+ \quad \theta_{1234}(1 - 2G_1(x_1))(1 - 2G_2(x_2))(1 - 2G_3(x_3))$$

$$b(x_1, x_2, x_3) \quad = \quad 1 + \theta_{23}(1 - 2G_1(x_1))(1 - 2G_2(x_2))$$

$$+ \quad \theta_{24}(1 - 2G_1(x_1))(1 - 2G_3(x_3))$$

$$+ \quad \theta_{34}(1 - 2G_2(x_2))(1 - 2G_3(x_3))$$

$$+ \quad \theta_{234}(1 - 2G_1(x_1))(1 - 2G_2(x_2))(1 - 2G_3(x_3))$$

Proposition 2.5. *If $\theta_{j_1 j_2 \ldots j_k} = \theta \; \forall j_i, i = 1, 2, \ldots, n+1$, then θ has a unique range of parameter, given by*

$$(20) \qquad\qquad\qquad \theta \in \left[-\frac{1}{2^d - d - 1}, \frac{1}{d - 1} \right]$$

for d-dimensions, where $d = n + 1$.

Proof. The proof is the extension of Johnson and Kott's work [10]. Suppose we have $n + 1$ variables consist of $Y, X_1, \ldots, X_n$ with $u_1 = F(y), u_2 = G_1(x_1), \ldots, u_{n+1} = G_n(x_n)$, the joint probability density function of $Y, X_1, \ldots, X_n$ is

$$h(y, x_1, \ldots, x_n) \quad = \quad \frac{\partial^{n+1}}{\partial y \partial x_1 \partial x_2 \ldots \partial x_n} H(y, x_1, x_2, \ldots, x_n)$$

$$= \quad \frac{\partial^{n+1}}{\partial y \partial x_1 \partial x_2 \ldots \partial x_n} C(F(y), G_1(x_1), G_2(x_2), \ldots, G_n(x_n))$$

$$= \quad \left(1 + \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1 j_2 \ldots j_k} (1 - 2F(y)) \prod_{i=2}^{k} (1 - 2u_{j_i}) \right)$$

$$\times f(y) \prod_{i=2}^{n} g(x_i)$$

$$= \quad \left(1 + \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq n+1} \theta_{j_1 j_2 \ldots j_k} \prod_{i=1}^{k} c_{j_i} \right) \times f(y) \prod_{i=2}^{n} g(x_i)$$

$$(21) \qquad = \quad \left(1 + \theta \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq n+1} \prod_{i=1}^{k} c_{j_i} \right) \times f(y) \prod_{i=2}^{n} g(x_i)$$

where $\theta_{j_1 j_2 \ldots j_k} = \theta \; \forall j_i, i = 1, 2, \ldots, n+1$, $c_{j_1} = (1 - 2F(y))$, and $c_{j_i} = (1 - 2G_{i-1}(x_{i-1}))$ for $i = 2, 3, \ldots, n+1$. Because both $0 \leq F(y) \leq 1$ and $0 \leq G_i(x_i) \leq 1$, then the possible value of

c_{j_i} is $-1 \leq c_{j_i} \leq 1$. Eq.(21) is said to be a probability density function if the value of $h \geq 0$, it means that the value in the bracket should be greater than or equal to zero.

$$(22) \qquad \left(1 + \theta \sum_{k=2}^{n+1} \sum_{1 \leq j_1 < ... < j_k \leq n+1} \prod_{i=1}^{k} c_{j_i}\right) \geq 0$$

For two dimensions $(d = 2)$, the result is quite obvious. We can refer to the case of bivariat FGM copula, and we obtain the range of parameter, i.e. $\theta \in [-1, 1]$.

For three dimensions consists of Y, X_1, and X_2, we have

$$1 + \theta \left(c_{j_1} c_{j_2} + c_{j_1} c_{j_3} + c_{j_2} c_{j_3} + c_{j_1} c_{j_2} c_{j_3}\right) \geq 0$$

The possible values of θ are

$(i) \qquad 1 + 4\theta \geq 0 \quad (c_{j_1} = c_{j_2} = c_{j_3} = 1)$

$(ii) \qquad 1 + 2\theta \geq 0 \quad (c_{j_1} = c_{j_2} = c_{j_3} = -1)$

$(iii) \qquad 1 - 2\theta \geq 0 \quad (c_{j_1} = -1; c_{j_2} = c_{j_3} = 1 \text{ and 2 other similar conditions})$

and trivial solution for the case $c_{j_1} = 1; c_{j_2} = c_{j_3} = -1$ and two other similar conditions. By completing the three inequalities, we have the range of parameter of θ for three dimensions that is $\theta \in \left[-\frac{1}{4}, \frac{1}{2}\right]$.

For four dimensions consists of Y, X_1, X_2, and X_3, we have

$$1 + \theta \left(c_{j_1} c_{j_2} + c_{j_1} c_{j_3} + c_{j_1} c_{j_4} + c_{j_2} c_{j_3} + c_{j_2} c_{j_4} + c_{j_3} c_{j_4}\right.$$

$$\left. + c_{j_1} c_{j_2} c_{j_3} + c_{j_1} c_{j_2} c_{j_4} + c_{j_1} c_{j_3} c_{j_4} + c_{j_2} c_{j_3} c_{j_4} + c_{j_1} c_{j_2} c_{j_3} c_{j_4}\right) \geq 0$$

Using similar way, we have possible values for θ that are

$(i) \qquad 1 + 11\theta \geq 0 \quad (c_{j_1} = c_{j_2} = c_{j_3} = c_{j_4} = 1)$

$(ii) \qquad 1 + 3\theta \geq 0 \quad (c_{j_1} = c_{j_2} = c_{j_3} = c_{j_4} = -1)$

$(iii) \qquad 1 - 3\theta \geq 0 \quad (c_{j_1} = -1; c_{j_2} = c_{j_3} = c_{j_4} = 1 \text{ and 3 other similar conditions})$

$(iv) \qquad 1 + \theta \geq 0 \quad (c_{j_1} = 1; c_{j_2} = c_{j_3} = c_{j_4} = -1 \text{ and 3 other similar conditions})$

$(v) \qquad 1 - \theta \geq 0 \quad (c_{j_1} = c_{j_2} = -1; c_{j_3} = c_{j_4} = 1 \text{ and 2 other similar conditions})$

Solution set for the five inequalities is $\theta \in \left[-\frac{1}{11}, \frac{1}{3}\right]$.

This method can be expanded into d dimensions, and the general solution is

$$\theta \in \left[-\frac{1}{2^d - d - 1}, \frac{1}{d - 1} \right]$$

$\square$

2.2. Multivariate Elliptical Copula.

In class of elliptical copula, two popular copulas are often to be used, that are Gaussian and Student-t copula. Multivariate copula for this class is defined by

$$(23) \qquad C(u_1, \ldots, u_n) = \begin{cases} \Phi_n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)), & \text{Gaussian copula} \\ t_n(t^{-1}(u_1), \ldots, t^{-1}(u_n)), & \text{Student-t copula} \end{cases}$$

where $\Phi_n(\cdot)$ and $t_n(\cdot)$ are the joint distribution functions of standard normal and Student-t, respectively.

We obtain the conditional expectation formula for multivariate elliptical copula as follows.

Theorem 2.6. *Suppose that Y is a random variable represents a dependent variable and a sequence $\{X_1, X_2, \ldots, X_n\}$ represent the independent variables, with $u_1 = F(y)$, and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, then the conditional expectation of Y given by $\{X_1, X_2, \ldots, X_n\}$ for multivariate elliptical copula is defined by*

$$
\begin{aligned}
& E(Y|X_1 = x_1, \ldots, X_n = x_n) \\
(24) \qquad & = \int_R y \frac{\psi_{n+1}(\Psi^{-1}(F(y)), \Psi^{-1}(G_1(x_1)), \ldots, \Psi^{-1}(G_n(x_n)))}{\psi_n(\Psi^{-1}(G_1(x_1)) \ldots \Psi^{-1}(G_n(x_n)))} \, dy
\end{aligned}
$$

where $\Psi_n(\cdot)$ dan $\psi_n(\cdot)$ are the joint distribution and density functions of the elliptical distribution with

$$(25) \qquad \Psi_n(\cdot) = \begin{cases} \Phi_n(\cdot), & \text{joint distribution function of standard normal} \\ t_n(\cdot), & \text{joint distribution function of Student-t} \end{cases}$$

$$(26) \qquad \psi_n(\cdot) = \begin{cases} \phi_n(\cdot), & \text{joint density function of standard normal} \\ f_{t,n}(\cdot), & \text{joint density function of Student-t} \end{cases}$$

Proof. Suppose that $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, then

$$(27) \qquad C(u_1, u_2, \ldots, u_{n+1}) = \Psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))$$

and

$$
\begin{aligned}
C(1, u_2, \ldots, u_{n+1}) &= \Psi_{n+1}(\Psi^{-1}(1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1})) \\
&= \Psi_{n+1}(\infty, \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1})) \\
(28) \qquad &= \Psi_n(\Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))
\end{aligned}
$$

Furthermore, Eq. (27) and (28) is derived partially to $n+1$ and n-th order such that

$$
\begin{aligned}
C_{123\ldots n(n+1)}(u_1, \ldots, u_{n+1}) &= \frac{\partial^{n+1}}{\partial u_1 \ldots \partial u_{n+1}} C(u_1, u_2, \ldots, u_{n+1}) \\
&= \frac{\partial^{n+1}}{\partial u_1 \ldots \partial u_{n+1}} \Psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1})) \\
(29) \qquad &= \frac{\psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi(\Psi^{-1}(u_1)) \psi(\Psi^{-1}(u_2)) \ldots \psi(\Psi^{-1}(u_{n+1}))}
\end{aligned}
$$

and

$$
\begin{aligned}
C_{23\ldots n(n+1)}(1, u_2, \ldots, u_{n+1}) &= \frac{\partial^n}{\partial u_2 \ldots \partial u_{n+1}} C(1, u_2, \ldots, u_{n+1}) \\
&= \frac{\partial^n}{\partial u_2 \ldots \partial u_{n+1}} \Psi_n(\Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1})) \\
(30) \qquad &= \frac{\psi_n(\Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi(\Psi^{-1}(u_2)) \ldots \psi(\Psi^{-1}(u_{n+1}))}
\end{aligned}
$$

Finally, substitute Eq. (29) and (30) into Eq. (8) so that we obtain

$$
\begin{aligned}
&E(Y | X_1 = x_1, \ldots, X_n = x_n) \\
&= \int_R y \frac{C_{123\ldots n(n+1)}(F(y), G_1(x_1), \ldots, G_n(x_n))}{C_{23\ldots n(n+1)}(1, G_1(x_1), \ldots, G_n(x_n))} F'(y) dy \\
&= \int_R y \frac{C_{123\ldots n(n+1)}(u_1, u_2, \ldots, u_{n+1})}{C_{23\ldots n(n+1)}(1, u_2, \ldots, u_{n+1})} F'(y) dy \\
&= \int_R y \frac{\frac{\psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi(\Psi^{-1}(u_1)) \psi(\Psi^{-1}(u_2)) \ldots \psi(\Psi^{-1}(u_{n+1}))}}{\frac{\psi_n(\Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi(\Psi^{-1}(u_2)) \ldots \psi(\Psi^{-1}(u_{n+1}))}} \psi(\Psi^{-1}(u_1)) dy \\
&= \int_R y \frac{\psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi(\Psi^{-1}(u_1)) \psi_n(\Psi^{-1}(u_2) \ldots \Psi^{-1}(u_{n+1}))} \psi(\Psi^{-1}(u_1)) dy
\end{aligned}
$$

$$= \int_R y \frac{\psi_{n+1}(\Psi^{-1}(u_1), \Psi^{-1}(u_2), \ldots, \Psi^{-1}(u_{n+1}))}{\psi_n(\Psi^{-1}(u_2) \ldots \Psi^{-1}(u_{n+1}))} dy$$

$$= \int_R y \frac{\psi_{n+1}(\Psi^{-1}(F(y)), \Psi^{-1}(G_1(x_1)), \ldots, \Psi^{-1}(G_n(x_n)))}{\psi_n(\Psi^{-1}(G_1(x_1)) \ldots \Psi^{-1}(G_n(x_n)))} dy$$

$$\square$$

2.2.1. *Gaussian Copula.* We provide the two and three dimensional examples of conditional expectation formula for Gaussian copula.

Example 2.7. For two-dimensional case, the conditional expectation formula of Y given by $X = x$, where $u_1 = F(y)$ and $u_2 = G(x)$, for Gaussian copula is

$$E(Y|X=x)$$

$$= \int_R y \frac{\phi_2(\Phi^{-1}(F(y)), \Phi^{-1}(G(x)))}{\phi(\Phi^{-1}(G(x)))} dy$$

$$= \int_R y \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{(\Phi^{-1}(F(y)))^2 - 2\rho\Phi^{-1}(F(y))\Phi^{-1}(G(x)) + (\Phi^{-1}(G(x)))^2}{2(1-\rho^2)}\right\}}{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(\Phi^{-1}(G(x)))^2\right\}} dy$$

$$= \int_R y \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left\{\frac{2\rho\Phi^{-1}(F(y))\Phi^{-1}(G(x)) - (\Phi^{-1}(F(y)))^2 - \rho^2(\Phi^{-1}(G(x)))^2}{2(1-\rho^2)}\right\} dy$$

$$= \frac{1}{\sqrt{1-\rho^2}} \int_R y\phi\left(\frac{\Phi^{-1}(F(y)) - \rho\Phi^{-1}(G(x))}{\sqrt{1-\rho^2}}\right) dy$$

(31)

Example 2.8. For three-dimensional case, the conditional expectation formula of Y given by $X_1 = x_1$ dan $X_2 = x_2$, where $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$, for Gaussian copula is

$$E(Y|X_1 = x_1, X_2 = x_2)$$

$$= \int_R y \frac{\phi_3(\Phi^{-1}(F(y)), \Phi^{-1}(G_1(x_1)), \Phi^{-1}(G_2(x_2)))}{\phi_2(\Phi^{-1}(G_1(x_1)), \Phi^{-1}(G_2(x_2)))} dy$$

$$= \int_R y \frac{\frac{(2\pi)^{-\frac{3}{2}}}{\sqrt{1-(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2)+2\rho_{12}\rho_{13}\rho_{23}}} \exp\left\{-\frac{w_1}{2(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1)}\right\}}{\frac{(2\pi)^{-1}}{\sqrt{1-\rho_{23}^2}} \exp\left\{-\frac{w_2}{2(1-\rho_{23}^2)}\right\}} dy$$

$$= \int_R y \frac{\sqrt{1-\rho_{23}^2}}{\sqrt{2\pi(1-(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2)+2\rho_{12}\rho_{13}\rho_{23})}}$$

$$(32) \qquad \exp\left\{\frac{-w_1(1-\rho_{23}^2)+w_2(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1)}{2(1-\rho_{23}^2)(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1)}\right\}dy$$

with

$$
\begin{aligned}
w_1 &= (\Phi^{-1}(F(y)))^2(\rho_{23}^2-1)+(\Phi^{-1}(G_1(x_1))(\rho_{13}^2-1)+(\Phi^{-1}(G_2(x_2)))^2(\rho_{12}^2-1)\\
&+ 2\left[\Phi^{-1}(F(y))\Phi^{-1}(G_1(x_1))(\rho_{12}-\rho_{13}\rho_{23})+\Phi^{-1}(F(y))\Phi^{-1}(G_2(x_2))(\rho_{13}-\rho_{12}\rho_{23})\right.\\
&+ \left.\Phi^{-1}(G_1(x_1))\Phi^{-1}(G_2(x_2))(\rho_{23}-\rho_{12}\rho_{13})\right]\\
w_2 &= (\Phi^{-1}(G_1(x_1)))^2-2\rho_{23}\Phi^{-1}(G_1(x_1))\Phi^{-1}(G_2(x_2))+(\Phi^{-1}(G_2(x_2)))^2
\end{aligned}
$$

2.2.2. *Student-t Copula.* We also provide two and three-dimensional example of conditional expectation formula of Student-*t* copula.

Example 2.9. For two-dimensional case, conditional expectation of Y given by $X = x$, where $u_1 = F(y)$ and $u_2 = G(x)$, for Student-*t* copula is

$$
\begin{aligned}
&E(Y|X=x)\\
&= \int_R y\frac{f_{t,2}(t^{-1}(F(y)),t^{-1}(G(x)))}{f_t(t^{-1}(G(x)))}dy\\
&= \int_R y\frac{\frac{\Gamma\left(\frac{v+2}{2}\right)}{v\pi\Gamma\left(\frac{v}{2}\right)\sqrt{1-\rho^2}}\left(1+\frac{t^{-1}(F(y))^2+t^{-1}(G(x))^2-2\rho t^{-1}(F(y))t^{-1}(G(x))}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}}}{\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v}{2}\right)}\left(1+\frac{t^{-1}(G(x))^2}{v}\right)^{-\frac{v+1}{2}}}dy\\
&= \frac{\Gamma\left(\frac{v+2}{2}\right)}{\sqrt{v\pi}\Gamma\left(\frac{v+1}{2}\right)\sqrt{1-\rho^2}}\int_R y\frac{\left(1+\frac{t^{-1}(F(y))^2+t^{-1}(G(x))^2-2\rho t^{-1}(F(y))t^{-1}(G(x))}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}}}{\left(1+\frac{t^{-1}(G(x))^2}{v}\right)^{-\frac{v+1}{2}}}dy
\end{aligned}
$$

(33)

Example 2.10. For three-dimensional case, conditional expectation of Y given by $X_1 = x_1$ and $X_2 = x_2$, where $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$ is

$$
\begin{aligned}
&E(Y|X_1=x_1,X_2=x_2)\\
&= \int_R y\frac{f_{t,3}(t^{-1}(F(y)),t^{-1}(G_1(x_1)),t^{-1}(G_2(x_2)))}{f_{t,2}(t^{-1}(G_1(x_1)),t^{-1}(G_2(x_2)))}dy
\end{aligned}
$$

$$= \int_R y \frac{\frac{\Gamma\left(\frac{v+3}{2}\right)}{v^{\frac{3}{2}}\pi^{\frac{3}{2}}\Gamma\left(\frac{v}{2}\right)\sqrt{1-(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2)+2\rho_{12}\rho_{13}\rho_{23}}} \left(1+\frac{s_1}{v(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1)}\right)^{-\frac{v+3}{2}}}{\frac{\Gamma\left(\frac{v+2}{2}\right)}{v\pi\Gamma\left(\frac{v}{2}\sqrt{1-\rho_{23}^2}\right)}\left(1+\frac{s_2}{v(1-\rho_{23}^2)}\right)^{-\frac{v+2}{2}}} \, dy$$

$$= \frac{\Gamma\left(\frac{v+3}{2}\right)\sqrt{1-\rho_{23}^2}}{\sqrt{v\pi}\Gamma\left(\frac{v+2}{2}\right)\sqrt{1-(\rho_{12}^2\rho_{13}^2+\rho_{23}^2)+2\rho_{12}\rho_{13}\rho_{23}}} \int_R y \frac{\left(1+\frac{s_1}{v(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}\rho_{23}-1)}\right)^{-\frac{v+3}{2}}}{\left(1+\frac{s_2}{v(1-\rho_{23}^2)}\right)^{-\frac{v+2}{2}}} \, dy$$

(34)

with

$$
\begin{aligned}
s_1 &= (t^{-1}(F(y)))^2(\rho_{23}^2-1)+(t^{-1}(G_1(x_1))(\rho_{13}^2-1)+(t^{-1}(G_2(x_2)))^2(\rho_{12}^2-1) \\
&\quad + 2\left[t^{-1}(F(y))t^{-1}(G_1(x_1))(\rho_{12}-\rho_{13}\rho_{23})+t^{-1}(F(y))t^{-1}(G_2(x_2))(\rho_{13}-\rho_{12}\rho_{23})\right. \\
&\quad + \left. t^{-1}(G_1(x_1))\Phi^{-1}(G_2(x_2))(\rho_{23}-\rho_{12}\rho_{13})\right] \\
s_2 &= (t^{-1}(G_1(x_1)))^2-2\rho_{23}t^{-1}(G_1(x_1))t^{-1}(G_2(x_2))+(t^{-1}(G_2(x_2)))^2
\end{aligned}
$$

2.3. Multivariate Archimedean Copula.

Another popular type of copula is Archimedean copula which is defined by [12]

$$C(u_1,u_2,\ldots,u_d) = \phi^{-1}(\phi(u_1)+\phi(u_2)+\ldots+\phi(u_d)) \tag{35}$$

where $\phi(\cdot)$ is generator function of Archimedean copula consists of Clayton and Gumbel. Generator functions of Archimedean copula with the range of parameter which satisfy the completely monotonic properties for d-copula in $[0,\infty)$ are shown in Table 1 [3, 12]

Table 1. Generator Function of Archimedean Copula

Copula	Generator Function	Range of Parameter*
Clayton	$\frac{1}{u^\theta}-1$	$\theta > 0$
Gumbel	$(-\ln u)^\theta$	$\theta \geq 1$

* Monotonic for d-copula

2.3.1. *Clayton Copula.*

Based on Table 1, multivariate Clayton copula is defined by

$$C(u_1,u_2,\ldots,u_d) = (u_1^{-\theta}+u_2^{-\theta}+\ldots+u_n^{-\theta}-d+1)^{-\frac{1}{\theta}} \tag{36}$$

Theorem 2.11. *Suppose that Y represents a dependent variable and a sequence $\{X_1, X_2, \ldots, X_n\}$ represents the independent variables, with $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, then the conditional expectation of Y given by $\{X_1, X_2, \ldots, X_n\}$ of multivariate Clayton copula is defined by*

$$
E(Y|X_1 = x_1, \ldots, X_n = x_n)
$$

$$
(37) \qquad = (1+n\theta)\gamma(x_1, \ldots, x_n)^{-(1+n\theta)} \int_R y F(y)^{-(1+\theta)} \beta(y, x_1, \ldots, x_n)^{(1+(n+1)\theta)} F'(y) dy
$$

with

$$
\gamma(x_1, \ldots, x_n) = C(1, G_1(x_1), \ldots, G_n(x_n))
$$

$$
= (G_1(x_1)^{-\theta} + G_2(x_2)^{-\theta} + \ldots + G_n(x_n)^{-\theta} - n + 1)^{-\frac{1}{\theta}}
$$

$$
\beta(y, x_1, \ldots, x_n) = C(F(y), G_1(x_1), \ldots, G_n(x_n))
$$

$$
= (F(y)^{-\theta} + G_1(x_1)^{-\theta} + \ldots + G_n(x_n)^{-\theta} - (n+1) + 1)^{-\frac{1}{\theta}}
$$

Proof. We have $(n+1)$ variables consist of Y and $\{X_1, X_2, \ldots, X_n\}$ with $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2), \ldots, u_{n+1} = G_n(x_n)$, therefore, based on Eq.(36) we have the following equations

$$
(38) \qquad C(u_1, u_2, \ldots, u_{n+1}) = (u_1^{-\theta} + u_2^{-\theta} + \ldots + u_{n+1}^{-\theta} - (n+1) + 1)^{-\frac{1}{\theta}}
$$

and

$$
(39) \qquad C(1, u_2, \ldots, u_{n+1}) = (u_2^{-\theta} + \ldots + u_{n+1}^{-\theta} - n + 1)^{-\frac{1}{\theta}}
$$

Using Eq.(8), we have to derive Eq.(38) and (39) partially to $(n+1)$ and n-th order and we obtain

$$
C_{123\ldots n(n+1)}(u_1, u_2, \ldots, u_{n+1})
$$

$$
(40) \qquad = \prod_{k=1}^{n+1}(1+(k-1)\theta)u_k^{-(1+\theta)}\{C(u_1, u_2, \ldots, u_{n+1})\}^{(1+(n+1)\theta)}
$$

and

$$
C_{23\ldots n(n+1)}(1, u_2, \ldots, u_{n+1})
$$

$$(41) \qquad = \prod_{k=1}^{n}(1+(k-1)\theta)\,u_{k+1}^{-(1+\theta)}\,\{C(1,u_2,\ldots,u_{n+1})\}^{(1+n\theta)}$$

Then, substitute Eq.(40) and (41) to Eq.(8) and we obtain

$$E(Y|X_1 = x_1, \ldots, X_n = x_n)$$

$$= \int_R y \frac{C_{123\ldots n(n+1)}(F(y),G_1(x_1),\ldots,G_n(x_n))}{C_{23\ldots n(n+1)}(1,G_1(x_1),\ldots,G_n(x_n))} F'(y)dy$$

$$= \int_R y \frac{C_{123\ldots n(n+1)}(u_1,u_2,\ldots,u_{n+1})}{C_{23\ldots n(n+1)}(1,u_2,\ldots,u_{n+1})} F'(y)dy$$

$$= \int_R y \frac{\prod_{k=1}^{n+1}(1+(k-1)\theta)\,u_k^{-(1+\theta)}\{C(u_1,u_2,\ldots,u_{n+1})\}^{(1+(n+1)\theta)}}{\prod_{k=1}^{n}(1+(k-1)\theta)\,u_{k+1}^{-(1+\theta)}\{C(1,u_2,\ldots,u_{n+1})\}^{(1+n\theta)}} F'(y)dy$$

$$= \int_R y(1+(n+1-1)\theta)\,u_1^{-(1+\theta)}\frac{\{C(u_1,u_2,\ldots,u_{n+1})\}^{(1+(n+1)\theta)}}{\{C(1,u_2,\ldots,u_{n+1})\}^{(1+n\theta)}} F'(y)dy$$

$$= (1+n\theta)\{C(1,u_2,\ldots,u_{n+1})\}^{-(1+n\theta)}\int_R y u_1^{-(1+\theta)}\{C(u_1,\ldots,u_{n+1})\}^{(1+(n+1)\theta)}$$

$$F'(y)dy$$

$$= (1+n\theta)\gamma(x_1,\ldots,x_n)^{-(1+n\theta)}\int_R yF(y)^{-(1+\theta)}\beta(y,x_1,\ldots,x_n)^{(1+(n+1)\theta)}F'(y)dy$$

with

$$\gamma(x_1,\ldots,x_n) \;=\; C(1,G_1(x_1),\ldots,G_n(x_n))$$

$$= (G_1(x_1)^{-\theta}+G_2(x_2)^{-\theta}+\ldots+G_n(x_n)^{-\theta}-n+1)^{-\frac{1}{\theta}}$$

$$\beta(y,x_1,\ldots,x_n) \;=\; C(F(y),G_1(x_1),\ldots,G_n(x_n))$$

$$= (F(y)^{-\theta}+G_1(x_1)^{-\theta}+\ldots+G_n(x_n)^{-\theta}-(n+1)+1)^{-\frac{1}{\theta}}$$

$$\square$$

Example 2.12. Using the formula of conditional expectation of Clayton in Theorem 2.11, for two-dimensional case we obtain

$$E(Y|X = x)$$

(42) $$= (1+\theta)G(x)^{-(1+\theta)} \int_R yF(y)^{-(1+\theta)}(F(y)^{-\theta} + G(x)^{-\theta} - 1)^{-(\frac{1+2\theta}{\theta})}F'(y)dy$$

Eq. (42) is as the same as provided by [5].

Example 2.13. Suppose that there are three variables; Y that represents a dependent variable and $\{X_1, X_2\}$ that represent independent variables, with $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$. The Clayton 3-copula is defined by

(43) $$C(u_1, u_2, u_3) = (u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} - 2)^{-\frac{1}{\theta}}$$

By using Theorem 2.11, the conditional expectation of Y given by $X_1 = x_1$ and $X_2 = x_2$ given by

$$E(Y|X_1 = x_1, X_2 = x_2)$$

(44) $$= (1+2\theta)\gamma(x_1, x_2)^{-(1+2\theta)} \int_R yF(y)^{-(1+\theta)}\beta(y, x_1, x_2)^{(1+3\theta)}F'(y)dy$$

with

$$\gamma(x_1, x_2) = (G_1(x_1)^{-\theta} + G_2(x_2)^{-\theta} - 1)^{-\frac{1}{\theta}}$$
$$\beta(y, x_1, x_2) = (F(y)^{-\theta} + G_1(x_1)^{-\theta} + G_2(x_2)^{-\theta} - 2)^{-\frac{1}{\theta}}$$

Example 2.14. For four-dimensional case, suppose that Y represents a dependent variable and a set of $\{X_1, X_2, X_3\}$ represents independent variables, with $u_1 = F(y), u_2 = G_1(x_1), u_3 = G_2(x_2)$, and $u_4 = G_3(x_3)$, then the Clayton 4-copula is defined by

(45) $$C(u_1, u_2, u_3, u_4) = (u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} + u_4^{-\theta} - 3)^{-\frac{1}{\theta}}$$

The conditional expectation of Y given by $X_1 = x_1, X_2 = x_2$, and $X_3 = x_3$ given by

$$E(Y|X_1 = x_2, X_2 = x_2, X_3 = x_3)$$

(46) $$= (1+3\theta)\gamma(x_1, x_2, x_3)^{-(1+3\theta)} \int_R yF(y)^{-(1+\theta)}\beta(y, x_1, x_2, x_3)^{(1+4\theta)}F'(y)dy$$

with

$$\gamma(x_1,x_2,x_3) = (G_1(x_1)^{-\theta} + G_2(x_2)^{-\theta} + G_3(x_3)^{-\theta} - 2)^{-\frac{1}{\theta}}$$

$$\beta(y,x_1,x_2,x_3) = (F(y)^{-\theta} + G_1(x_1)^{-\theta} + G_2(x_2)^{-\theta} + G_3(x_3)^{-\theta} - 3)^{-\frac{1}{\theta}}$$

2.3.2. *Gumbel Copula.* Multivariate Gumbel copula is defined by

$$(47) \qquad C(u_1,u_2,\ldots,u_d) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta + \ldots + (-\ln u_d)^\theta\right]^{\frac{1}{\theta}}\right)$$

Suppose that Y represents a dependent variable and a sequence of $\{X_1,\ldots,X_n\}$ represent the independent variables, with $u_1 = F(y)$ and $u_2 = G_1(x_1), u_3 = G_2(x_2),\ldots,u_{n+1} = G_n(x_n)$, then the conditional expectation of Y given by $\{X_1,X_2,\ldots,X_n\}$ of multivariate Gumbel copula is defined by

$$E(Y|X_1 = x_1,\ldots,X_n = x_n)$$

$$(48) \qquad = \alpha_1(\theta)\alpha_2(x_1,\ldots,x_n) \int_R y(F(y))^{-1}(-\ln F(y))^{\theta-1}\alpha_3(y,x_1,\ldots,x_n)F'(y)dy$$

with

$$\alpha_2(x_1,\ldots,x_n)$$

$$= \frac{\left[(-\ln G_1(x_1))^\theta + \ldots + (\ln G_n(x_n))^\theta\right]^{\frac{n}{\theta}}}{\exp\left(-\left[(-\ln G_1(x_1))^\theta + \ldots + (-\ln G_n(x_n))^\theta\right]^{\frac{1}{\theta}}\right)}$$

$$\alpha_3(y,x_1,\ldots,x_n)$$

$$= \frac{\exp\left(-\left[(-\ln F(y))^\theta + (-\ln G_1(x_1))^\theta + \ldots + (-\ln G_n(x_n))^\theta\right]^{\frac{1}{\theta}}\right)}{\left[(-\ln F(y))^\theta + (-\ln G_1(x_1))^\theta + \ldots + (\ln G_n(x_n))^\theta\right]^{\frac{n+1}{\theta}}}$$

There is no closed form for $\alpha_1(\theta)$. However we provide some extensions of conditional expectation formula of Gumbel copula up to four dimensions, i.e. $\alpha_1(\theta) = 1$, $\alpha_1(\theta) = \frac{2\theta^2-4\theta+1}{(\theta-1)^2}$, and $\alpha_1(\theta) = \frac{6\theta^3-18\theta^2+12\theta+1}{(\theta-1)^2(2\theta^2-4\theta+1)}$, each for two, three, and four dimensions, respectively.

Example 2.15. Conditional expectation of Y given by $X = x$ for two-dimensional case of Gumbel copula is as the same as provided by [5], i.e.

$$E(Y|X = x)$$

$$= \frac{-\ln G(x)}{G(x)} \int_R y(F(y))^{-1} (-\ln F(y))^{\theta-1} \frac{\exp\left(-\left[(-\ln F(y))^\theta + (-\ln G(x))^\theta\right]^{\frac{1}{\theta}}\right)}{\left[(-\ln F(y))^\theta + (-\ln G(x))^\theta\right]^{\frac{2}{\theta}}}$$

(49)

Example 2.16. For three-dimensional case, the Gumbel 3-copula is defined by

(50)
$$C(u_1, u_2, u_3) = \exp\{-[(-\ln u_1)^\theta + (-\ln u_2)^\theta + (-\ln u_3)^\theta]^{\frac{1}{\theta}}\}$$

Suppose that $u_1 = F(y)$, $u_2 = G_1(x_1)$, and $u_3 = G_2(x_2)$, then the conditional expectation of Y given by X_1 and X_2 is

(51)
$$\begin{aligned}
& E(Y|X_1 = x_1, X_2 = x_2) \\
& = \frac{2\theta^2 - 4\theta + 1}{(\theta-1)^2} \alpha_2(x_1, x_2) \int_R y(F(y))^{-1} (-\ln F(y))^{\theta-1} \alpha_3(y, x_1, x_2) F'(y) dy
\end{aligned}$$

with

$$\alpha_2(x_1, x_2) = \frac{\left[(-\ln G_1(x_1))^\theta + (\ln G_2(x_2))^\theta\right]^{\frac{2}{\theta}}}{\exp\left(-\left[(-\ln G_1(x_1))^\theta + (-\ln G_2(x_2))^\theta\right]^{\frac{1}{\theta}}\right)}$$

$$\alpha_3(y, x_1, x_2) = \frac{\exp\left(-\left[(-\ln F(y))^\theta + (-\ln G_1(x_1))^\theta + (-\ln G_2(x_2))^\theta\right]^{\frac{1}{\theta}}\right)}{\left[(-\ln F(y))^\theta + (-\ln G_1(x_1))^\theta + (\ln G_2(x_2))^\theta\right]^{\frac{3}{\theta}}}$$

3. ESTIMATION PROCEDURES

Suppose that a joint distribution function of the marginal variables $\{X_1, \ldots, X_n\}$ with the parameter spaces of $\{\Omega_1, \ldots, \Omega_n\}$ is written by

(52)
$$H(x_1, \ldots, x_n; \Theta) = C(F_{X_1}(x_1; \Omega_1), \ldots, F_{X_n}(x_n; \Omega_n); \Theta)$$

where $C(\cdot)$ is the copula function and $\Theta = \{\theta^{FGM}, \rho^{Ga}, \{\rho^t, v^t\}, \theta^{Cl}, \theta^{Gb}\}$ is a set of parameter of copula function.

Parameter estimation of the marginal variables and the copula function can be solved by the maximum likelihood estimation (MLE) method. The log-likelihood function of the copula model is given by

$$\mathscr{L}(\Omega_1, \ldots, \Omega_n, \Theta)$$

$$= \log\left(\prod_{i=1}^{m} h(x_{1i},\ldots,x_{ni};\Theta)\right)$$

$$= \log\left(\prod_{i=1}^{m} \frac{\partial^n H(x_{1i},\ldots,x_{ni};\Theta)}{\partial x_1 \ldots \partial x_n}\right)$$

$$= \log\left(\prod_{i=1}^{m} \frac{\partial^n C(F_{X_1}(x_{1i};\Omega_1),\ldots,F_{X_n}(x_{ni};\Omega_n);\Theta)}{\partial x_1 \ldots \partial x_n}\right)$$

$$= \log\left(\prod_{i=1}^{m} [f_{X_1}(x_{1i};\Omega_1)\cdot\ldots\cdot f_{X_n}(x_{ni};\Omega_n)\cdot c(F_{X_1}(x_{1i};\Omega_1),\ldots,F_{X_n}(x_{ni};\Omega_n);\Theta)]\right)$$

$$= \sum_{i=1}^{m} (\log f_{X_1}(x_{1i},\Omega_1) + \ldots + \log f_{X_n}(x_{ni};\Omega_n))$$

$$+ \sum_{i=1}^{m} \log c(F_{X_1}(x_{1i};\Omega_1),\ldots,F_{X_n}(x_{ni};\Omega_n);\Theta)$$

$$= \sum_{i=1}^{m} \log f_{X_1}(x_{1i},\Omega_1) + \ldots + \sum_{i=1}^{m} \log f_{X_n}(x_{ni};\Omega_n)$$

$$(53) \quad + \sum_{i=1}^{m} \log c(F_{X_1}(x_{1i};\Omega_1),\ldots,F_{X_n}(x_{ni};\Omega_n);\Theta)$$

By considering the complexity of parameter estimation using the maximum likelihood method directly due to the number of unknown parameters, the estimation procedure can be carried out in two steps procedure [2, 4, 1]. First, estimate the marginal variable parameter $(\Omega_1,\ldots,\Omega_n)$ and then estimate the parameter of the copula function Θ. Therefore, Eq. (53) can be written as

$$(54) \quad \mathscr{L}(\Omega_1,\ldots,\Omega_n,\Theta) = \sum_{j=1}^{n} \mathscr{L}_{f_j}(\Omega_j) + \mathscr{L}_{c_\Theta}(\Theta)$$

where $\mathscr{L}_{f_j}(\Omega_j), j = 1,\ldots,n$ is the log-likelihood function of the marginal variables and $\mathscr{L}_{c_\Theta}(\Theta)$ is the log-likelihood function of the copula.

The last, the copula parameter Θ can be estimated by maximizing $\mathscr{L}_{c_\Theta}(\Theta)$ and select the best copula by choosing the smallest *AIC* value.

$$(55) \quad \hat{\Theta} = \arg\max_{\Theta} \mathscr{L}_{c_\Theta}(\Theta)$$

4. APPLICATION

In this section, we apply the general formula of the conditional expectation of copulas for higher dimensions to estimate the exchange rate value. We use KRW/USD, CNY/USD, and JPY/USD from 1st January, 2018 to 11th December, 2020. We gather the data from Yahoo Finance. First, we conduct a distribution fitting for the marginal variables. We then estimate the parameter of the copula for FGM, Gaussian, Student-t, Clayton, and Gumbel and select the best copula which describe the relationship between the three marginal variables by selecting the smallest *AIC* value. The last, we use the formula of conditional expectation of copula for three-dimensional case to estimate the value of KRW/USD based on its association with CNY/USD and JPY/USD.
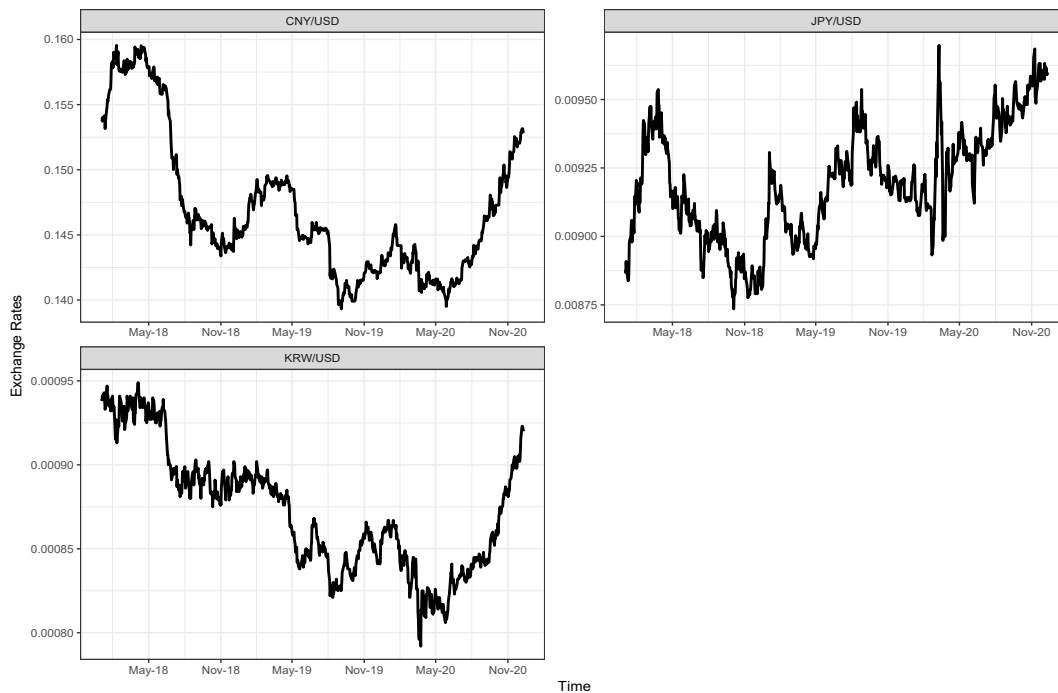
Figure 2. Exchange Rate Data

Figure 2 shows the original data of the exchange rates of KRW, CNY, and JPY against US Dollar. In general, CNY/USD and KRW/USD experience a decreasing trend, while JPY/USD experiences an increasing trend at the period of observations. Table 2 provides the parameter estimates of the marginal distribution of each exchange rate.

Table 2. Parameter Estimates of Marginal Variables

CNY/USD					
Distribution	Parameter Estimates		LL	AIC	BIC
Normal	$\hat{\mu}$	0.1470	2925.6590	-5847.3180	-5838.0250
	$\hat{\sigma}$	0.0054			
Log-Normal	$\hat{\mu}$	**0.1469**	**2938.0190**	**-5872.0380**	**-5862.7460**
	$\hat{\sigma}$	**1.0370**			
Exponential	$\hat{\lambda}$	6.8040	706.4881	-1410.9760	-1406.3300
Gamma	$\hat{\alpha}$	752.1571	2934.0100	-5864.0200	-5854.7270
	$\hat{\beta}$	5117.6646			
Weibull	$\hat{k}$	25.0876	2811.8860	-5619.7710	-5610.4790
	$\hat{\lambda}$	0.1498			
JPY/USD					
Distribution	Parameter Estimates		LL	AIC	BIC
Normal	$\hat{\mu}$	-	-	-	-
	$\hat{\sigma}$	-			
Log-Normal	$\hat{\mu}$	**0.0092**	**5448.0980**	**-10892.2000**	**-10882.9000**
	$\hat{\sigma}$	**1.0225**			
Exponential	$\hat{\lambda}$	108.7794	2840.7780	-5679.5550	-5674.9090
Gamma	$\hat{\alpha}$	2016.9440	5448.0440	-10892.0900	-10882.8000
	$\hat{\beta}$	219401.8300			
Weibull	$\hat{k}$	47.4976	5407.0210	-10810.0400	-10800.7500
	$\hat{\lambda}$	0.0093			
KRW/USD					
Distribution	Parameter Estimates		LL	AIC	BIC
Normal	$\hat{\mu}$	-	-	-	-
	$\hat{\sigma}$	-			
Log-Normal	$\hat{\mu}$	**0.0009**	**6777.1450**	**-13550.2900**	**-13541.0000**
	$\hat{\sigma}$	**1.0427**			
Exponential	$\hat{\lambda}$	1146.2570	4654.0780	-9306.1560	-9301.5090
Gamma	$\hat{\alpha}$	571.7798	6776.1160	-13548.2300	-13538.9400
	$\hat{\beta}$	655419.9853			
Weibull	$\hat{k}$	-	-	-	-
	$\hat{\lambda}$	-			

Based on Table 2, the best distribution for each marginal variable is log-normal distribution. The next step is to identify the structure of dependence of the three marginal variables using copula modeling. Table 3 provides the parameter estimates of the copula functions.

Table 3. Parameter Estimates of Copula Functions

Copula	Parameter Estimates		LL	AIC
FGM	$\hat{\theta}_{12}$	-	-	-
	$\hat{\theta}_{13}$	-		
	$\hat{\theta}_{23}$	-		
Gaussian	$\hat{\rho}_{12}$	**-0.04691**	**559.9**	**-1113.8**
	$\hat{\rho}_{13}$	**0.8593**		
	$\hat{\rho}_{23}$	**-0.21537**		
Student-*t*	$\hat{\theta}_{12}$	-0.0471	559.7	-1111.4
	$\hat{\theta}_{13}$	0.8593		
	$\hat{\theta}_{23}$	-0.2157		
	$\hat{v}$	769.0019		
Clayton	$\hat{\theta}$	0.14	11.65	-21.3
Gumbel	$\hat{\theta}$	1.134	47.62	-93.24

Parameter estimation for FGM copula cannot be done because the value of θ_{12}, θ_{13}, and/or θ_{23} may be out of the range parameter determined based on Proposition 2.5, i.e. $\theta \in \left[-\frac{1}{4}, \frac{1}{2}\right]$. Based on *AIC* value, Gaussian copula is the best copula to model the three-dimensional exchange rate data. The parameter estimates of ρ_{12}, ρ_{13}, and ρ_{23} describe the relationship between CNY/USD-JPY/USD, CNY/USD-KRW/USD, and JPY/USD-KRW/USD, respectively. The negative and positive value of ρ shows that it is in line with the trend of each exchange rate data showed by Figure 2. For example, the value of ρ_{12} is negative because CNY/USD and JPY/USD have different trend direction.

According to the best copula selected, i.e. Gaussian copula, we estimate the value of KRW/USD based on its association with CNY/USD and JPY/USD using Eq. (32). The estimation value of KRW/USD which can be determined by calculating $E(Y|X_1 = x_1, X_2 = x_2)$

where Y represents KRW/USD, while X_1 and X_2 represent CNY/USD and JPY/USD, respectively, is provided in Figure 3.
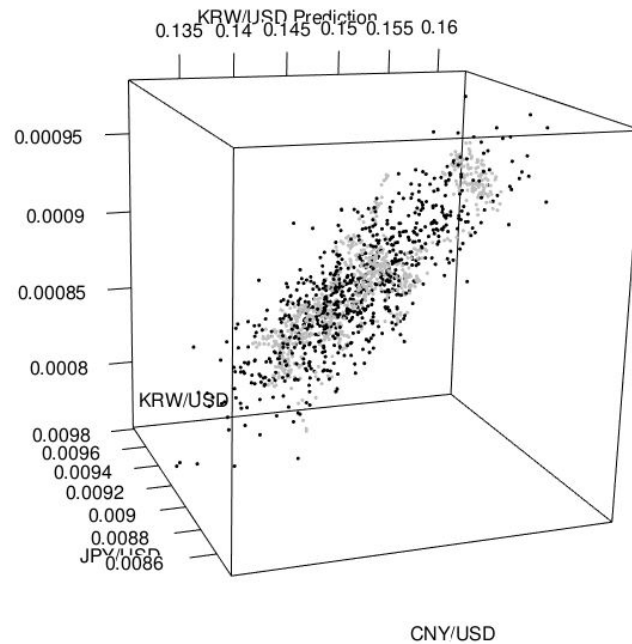
Figure 3. KRW/USD Estimates based on Its Association with CNY/USD and JPY/USD

Gray and black dots in Figure 3 represent the original data and the estimation value of the (CNY/USD,JPY/USD,KRW/USD) pairs, respectively. Distribution value of the estimates are quite close with the original data. It is also indicated by the value of the mean absolute percentage error (MAPE) of the original and estimated data which has a value of 0.0476%. Therefore, we can say that the general formula of the conditional expectation of copulas for higher dimensions can be used for estimation procedure of more than two-dimensional cases.

5. CONCLUSION

We have extended some general formula of conditional expectation of copulas for higher dimensions as the definition of copula-based regression model and its basic properties stretched from Crane and Hoek's work [5]. We obtained the closed-form of the conditional expectation formula for higher dimensions of Farlie-Gumbel-Morgenstren, Gaussian, Student-t, and Clayton copula, and provided some examples of Gumbel copula.

We have also employed the conditional expectation formula of copula for three-dimensional case to estimate the value of KRW/USD exchange rate based on its association with CNY/USD and JPY/USD. The result gives a quite small MAPE of 0.0476%. The estimation results show that the conditional expectation formula of copula for higher dimension can be used to estimate more than two-dimensional case quite accurately. In addition, the visualization shows that the estimation results are distributed around the actual value.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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